TAIL VARIANCE PREMIUM WITH APPLICATIONS FOR ELLIPTICAL PORTFOLIO OF RISKS

BY

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ABSTRACT

In this paper we consider the important circumstances involved when risk managers are concerned with risks that exceed a certain threshold. Such conditions are well-known to insurance professionals, for instance in the context of policies involving deductibles and reinsurance contracts. We propose a new premium called *tail variance premium* (TVP) which answers the demands of these circumstances. In addition, we suggest a number of risk measures associated with TVP. While the well-known tail conditional expectation risk measure provides a risk manager with information about the average of the tail of the loss distribution, *tail variance risk measure* (TV) estimates the variability along such a tail. Furthermore, given a multivariate setup, we offer a number of allocation techniques which preserve different desirable properties (sub-additivity and full-additivity, for instance). We are able to derive explicit expressions for TV and TVP, and risk capital decomposition rules based on them, in the general framework of multivariate elliptical distributions. This class is very popular among actuaries and risk managers because it contains distributions with marginals whose tails are heavier than those of normal distributions. This distinctive feature is desirable when modeling financial datasets. Moreover, according to our results, in some cases there exists an optimal threshold, such that by choosing it, an insurance company minimizes its risk.

KEYWORDS

Premium Calculation Principle, Risk Measure, Risk Capital Allocation, Tail Variance Premium, Tail Standard Deviation Premium, Tail Covariance Premium, Tail Conditional Expectation, Elliptical Distributions.

1. INTRODUCTION

Measuring risk is a necessary precursor to managing it. Increasingly, a major aim of financial regulators around the world is to encourage banks, insurance companies and investment firms to realize the self-assessment of the risks that may threaten their solvency.

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The above trend to risk-based supervision is best exemplified at the international level by the Basel II bank capital adequacy accord that the Basel Committee on Banking Supervision, the body that in effect regulates international banking, intends to bring into effect in the very near future. The European Commission plans to apply Basel II to all banks and investment firms in the European Union. The Commission's plans for the risk-based supervision of EU insurance companies, known as the Solvency II project, will be closely modelled on the principles embodied in Basel II. In the light of this, the tendency toward the so-called risk-based or risk-focused approach seems to be steadily replacing the more traditional regime, in which regulators simply dictate the protective capital levels to banks and insurance companies on a "one size fits all" basis.

Consider risk X to be a random variable with cumulative distribution (cdf) and density (df) functions $F_X(x)$ and $f_X(x)$, respectively. This may refer to the total claims for an insurance company or to the total loss in a portfolio of investment for an individual or institution. A premium principle assigns to the risk X a real number used as a financial compensation for the one who assumes this risk. Consider a situation when, for some reason, we are concerned only with risks that are bigger than a certain threshold, x_q. Such a case is very familiar to actuaries since many insurance policies include a deductible, and reinsurance contracts always involve some level of retention from the ceding insurer.

When dealing with the situations described above, the popular *tail conditional expectation* (TCE), which coincides with the *expected shortfall* (ES) and the *conditional value-at-risk* (CVaR) under the assumption of continuous distributions (see Hürlimann (2003), McNeil, Frei, Embrechts (2005, Lemma 2.16)),

$$TCE_q(X) = E(X \mid X > x_q) \tag{1.1}$$

is very useful in estimating the right-tail risk. It is interpreted as the expected worst possible loss, given that this loss exceeds a particular value x_q. The latter is in general referred to as the q-th quantile or Value-at-Risk (VaR) such that

$$VaR_q(X) = \inf\{x_q : F(x_q) \geq q\}. \tag{1.2}$$

The tail conditional expectation risk measure shares properties that are considered desirable in a variety of situations. For instance, due to the additivity of expectations, TCE allows for a natural decomposition of risk capital among its various constituents.

Expected shortfall has been studied thoroughly by various authors. An incomplete list is: Hürlimann (2001a) considered this risk measure in computing risk capitals for the sums of independent gamma risks; Panjer (2002) examined it in the context of the normal distribution; and Landsman and Valdez (2003, 2005) extended Panjer's results for the broader class of elliptical family and considered TCE for the Exponential Dispersion Models (EDF).

Although (1.1) provides risk experts with some necessary information about the riskiness of the loss distribution tail, very often it is not sufficient. This is especially true in today's competitive and investment-oriented marketplace, which requires that insurance directors exploit all the advantages of investing the risk capitals of their enterprises. Consider the following example.

Example 1. *Let X and Y be Pareto and normal risks respectively possessing expectations $E(X) = 490.5763$ and $E(Y) = 200$ and variances $Var(X) = 215.123^2$ and $Var(Y) = 500^2$. Regardless of the shape of the cdf of X and of Y, the well-known variance premium calculation principle (VP) finds Y to be more dangerous than X for all $\alpha > 0.0014264$, i.e., in such a case*

$$VP(Y) = E(Y) + \alpha Var(Y) > E(X) + \alpha Var(X) = VP(X).$$

The above ordering appears to be counter-intuitive, because the Pareto distribution has a heavier tail than the normal distribution (see Figure 1). Unlike the variance premium, the popular Value-at-Risk risk measure takes into consideration the shape of the cdf of the underlying risks. Let $q = 0.97$ in (1.2), then

$$VaR_q(Y) = 1140.4 > 955.9505 = VaR_q(X).$$

In other words, according to VaR, Y bears more risk than X, which is again somehow not reasonable. Finally, the tail conditional expectation risk measure, which often serves as an alternate to VaR, also fails to produce a useful ordering of X and Y. Indeed,

$$TCE_{0.97}(X) = 1334 = TCE_{0.97}(Y).$$

Therefore a need for a different risk measure is apparent.

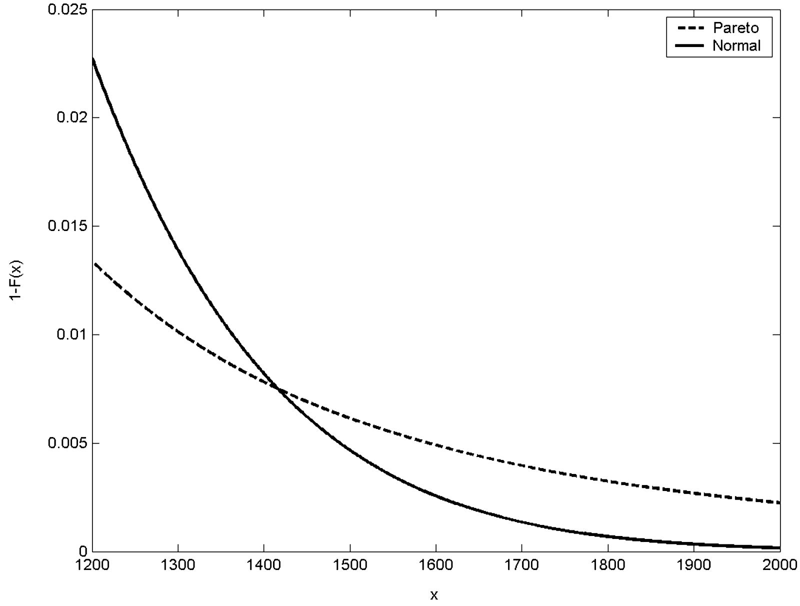

FIGURE 1: The decumulative distribution functions of Pareto and normal risks.

In order to resolve the inconvenience described in Example 1 and many other problems, including classical risk ordering process and conditional Chebyshev's inequality, we propose a measure of variability on the right tail $\{X > x_q\}$. We refer to this measure as *tail variance* (TV), and it is merely the conditional variance of the risk X, i.e.,

$$TV_q(X) = Var(X \mid X > x_q) = E\big((X - \tau_q(X))^2 \mid X > x_q\big), \tag{1.3}$$

where $\tau_q(X) = TCE_q(X)$. As (1.3) possesses the following property

$$TV_q(X) = \inf_c E\big((X-c)^2 \mid X > x_q\big),$$

it presents a natural measure of dispersion of X along the right tail. Moreover, in order to calculate the tail variance for X, only the information about its decumulative distribution function (ddf) $\overline{F}_X(x) = 1 - F_X(x)$, $x \geq x_q$ is required.

We note that Valdez (2004) suggested the *tail conditional variance* (TCV) risk measure for measuring the variability of risk X along the right tail of its distribution. However, TCV, which is given by

$$TCV_q(X) = E\big((X - E(X))^2 \mid X > x_q\big), \tag{1.4}$$

does not indeed imply the right-tail deviation of X. In effect, (1.4) simply projects the squared deviance of X from $E(X)$ to the right tail. In the light of this, equation (1.4) is unable to serve as a measure of the tail variability. We also note that it is always positive, i.e.,

$$TCV_q(X) = TV_q(X) + \big(TCE_q(X) - E(X)\big)^2 > 0,$$

and this is notwithstanding the very definition of a measure.

Further, in the context of Example 1, one has that

$$TCV_{0.97}(Y) = 1286100 > 847700 = TCV_{0.97}(X),$$

which again leads to a counter intuitive ordering of X and Y. At the same time, the risk measure in equation (1.3) provides a proper ordering of these risks, i.e., the tail variance of X is much greater than the tail variance of Y:

$$TV_{0.97}(X) = 126400 > 60.8216 = TV_{0.97}(Y),$$

as one would actually expect.

Although searching for original ways to quantify insurance and financial risks is a very important issue, the subsequent application of these theoretical approaches to real world problems is not less essential. In this paper, we show that for the univariate normal distributions tail variance is proportional to the variance of X, and is of the following form

$$TV_q(X) = Var(X)\,[1 + h(z_q)(z_q - h(z_q))], \tag{1.5}$$

where $h(z) = \frac{\varphi(z)}{1-\Phi(z)}$ is the hazard function corresponding to a standard normal $N(0,1)$ random variable ($\varphi(z)$ and $\Phi(z)$ are the df and the cdf of $N(0,1)$).

In the more complicated elliptical frameworks, the expression for $TV_q(X)$ can still be formulated. However, the hazard function of the standardized (spherical) random variable $Z = (X-\mu_X)/\sigma_X$ should be distorted. Such a distortion is supplied by using the spherical random variable Z^* associated with the underlying elliptical family, i.e.,

$$h_{Z,Z^*}(z) = \frac{f_{Z^*}(z)}{\overline{F}_Z(z)}. \tag{1.6}$$

The associated random variable Z^* was introduced in Landsman and Valdez (2003) and further developed in Landsman (2006). The intuitive interpretation of Z^* lies in the distortion it provides when extending many well-known results, obtained for normal distributions, to the non-normal elliptical context. We stress that in the case of normal distributions the distortion disappears, Z^* equals Z in distribution, and therefore $h_{Z,Z^*}(z) = h(z)$. A more precise definition of Z^* will be given in Section 4.

In the light of the above, (1.5) extends to

$$TV_q(X) = Var(X)\big[r(z_q) + h_{Z,Z^*}(z_q)\big(z_q - h_{Z,Z^*}(z_q)\cdot\sigma_Z^2\big)\big], \tag{1.7}$$

where

$$r(z) = \frac{\overline{F}_{Z^*}(z)}{\overline{F}_Z(z)} \tag{1.8}$$

is the distorted ratio. For the normal case $r(z) = 1$, since $Z^* \stackrel{D}{=} Z$ (no distortion), and therefore (1.5) follows from (1.7).

In a more complicated situation pertaining to an insurance company with n business lines, which results in considering an n-variate random vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)^T$ with some dependence structure, we are concerned with the contribution of the variability of each marginal risk X_k, $k = 1, 2, \ldots, n$ to the tail variance of the total sum $S = X_1 + X_2 + \cdots + X_n$. The expression

$$TV_q(X_k|S) = Var(X_k|S > s_q) = E\big((X_k - \tau_q(X_k|S))^2\,|\,S > s_q\big), \tag{1.9}$$

quantifies this phenomenon, where

$$\tau_q(X_k|S) = TCE_q(X_k|S) = E(X_k|S > s_q). \tag{1.10}$$

We investigate (1.9) in the general context of the multivariate elliptical distributions. The latter has a long history of numerous applications in the analysis of both non-life insurance and financial data: Panjer (2002) considers the

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multivariate normal distribution, one of the most important members of the elliptical class, as an appropriate model for the losses of the real insurance company; Wang (2002) considers intra-company allocation of the cost of capital in the framework of the multivariate normal distributions; Waldez and Chernih (2003) extend Wang's risk capital allocation to the elliptical family; and Hürlimann (2001b, Section 5) explores some elliptical distributions for fitting non-life insurance data. In general, the great importance of elliptical distributions in non-life insurance is explained by the fact that the distributions of the underlying risks generally possess tails, which are more leptocurtic than those of normal distributions. Owen and Rabinovitch (1982) seem to have been the first to point out that the class of elliptical distributions extends the Tobin (1958) separation theorem, Bawa's (1975) rule of ordering uncertain projects, and Ross's (1978) mutual fund separation theorems, and they applied the elliptical setup to the capital asset pricing modeling (CAPM). Landsman and Sherris (2005) suggested a model for pricing asset and insurance risks in incomplete markets using prices for traded assets and based on elliptical, in particular, multivariate Student-t, distributions.

In this paper we demonstrate that when $\mathbf{X} = (X_1, X_2, ..., X_n)^T$ has an elliptical dependence structure, the tail variance risk measure equals

$$TV_q(X_k|S) = Var(X_k)\,[r(z_{S,q}) + \gamma(z_{S,q}) \cdot \rho_{k,S}^2], \tag{1.11}$$

where $\rho_{k,S} = \frac{\sigma_{k,S}}{\sigma_k \sigma_s}$ is the correlation coefficient and $z_{S,q} = (x_{S,q} - \mu_S)/\sigma_S$. We would like to draw the reader's attention to the fact that the contribution of the variability of the marginal risk X_k is stipulated by its own variance and the squared correlation between X_k and the aggregate risk S.

We also show that if $\mathbf{X} \backsim N_n(\boldsymbol{\mu},\Sigma)$, then (1.9) reduces to

$$TV_q(X_k|S) = Var(X_k)\left[1 + \frac{\varphi\left(z_{S,q}\right)}{1-\Phi\left(z_{S,q}\right)}\left(z_{S,q} - \frac{\varphi\left(z_{S,q}\right)}{1-\Phi\left(z_{S,q}\right)}\right) \cdot \rho_{k,S}^2\right].$$

While (1.3) enables one to compute capital requirements in terms of variation for some financial institution, (1.9) is useful when the uncertainty has different sources and the decomposition of the total level of such uncertainty to these sources is important. Although the decomposition form of tail variance (1.9) is indeed a natural measure of the contribution of the variability of risk X_k to the total risk measure of an enterprise, one may encounter some difficulty applying it as a basis for risk capital allocation. This is because such an allocation, when derived from the expression in (1.9), is in general not additive (although the squared root of $TV_q(X_k|S)$ preserves sub-additivity, see Section 2), i.e.,

$$TV_q(S) \neq \sum_{k=1}^{n} TV_q(X_k|S)$$

but

$$\sqrt{TV_q(S)} \leq \sum_{k=1}^{n} \sqrt{TV_q(X_k|S)}.$$

In order to resolve this sometimes undesirable property (balance sheet computations must sum up) and thus provide a solution for those who are concerned with the full-additivity of the allocation rule, we offer the so-called *tail covariance allocation*. The latter is defined as follows

$$TCov_q(X_k|S) = Cov(X_k, S|S > s_q) = E((X_k - \tau_q(X_k|S))(S - \tau_q(S))|S > s_q), \quad (1.12)$$

where $\tau_q(X_k|S)$ is given in (1.10) and $\tau_q(S) = E(S|S > s_q)$. Moreover, notice that (1.12) is indeed additive

$$\begin{aligned}\sum_{k=1}^{n} TCov_q(X_k|S) &= \sum_{k=1}^{n} Cov\left(X_k, S \middle| S > s_q\right) \\ &= Cov\left(\sum_{k=1}^{n} X_k, S \middle| S > s_q\right) = Var\left(S \middle| S > s_q\right) = TV_q(S).\end{aligned}$$

Careful investigation of the above risk measures and the desire to combine the information given by the tail conditional expectation, whose importance is well known, and the tail variance introduced in this paper, stimulated us to propose the following premium principle. *Tail variance premium* (TVP), defined as

$$TVP_q(X) = TCE_q(X) + \alpha \cdot TV_q(X), \quad (1.13)$$

where α is some non-negative constant, considers the important case when insurance directors are concerned only with significant "right-tail" risks, i.e., risks that are bigger than x_q. Similarly to variance premium, which is based on the net premium principle, TVP builds on TCE, but at the same time takes into account the risk load, which is proportional to the conditional variance of X. In other words (1.13) extends the classical variance premium to the so-called conditional context, and it provides a kind of stochastic ordering for relatively large losses. It is also able to order different risks with equal first and second moments, although the variance premium fails to perform this task. We note that TVP builds on the information of the right tail of the loss distribution only, and therefore it seems to provide a fair response to the situations, when the decision makers are concerned with risks that exceed a certain threshold. The latter has received extensive consideration both in the theoretical and practical actuarial sciences.

In the same manner, we introduce the tail variance premium for the cases when the decomposition of the total risk to its constituents is needed:

$$TVP_q(X_k|S) = TCE_q(X_k|S) + \alpha \cdot TV_q(X_k|S). \quad (1.14)$$

Certainly, both (1.13) and (1.14) are particularly useful when the variability along the right tail is crucial for decision makers.

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The rest of the paper is organized as follows. We investigate in detail the proposed premium principle in Section 2. Section 3 provides a preliminary discussion of elliptical distributions. In Section 4, we derive the expressions for tail variance risk measure in the univariate context, and we advance these expressions to the multivariate framework in Section 5. Section 6 concludes the paper and we provide the proofs of our results in the Appendix.

2. Tail Variance Premium

In order to determine a premium for a risk, it is necessary to convert the random loss into financial terms. Both the probability distribution of the losses and a pricing principle are required. In this section we introduce the tail variance premium principle and we underline its most important properties.

Definition 1. *Tail variance premium is*

$$TVP_q(X) = TCE_q(X) + \alpha \cdot TV_q(X), \tag{2.1}$$

where $\alpha \geq 0$, x_q *is defined in (1.2)*, $TCE_q(X)$*, and* $TV_q(X)$ *are given in (1.1) and (1.3), respectively.*

The above premium satisfies some important properties. While the first two properties are traditional and well explained in Kaas et al. (2001), the third has not been much studied, and would seem to be very useful in the case of reinsurance contracts and policies with deductibles.

1. *Non-negative loading.*

$$TVP_q(X) \geq E(X).$$

   Tail variance premium is not smaller than the well-known net premium.

2. *Translation invariance.* If c is some constant risk, then

$$TVP_q(X + c) = TVP_q(X) + c.$$

   Raising the risk by some constant amount c increases the premium by the same amount. Kaas et al. (2001) refer to this property as *consistency*.

3. *Tail parity*. We call X and Y tail equivalent if some q exists such that $\overline{F}_X(x_t) = \overline{F}_Y(x_t)$ for every level $t \geq q$, and then

$$TVP_t(X) = TVP_t(Y).$$

   Tail variance premium is dependent only on the tail of the loss distribution. Parity of these tails implies equality of tail variance premiums.

Remark 1. *For the special loss function* $L(x,P) = x(x-P)^2$*, the tail variance premium (2.1) with* $\alpha = 1/TCE_X(x_q)$ *minimizes the expected loss along the right tail* $\{X > x_q\}$*, i.e.,*

$$TVP_q(X) = TCE_q(X) + \frac{TV_q(X)}{TCE_q(X)} = \arg\inf_P E(L(X,P) \mid X > x_q). \quad (2.2)$$

In fact, after differentiating in P under the integral sign of the expected loss, one straightforwardly obtains TVP as the solution of the equation

$$\int_{x_q}^{\infty} x(x-P)\,dF_X = 0.$$

Similarly to Definition 1 we may propose the *tail standard deviation premium*.

Definition 2. *Tail standard deviation premium is*

$$TSDP_q(X) = TCE_q(X) + \alpha \cdot \sqrt{TV_q(X)}, \quad (2.3)$$

where $a \geq 0$, x_q *is defined in (1.2),* $TCE_q(X)$ *and* $TV_q(X)$ *are given in (1.1) and (1.3), respectively.*

Certainly, the tail standard deviation premium shares all three properties of the tail variance premium. However, we draw the reader's attention to the fact that it satisfies the so-called *positive homogeneity* property as well.

4. *Positive homogeneity*. For any risk X and any positive constant β

$$TSDP_q(\beta X) = \beta \cdot TSDP_q(X).$$

   If the risk exposure of a company is proportionally increased/decreased then its risk measure must also increase/decrease correspondingly.

Further, let us consider an n-variate random vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)^T$, where each marginal random variable X_k represents a risk associated with k-th business line for an insurance company or a loss from the k-th asset in a portfolio of investment for an individual or an enterprise. The aggregate risk or loss is then $S = X_1 + X_2 + \cdots + X_n$.

Definition 3. *Tail variance premium for the marginal risk* X_k *is*

$$TVP_q(X_k|S) = TCE_q(X_k|S) + \alpha \cdot TV_q(X_k|S), \quad (2.4)$$

and consequently the tail standard deviation premium is

$$TSDP_q(X_k|S) = TCE_q(X_k|S) + \alpha \cdot \sqrt{TV_q(X_k|S)}, \quad (2.5)$$

where $\alpha \geq 0$, s_q *follows from (1.2),* $TCE_q(X_k|S)$ *and* $TV_q(X_k|S)$ *are defined in (1.10) and (1.9), respectively.*

Equations (2.4) and (2.5) may also serve as allocation rules, although not fully additive ones. Note that, for instance, (2.5) preserves the sub-additivity property, i.e.,

5. *Sub-additivity*. For any random risks X, Y and the aggregate sum $S = X + Y$

$$TSDP_q(S) \le TSDP_q(X|S) + TSDP_q(Y|S).$$

Nothing is gained by disaggregation.

Moreover, we define *tail covariance premium* (TCovP) for situations when the full-additivity of the allocation rule is crucial.

Definition 4. *Tail covariance premium is*

$$TCovP_q(X_k|S) = TCE_q(X_k|S) + \alpha \cdot TCov_q(X_k|S), \tag{2.6}$$

where α is again some non-negative constant, s_q is given in (1.2), and $TCE_q(X_k|S)$ and $TCov_q(X_k|S)$ are defined in (1.10) and (1.12), respectively.

In current research we found an appealing way to characterize the proposed premium principles in the general frameworks of elliptical distributions. The next section provides a brief discussion of this family.

3. The Class of Elliptical Distributions

The class of elliptical distributions provides a rich range of symmetrical multivariate distributions, which are becoming widely popular in actuarial sciences and finance. Many members of this class are more leptokurtic than the normal distributions, and this property allows one to model tails that are frequently observed in financial data (see Embrechts et al. (2001)). A helpful and extensive discussion of these distributions may be found in Fang et al. (1990).

Let Ψ_n be a class of functions $\psi(t): [0,\infty) \to \mathbf{R}$ such that function $\psi\left(\sum_{i=1}^n t_i^2\right)$ is an n dimensional characteristic function (Fang et al., 1987). It is clear that

$$\Psi_n \subset \Psi_{n-1} \cdots \subset \Psi_1.$$

Further, consider an n-dimensional random vector $\mathbf{X} = (X_1, X_2, ..., X_n)^T$.

Definition 5. *The random vector* $\mathbf{X}$ *has a multivariate elliptical distribution, written as* $\mathbf{X} \backsim E_n(\boldsymbol{\mu}, \Sigma, \psi)$, *if its characteristic function can be expressed as*

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \exp(i\mathbf{t}^T\boldsymbol{\mu})\,\psi(\tfrac{1}{2}\mathbf{t}^T\Sigma\mathbf{t}) \tag{3.1}$$

for some column-vector $\boldsymbol{\mu}$, *$n \times n$ positive-defined matrix* $\boldsymbol{\Sigma} = \left\|s_{ij}\right\|_{i,j=1}^n$, *and for some function $\psi(t) \in \Psi_n$, which is called the characteristic generator.*

In general, from $\mathbf{X} \backsim E_n(\boldsymbol{\mu}, \Sigma, \psi)$ it does not follow that $\mathbf{X}$ has a density $f_{\mathbf{X}}(\mathbf{x})$, but if that density does exist, it has the following form

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\Sigma|}} g_n\left(\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right). \tag{3.2}$$

In the above equation $g_n(\cdot)$ is called the density generator and the normalizing constant c_n is

$$c_n = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left(\int_0^\infty x^{n/2-1} g_n(x)\,dx\right)^{-1}, \tag{3.3}$$

which is subject to the convergence of the integral

$$\int_0^\infty x^{n/2-1} g_n(x)\,dx < \infty. \tag{3.4}$$

While Definition 5 presents the elliptical class in terms of the characteristic generator, one can similarly introduce the elliptical distributions by the density generator and then write $\mathbf{X} \backsim E_n(\boldsymbol{\mu}, \Sigma, g_n)$.

From (3.1), it follows that, if $\mathbf{X} \backsim E_n(\boldsymbol{\mu}, \Sigma, g_n)$, A is some $m \times n$ matrix of rank $m \leq n$ and $\mathbf{b}$ some m-dimensional column-vector, then

$$A\mathbf{X} + \mathbf{b} \backsim E_m(A\boldsymbol{\mu} + \mathbf{b}, A\Sigma A^T, g_m). \tag{3.5}$$

In other words, any linear combination of elliptical distributions is another elliptical distribution with the same characteristic generator ψ or from the same sequence of density generators $g_1, \ldots, g_n$, corresponding to ψ.

The following condition guarantees the existence of the mean

$$\int_0^\infty g_1(x)\,dx < \infty \tag{3.6}$$

and then the mean vector for $\mathbf{X} \backsim E_n(\boldsymbol{\mu}, \Sigma, g_n)$ is $E(\mathbf{X}) = \boldsymbol{\mu}$. Additionally, the next condition guarantees the existence of the covariance matrix

$$\left|\psi'(0)\right| < \infty \tag{3.7}$$

and the former is equal to

$$Cov(\mathbf{X}) = -\psi'(0)\,\Sigma \tag{3.8}$$

(Cambanis et al., 1981). Afterwards, the characteristic generator can be chosen such that

$$\psi'(0) = -1 \tag{3.9}$$

and one automatically obtains

$$Cov(\mathbf{X}) = \Sigma.$$

For the detailed list of examples and properties of particular members of the elliptical class of distributions (normal, Student-t, Logistic and more) see Fang et al. (1990).

4. Tail Variance and Tail Variance Premium for Univariate Elliptical Risks

In this section, we develop tail variance formulas for univariate elliptical distributions, which as a matter of fact coincide with the class of symmetric distributions on the real line $\mathbf{R}$. Recall that we denote by x_q the q-th quantile of the loss distribution $F_X(x)$. As we are interested in considering the tails of symmetric distributions, we suppose that $q > 1/2$. Then clearly

$$x_q > \mu. \tag{4.1}$$

Now suppose $g(x)$ is a non-negative function on $[0,\infty)$, satisfying the following condition

$$\int_0^\infty x^{-1/2} g(x)\,dx < \infty.$$

Then (see Section 3) $g(x)$ can be a density generator of an univariate elliptical distribution of the random variable $X \backsim E_1(\mu,\sigma^2,g)$ whose density can be expressed as

$$f_X(x) = \frac{c}{\sigma}\, g\left(\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right), \tag{4.2}$$

where c is the normalizing constant. (We sometimes omit the dimension index, when the univariate density generator is considered).

Note that, because X has an elliptical distribution, the standardized random variable $Z = (X-\mu)/\sigma$ will have a standard elliptical (often called spherical) distribution function

$$F_Z(z) = c\int_{-\infty}^{z} g\left(\tfrac{1}{2}u^2\right) du$$

with mean 0 and variance

$$\sigma_Z^2 = 2c\int_0^\infty u^2 g\left(\tfrac{1}{2}u^2\right) du = \psi'(0),$$

if condition (3.7) holds. Furthermore, if the generator of the elliptical family is chosen such that condition (3.9) holds, then $\sigma_Z^2 = 1$.

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In this paper we assume that the covariance matrix of Z is finite; then Landsman and Valdez (2003) showed that

$$f_{Z^*}(z) = \frac{1}{\sigma_Z^2}\,\overline{G}\left(\tfrac{1}{2}z^2\right) \tag{4.3}$$

is a density of some spherical random variable Z^*. Here

$$\sigma_Z^2 = Var(Z) \tag{4.4}$$

and

$$\overline{G}(x) = c\int_x^{\infty} g(u)\,du. \tag{4.5}$$

Let us notice that up to a probability space, (4.3) defines the random variable Z^* uniquely up to the class of equivalences (see Loève (1977), Chapters 10.1, 10.2).

A significant number of important results obtained for the normal family of distributions, such as the tail conditional expectation formulas, the risk capital allocation based on it and Stein's Lemma, which is of special interest in financial economics and in actuarial sciences for its important application to capital asset pricing models (CAPM) (see Panjer et al. (1998), Sect. 4.5.), can be generalized to the elliptical context, and Z^* plays an important role in performing this generalization. The latter indicates the distortion from the normal distributions in the more complicated elliptical cases.

Let us recall that we denote by

$$r(z) = \frac{\overline{F}_{Z^*}(z)}{\overline{F}_Z(z)} \tag{4.6}$$

and

$$h_{Z,Z^*}(z) = \frac{f_{Z^*}(z)}{\overline{F}_Z(z)} \tag{4.7}$$

the elliptical distortion ratio function and the elliptical distorted hazard function, respectively. The following theorem derives the expression for the tail variance risk measure in the univariate elliptical case.

Theorem 1. *Let* $X \backsim E_1(\mu, \sigma^2, g)$. *Under condition (3.7), the tail variance of* X *is given by*

$$TV_q(X) = Var(X)\,[r(z_q) + \gamma(z_q)], \tag{4.8}$$

where

$$\gamma(z_q) = h_{Z,Z^*}(z_q)\,(z_q - h_{Z,Z^*}(z_q)\cdot\sigma_Z^2), \tag{4.9}$$

and $z_q = (x_q - \mu)/\sigma$.

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Corollary 1. *Under the conditions in Theorem 1, the tail variance premium for the univariate elliptical distributions is*

$$
\begin{aligned}
&TVP_q(X) \\
&= E(X) + \tfrac{1}{\sigma} h_{Z,Z^*}(z_q) \cdot Var(X) + \alpha \cdot Var(X)\,[r(z_q) + \gamma(z_q)] \\
&= E(X) + Var(X)\,\left[\tfrac{1}{\sigma} h_{Z,Z^*}(z_q) + \alpha \cdot (r(z_q) + \gamma(z_q))\right],
\end{aligned}
$$

where $r(z_q)$ and $\gamma(z_q)$ follow from (4.6) and (4.9), correspondingly.

We now illustrate Theorem 1 by considering useful examples for such well-known symmetric distributions as normal and student-t.

1. *Normal Distribution*. Let $X \backsim N(\mu, \sigma^2)$ so that the function in (4.2) has the form $g(u) = \exp(-u)$. Then

$$f_{Z^*}(z) = \frac{1}{\sigma_z^2} \overline{G}\left(\tfrac{1}{2} z^2\right) = \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2} z^2\right) = \varphi(z),$$

i.e., in the context of normal distributions $Z^* \overset{D}{=} Z$, and therefore the elliptical distortion ratio function and the elliptical distorted hazard function are

$$r(z) = 1 \tag{4.10}$$

and

$$h_{Z,Z^*}(z) = \frac{\varphi(z)}{\overline{F}_Z(z)}. \tag{4.11}$$

Finally, the tail variance risk measure in this case is

$$TV_q(X) = Var(X)\left[1 + \frac{\varphi(z_q)}{1 - \Phi(z_q)}\left(z_q - \frac{\varphi(z_q)}{1 - \Phi(z_q)}\right)\right].$$

The next table enables a comparison of the various known measures of risk and tail variance premium proposed in this paper. As a basis, we take some

TABLE 1

COMPARISON OF TCE, TV AND TVP ($\alpha = 0.2$)

q	x_q	$TCE_X(x_q)$	$TV_X(x_q)$	$TVP_X(x_q)$
0.5	500	525.2313	363.3802	597.9074
0.75	521.3292	540.1959	241.6370	588.5233
0.9	540.5262	555.4974	169.1352	589.3245
0.95	552.0148	565.2287	138.0765	592.8440
0.975	561.9795	573.9278	116.6874	597.2653
0.999	597.7217	606.4767	67.7949	620.0357

risk X distributed normally with $\mu = 500$ and $\sigma^2 = 1000$, the parameter α equals 0.2.

From the above table, one can deduce that for the chosen α, TVP is not a monotonic function in q. Actually, while the well-known TCE and VaR always increase, when one moves along the right tail of the risk distribution, TVP decreases at the very beginning of such a movement. This phenomenon is easily explained by the opposite directions of $E(X|X > x_q)$ and $Var(X|X > x_q)$. In fact, for small q's, when the influence of the conditional variance is very significant, TVP goes down; at the same time, for relatively high probabilities, the conditional expectation takes over and therefore the tail variance premium begins to rise.

As for the insurance industry, our result seems to state an important fact. Assuming that the influence of tail variance is substantial, there exists some optimal q and therefore a quantile x_q that minimizes the value of TVP. In other words, an insurance company may bring its risk to minimum by choosing the x_q above to be the deductible in a policy with a deductible, or the retention level in the context of reinsurance contracts.

The next plot compares TVP's having different α's with other well-known risk measures.

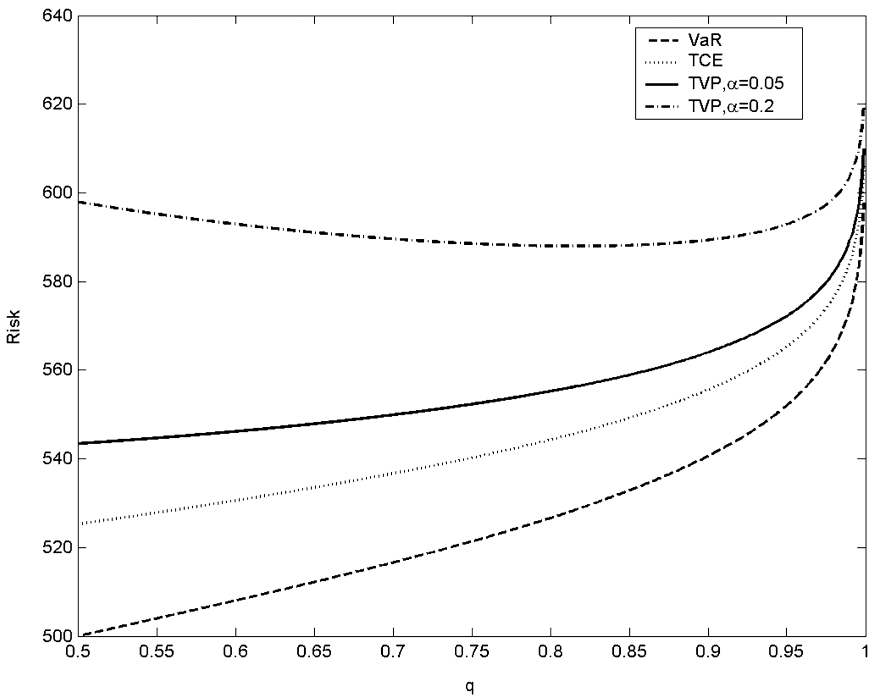

FIGURE 2: Comparison of *VaR*, *TCE* and *TVP*.

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2. *Generalized Student-t Distribution (GST)*. An elliptical random variable X is distributed *Student-t* with some power parameter $p > \frac{1}{2}$, i.e., $X \backsim t(\mu, \sigma^2 : p)$, if the density generator of X can be expressed as

$$g(u) = \left(1 + \frac{u}{k_p}\right)^p.$$

Therefore

$$\overline{G}(u) = c_p \int_u^{\infty} g(t)\, dt = c_p \frac{k_p}{p-1} \left(1 + \frac{u}{k_p}\right)^{-p+1}, \; p > 1.$$

Here, we denote the normalizing constant by c_p with the subscript p to emphasize that it depends on the parameter p. Recall that

$$c_p = \frac{\Gamma(p)}{\sqrt{2\pi k_p}\, \Gamma(p - 1/2)}. \tag{4.12}$$

Bian and Tiku (1997) and MacDonald (1996) suggest putting $k_p = (2p-3)/2$ if $p > 3/2$ to obtain the so-called Generalized Student-t (GST) univariate distribution with density

$$f_X(x) = \frac{1}{\sigma \sqrt{2k_p}\, B(1/2, p - 1/2)} \left[1 + \frac{(x-\mu)^2}{2k_p \sigma^2}\right]^{-p},$$

where $B(\cdot, \cdot)$ is the beta function. This parameterization leads to the very important property that $Var(X) = \sigma^2$, i.e., $\sigma_Z^2 = 1$. Recall that in the context of the GST family we have

$$k_p = \begin{cases} \dfrac{2p-3}{2}, & \text{if } p > 3/2 \\ \dfrac{1}{2}, & \text{if } 1/2 < p \le 3/2 \end{cases}. \tag{4.13}$$

In the case when $1/2 < p \le 3/2$ the variance does not exist and therefore TVP is inapplicable. Let us denote by $t(z : p) = f_Z(z)$ and $\overline{T}(z : p) = \overline{F}_Z(z)$ the density and the tail functions of the standardized student random variable Z. The expressions for the density of the random variable Z^* associated with Z readily follow after considering

$$\overline{G}_Z\left(\frac{1}{2} z^2\right) = \frac{c_p}{c_{p-1}} \frac{k_p}{p-1}\, t\left(\sqrt{\frac{k_{p-1}}{k_p}}\, z : p - 1\right) \tag{4.14}$$

and

$$\frac{c_p}{c_{p-1}} = \frac{\Gamma(p)\,\Gamma(p-3/2)}{\Gamma(p-1/2)\,\Gamma(p-1)}\sqrt{\frac{k_{p-1}}{k_p}} = \frac{(p-1)}{(p-3/2)}\sqrt{\frac{k_{p-1}}{k_p}}. \tag{4.15}$$

Then

$$f_{Z^*}(z) = \begin{cases} \sqrt{\frac{1}{2p-3}}\, t\left(\sqrt{\frac{1}{2p-3}}\, z : p-1\right), & \text{if } 3/2 < p \le 5/2 \\ \sqrt{\frac{2p-5}{2p-3}}\, t\left(\sqrt{\frac{2p-5}{2p-3}}\, z : p-1\right), & \text{if } p > 5/2 \end{cases}$$

and the distorted hazard functions are

$$h_{Z,Z^*}(z) = \begin{cases} \sqrt{\frac{1}{2p-3}}\, t\left(\sqrt{\frac{1}{2p-3}}\, z : p-1\right) / \overline{T}(z:p), & \text{if } 3/2 < p \le 5/2 \\ \sqrt{\frac{2p-5}{2p-3}}\, t\left(\sqrt{\frac{2p-5}{2p-3}}\, z : p-1\right) / \overline{T}(z:p), & \text{if } p > 5/2 \end{cases}$$

The formulas for the tail variance risk measure are now obtained by straightforward substitution.

The next two figures relate the distorted hazard function $h_{Z,Z^*}(z)$ with the power parameter p for different q's. We would like to draw the reader's attention

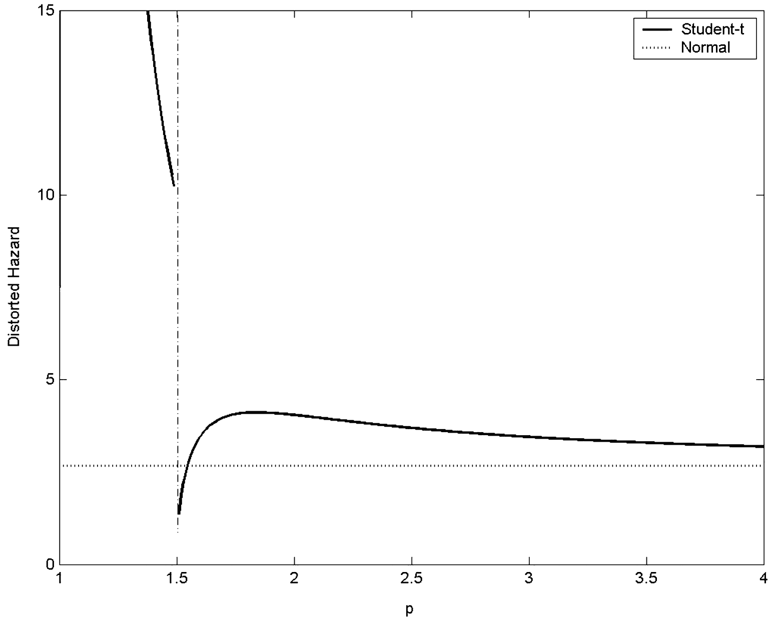

FIGURE 3: The Relation Between the Distorted Hazard $h_{Z,Z^*}(z_q)$ and the Parameter p for the GST Distributions, $q = 0.99$.

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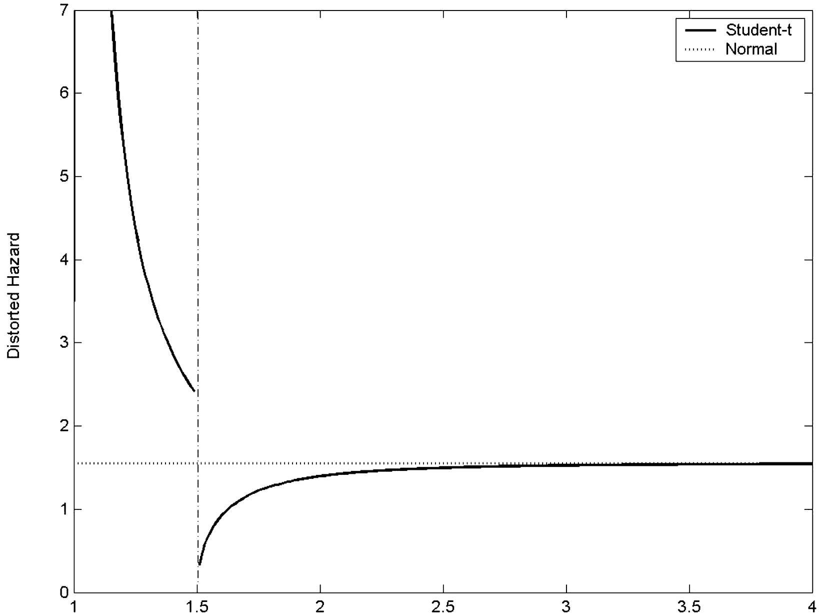

FIGURE 4: The Relation Between the Distorted Hazard $h_{Z,Z^*}(z_q)$ and the Parameter p for the GST Distributions, $q = 0.85$.

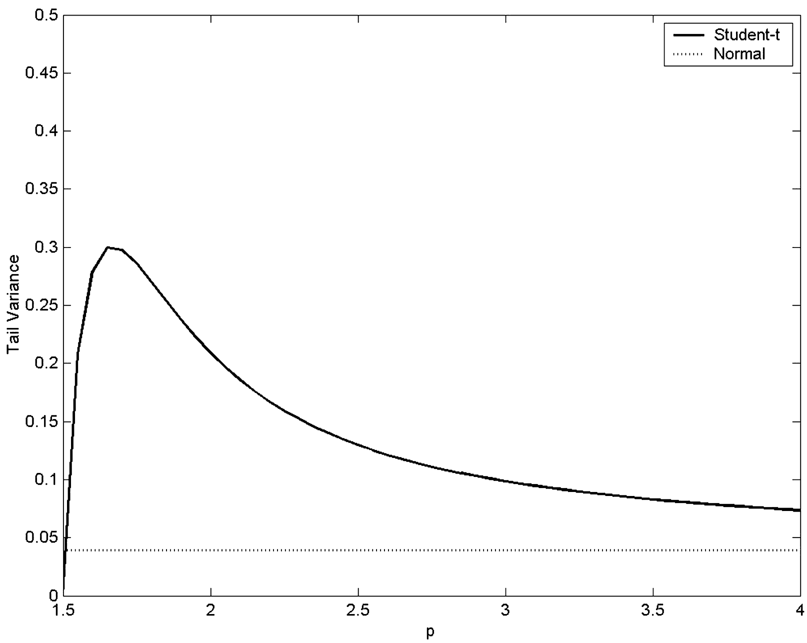

FIGURE 5: The Relation Between Tail Variance Risk Measure and various values of the Parameter p for GST Distributions, $q = 0.85$.

to the fact that, according to classical TCE risk measure, GST distributions are riskier than normal distributions for bigger q's, while the opposite is true for relatively small q's. For instance, Figure 4 implies that for $q = 0.85$ the normal $N(0,1)$ is riskier than $t(0,1:p)$ for every $p > 3/2$ and this in spite of the fact that the tail of the last is heavier.

On the other hand, the tail variance risk measure finds Student-t riskier than standard normal distribution for $q = 0.85$. (See Figure 5)

5. TV Risk Measure and TVP for Multivariate Elliptical Distributions

Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)^T$ be a multivariate elliptical vector, i.e., $\mathbf{X} \backsim E_n(\boldsymbol{\mu}, \Sigma, g_n)$, here $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)^T$, $\Sigma = \|\sigma_{ij}\|_{i,j=1}^{n}$ is some positive defined matrix and g_n is the density generator.

Let $Z = (X_1 - \mu_1)/\sigma_1$ be a standardized univariate marginal random variable, whose distribution does not depend on μ_1 and σ_1. Embrechts and Landsman (2004) introduced an absolutely continuous measure P^* and the measurable map $Z_n^* : (\Omega, \mathcal{F}, P^*) \to (\mathbf{R}^n, \mathcal{B}^{*n})$ from a measurable space $(\Omega, \mathcal{F}, P^*)$ into the n-dimensional Borel space $(\mathbf{R}^n, \mathcal{B}^{*n})$, associated with the n-variate elliptical family as follows

$$dP^* = f_{\mathbf{Z}_n^*}(\mathbf{z})\, d\mathbf{z},$$

where $\mathbf{z}^T = (z_1, z_2, \ldots, z_n) \in \mathbf{R}^n$, $d\mathbf{z} = (dz_1, dz_2, \ldots, dz_n)$ and under condition (3.7)

$$f_{\mathbf{Z}_n^*}(\mathbf{z}) = \frac{1}{\sigma_Z^2}\, \overline{G}_n\left(\frac{1}{2}\mathbf{z}^T\mathbf{z}\right)$$

is the n-variate density, i.e., P^* is a probability measure and $\mathbf{Z}_n^*$ is a random vector. Note that equivalently to (4.5), we introduce

$$\overline{G}_n(x) = c_n \int_x^{\infty} g_n(u)\, du. \tag{5.1}$$

Let $\mathbf{Z}_{n-1}$ and $\mathbf{Z}_{n-1}^*$ be $(n-1)$, dimensional marginals of $\mathbf{Z}_n$ and $\mathbf{Z}_n^*$, respectively. In the following lemma we prove a property that can in some sense be interpreted as a consistency of the associated measure.

Lemma 1. *The $n-1$ variate marginal distribution of $\mathbf{Z}_n^*$ coincides with the probability measure associated with $\mathbf{Z}_{n-1}$.*

The next lemma plays a central role in evaluating the formulas for the contribution of the variability of risk X_k to the total tail variance of the aggregate sum $S = X_1 + X_2 + \cdots + X_n$. Note, that if $\mathbf{e} = (1, \ldots, 1)^T$ is a column vector of ones

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with dimension n, and therefore we can define the sum $S = \mathbf{e}^T\mathbf{X}$, then it immediately follows from (3.5) that

$$S \backsim E_1(\mathbf{e}^T\boldsymbol{\mu}, \mathbf{e}^{\mathbf{T}}\Sigma\mathbf{e}, g_1).$$

Lemma 2. *Let* $\mathbf{Y} = (Y_1, Y_2)^T \backsim E_2(\boldsymbol{\mu}, \Sigma, g_2)$ *such that condition (3.7) holds. Then*

$$TV_q(Y_1 | Y_2) = Var(Y_1)\,[r(z_{2,q}) + \gamma(z_{2,q}) \cdot \rho_{12}^2],$$

where

$$\gamma(z_{2,q}) = h_{Z,Z^*}(z_{2,q})\,(z_{2,q} - h_{Z^*}(z_{2,q}) \cdot \sigma_Z^2)$$

and $\rho_{12} = \dfrac{\sigma_{12}}{\sigma_1 \sigma_2}$, $\sigma_1 = \sqrt{\sigma_{11}}$, $\sigma_2 = \sqrt{\sigma_{22}}$, $z_{2,q} = \dfrac{y_q - \mu_2}{\sigma_2}$.

Now, we are ready to obtain the formula for the contribution of the variability of X_k to the total tail variance risk measure of $S = X_1 + \cdots + X_n$. Here again the random variables $X_1, X_2, ..., X_n$ can be interpreted as all kinds of financial risks.

Theorem 2. Let $\mathbf{X} = (X_1, X_2, ..., X_n)^T \backsim E_n(\boldsymbol{\mu}, \Sigma, g_n)$ *such that condition (3.7) holds. Then*

$$TV_q(X_k | S) = Var(X_k)\,[r(z_{S,q}) + \gamma(z_{S,q}) \cdot \rho_{k,S}^2], \tag{5.3}$$

where $\rho_{k,S} = \dfrac{\sigma_{k,S}}{\sigma_k \sigma_S}$ and $z_{S,q} = (s_q - \mu_S)/\sigma_S$.

Proof. The result immediately follows from Lemma 2 by considering $\mathbf{Y} = (X_k, S)^T$ and afterwards recalling that due to (3.5) $\mathbf{Y}$ has an elliptical distribution, i.e.,

$$(X_k, S)^T \backsim E_2(\boldsymbol{\mu}_{k,S}, \Sigma_{k,S}, g_2), \tag{5.4}$$

where $\boldsymbol{\mu}_{k,S} = \left(\mu_k, \sum_{j=1}^n \mu_j\right)^T$,

$$\Sigma_{k,S} = \begin{pmatrix} \sigma_k^2 & \sigma_{kS} \\ \sigma_{kS} & \sigma_S^2 \end{pmatrix}$$

and $\sigma_k^2 = \sigma_{kk}$, $\sigma_{kS} = \sum_{j=1}^n \sigma_{kj}$, $\sigma_S^2 = \sum_{i,j=1}^n \sigma_{ij}$. ∎

Corollary 2. *Under the conditions in Theorem 2, the tail variance premium for the marginal risk* X_k *is*

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$$
\begin{aligned}
&TVP_q(X_k|S)\\
&= E(X_k) + Var(X_k)\, h_{Z,Z^*}\left(z_{S_q}\right)\frac{\rho_{k,S}}{\sigma_k} + \alpha \cdot Var(X_k)\left[r\left(z_{S,q}\right) + \gamma\left(z_{S,q}\right)\cdot \rho_{k,S}^2\right]\\
&= E(X_k) + Var(X_k)\left[\frac{h_{Z,Z^*}\left(z_{S_q}\right)}{\sigma_k}\rho_{k,S} + \alpha\cdot\left(r\left(z_{S,q}\right) + \gamma\left(z_{S,q}\right)\cdot \rho_{k,S}^2\right)\right].
\end{aligned}
$$

Lemma 2 allows for immediate evaluation of the tail covariance risk measure introduced in (1.12).

Corollary 3. *The tail covariance risk measure is*

$$TCov_q(X_k|S) = Cov(X_k, S)\,[r(z_{S,q}) + \gamma(z_{S,q})],$$

where $Cov(X_k, S) = \sigma_{k,S}\,\sigma_Z^2$.

The full-additivity of the allocation rule based on the tail covariance premium is straightforward:

$$
\begin{aligned}
&\sum_{k=1}^{n} TCovP_q(X_k|S)\\
&= \sum_{k=1}^{n} TCE_q(X_k|S) + \alpha\sum_{k=1}^{n} TCov_q(X_k|S)\\
&= TCE_S\left(s_q\right) + \alpha\sum_{k=1}^{n} Cov(X_k, S)\left[r\left(z_{S,q}\right) + \gamma\left(z_{S,q}\right)\right]\\
&= TCE_S\left(s_q\right) + \alpha\cdot Cov\left(\sum_{k=1}^{n} X_k, S\right)\left[r\left(z_{S,q}\right) + \gamma\left(z_{S,q}\right)\right]\\
&= TCE_S\left(s_q\right) + \alpha\cdot Var(S)\left[r\left(z_{S,q}\right) + \gamma\left(z_{S,q}\right)\right]\\
&= TCE_S\left(s_q\right) + \alpha\cdot TV_S\left(s_q\right) = TCovP_S\left(s_q\right).
\end{aligned}
$$

6. Conclusions

In today's competitive and investment-oriented marketplace every piece of information about the risk to which an insurance company is exposed may be of critical importance for decision makers. In this paper we proposed a new premium principle, named the tail variance premium principle or TVP, which may be considered some kind of generalization of the popular variance premium. As distinct from the latter, TVP builds on the tail conditional expectation and at the same time takes into account the risk load, that is proportional to the conditional variance of the risk. Hence, due to its very definition, tail variance

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premium's vocation is to provide a risk assessment in situations when risk managers are concerned with risks exceeding a certain threshold (policies with deductibles and reinsurance contracts may be considered). Insurance losses are known to have significant dispersions. We believe that the proposed premium is particularly useful when the variability along the right tail is crucial. Moreover, according to our results, in situations when the contribution of tail variance to TVP is substantial, there exists some optimal threshold, such that by choosing it, an insurance company minimizes its risk. In the current research, we were able to present a number of allocation rules, possessing different properties. One of them, for instance, bases the decomposition of the total uncertainty level on the tail standard deviation risk measure, and consequently, such an allocation rule preserves the desirable sub-additivity property. Unlike the allocation based on TCE, where the dependence between marginal and aggregate risks is presented by the correlation parameter, the decompositions derived from tail variance and tail standard deviation risk measures are stipulated by squared correlation. On the one hand, this fact reduces the influence of the dependence between X_k and S and, on the other, it results in violation of the full-additivity property by the constructed allocation rules. For those concerned with the full-additivity of allocation rule, we have offered another method of risk capital decomposition which is based on the tail covariance risk measure (TCov). We were able to derive exact expressions for every risk measure or premium principle that we proposed.

Our investigations were performed in the general framework of the multivariate elliptical distributions. This class consists of such well-known distributions as normal, Student-t, logistic, and exponential power distributions. Many members of the elliptical class have tails that are heavier than those of normal distributions, and this attractive property allows one to model financial datasets.

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7. APPENDIX

PROOF OF THEOREM 1.

Proof. Following the definition of the tail variance risk measure in (1.3)

$$\begin{aligned} TV_q(X) &= \frac{1}{\overline{F}_X(x_q)} \int_{x_q}^{\infty} x^2 \cdot \frac{c}{\sigma} g\left(\tfrac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx - \left[TCE_q(X)\right]^2 \\ &= I - \left[TCE_q(X)\right]^2 \end{aligned}$$

and after the transformation $z = (x-\mu)/\sigma$ we have

$$I = \frac{1}{\overline{F}_Z(z_q)} \int_{z_q}^{\infty} (z\sigma + \mu)^2 \, cg\left(\tfrac{1}{2} z^2\right) dz.$$

Further, let us divide the expression for I into three parts

$$\begin{aligned} I &= \frac{1}{\overline{F}_Z(z_q)} \left(\sigma^2 \int_{z_q}^{\infty} z^2 cg\left(\tfrac{1}{2} z^2\right) dz + 2\sigma\mu \int_{z_q}^{\infty} zcg\left(\tfrac{1}{2} z^2\right) dz\right) + \mu^2 \\ &= I_1 + I_2 + \mu^2. \end{aligned}$$

As for I_1, it is easily found by integration by parts

$$\begin{aligned} I_1 &= -\frac{\sigma^2}{\overline{F}_Z(z_q)} \int_{z_q}^{\infty} z d\,\overline{G}_Z\left(\tfrac{1}{2} z^2\right) dz \\ &= \frac{1}{\overline{F}_Z(z_q)} \left(z_q \sigma^2\, \overline{G}_Z\left(\tfrac{1}{2} z_q^2\right) + \sigma^2 \sigma_z^2 \int_{z_q}^{\infty} \frac{\overline{G}_Z\left(\tfrac{1}{2} z^2\right)}{\sigma_z^2} dz \right). \end{aligned}$$

Recalling (4.3) and taking into account that $Var(X) = \sigma^2\sigma_z^2$, the above result may be rewritten as

$$I_1 = Var(X)\left(\frac{f_{Z*}(z_q)}{\overline{F}_Z(z_q)}z_q + \frac{\overline{F}_{Z*}(z_q)}{\overline{F}_Z(z_q)}\right).$$

The expression for I_2 is straightforwardly obtained

$$I_2 = \frac{2\sigma\mu}{\overline{F}_Z(z_q)}\int_{\frac{1}{2}z_q^2}^{\infty} cg(u)\,du = 2\sigma\mu\frac{\overline{G}_Z\left(\frac{1}{2}z_q^2\right)}{\overline{F}_Z(z_q)}.$$

The formula for the tail conditional expectation in the case of elliptical distributions is given by

$$TCE_q(X) = \mu + \frac{\frac{1}{\sigma}f_{Z*}(z_q)}{\overline{F}_Z(z_q)}\sigma_Z^2\sigma^2,$$

(see Landsman and Valdez (2003)). Therefore recalling again that $Var(X) = \sigma^2\sigma_Z^2$, we get

$$\begin{aligned}
&TV_q(X)\\
&= Var(X)\left(\frac{\overline{F}_{Z*}(z_q)}{\overline{F}_Z(z_q)} + \frac{f_{Z*}(z_q)}{\overline{F}_Z(z_q)}z_q\right) + 2\sigma\mu\frac{\overline{G}_Z\left(\frac{1}{2}z_q^2\right)}{\overline{F}_Z(z_q)} + \mu^2 - \left(TCE_q(X)\right)^2\\
&= Var(X)\left(\frac{\overline{F}_{Z*}(z_q)}{\overline{F}_Z(z_q)} + \frac{f_{Z*}(z_q)}{\overline{F}_Z(z_q)}z_q\right) + 2\sigma\mu\frac{\overline{G}_Z\left(\frac{1}{2}z_q^2\right)}{\overline{F}_Z(z_q)} + \mu^2 - \left(\mu + \frac{\overline{G}_Z\left(\frac{1}{2}z_q^2\right)}{\overline{F}_Z(z_q)}\sigma\right)^2\\
&= Var(X)\left[\frac{\overline{F}_{Z*}(z_q)}{\overline{F}_Z(z_q)} + \frac{f_{Z*}(z_q)}{\overline{F}_Z(z_q)}\left(z_q - \frac{f_{Z*}(z_q)}{\overline{F}_Z(z_q)}\cdot\sigma_z^2\right)\right]
\end{aligned}$$

and (4.8) follows. ■

Proof of Lemma 1.

Proof. In fact, one has to prove that

$$\int_{\mathbf{R}}\overline{G}_n\left(\frac{1}{2}\mathbf{z}^T\mathbf{z}\right)dz_n = \overline{G}_{n-1}\left(\frac{1}{2}\mathbf{z}_{n-1}^T\mathbf{z}_{n-1}\right).$$

Taking into account (3.2) and according to the well-known property of elliptical marginals, we have

$$c_{n-1} \cdot g_{n-1}\left(\frac{1}{2}\mathbf{z}_{n-1}^T \mathbf{z}_{n-1}\right) = c_n \int_{\mathbf{R}} g_n\left(\frac{1}{2}\left(\mathbf{z}_{n-1}^T \mathbf{z}_{n-1} + z_n^2\right)\right) dz_n.$$

Then

$$\begin{aligned}
\overline{G}_{n-1}\left(\frac{1}{2}\mathbf{z}_{n-1}^T \mathbf{z}_{n-1}\right) &= c_{n-1}\int_{\frac{1}{2}\mathbf{z}_{n-1}^T \mathbf{z}_{n-1}}^{\infty} g_{n-1}(v)\,dv \\
&= c_n \int_{\frac{1}{2}\mathbf{z}_{n-1}^T \mathbf{z}_{n-1}}^{\infty} dv \int_{\mathbf{R}} g_n\left(v + \frac{1}{2}z_n^2\right) dz_n \\
&= c_n \int_{\mathbf{R}} dz_n \int_{\frac{1}{2}\mathbf{z}_{n-1}^T \mathbf{z}_{n-1}}^{\infty} g_n\left(\frac{1}{2}z_n^2 + v\right) dv
\end{aligned}$$

and after the change of variables $t = \frac{1}{2}z_n^2 + v$, we readily get

$$\overline{G}_{n-1}\left(\frac{1}{2}\mathbf{z}_{n-1}^T \mathbf{z}_{n-1}\right) = \int_{\mathbf{R}}\left(c_n \int_{\frac{1}{2}\mathbf{z}^T\mathbf{z}}^{\infty} g_n(t)\,dt\right) dz_n = \int_{\mathbf{R}} \overline{G}_n\left(\frac{1}{2}\mathbf{z}^T\mathbf{z}\right) dz_n. \qquad \blacksquare$$

PROOF OF LEMMA 2

Proof. First let us denote

$$\tau_q(Y_1 | Y_2) = TCE_q(Y_1 | Y_2),$$

then by definition and according to (3.2), we have

$$\begin{aligned}
&E\left(\left(Y_1 - \tau_q(Y_1|Y_2)\right)^2 \,\middle|\, Y_2 > y_q\right) \\
&= E\left(Y_1^2 \,\middle|\, Y_2 > y_q\right) - \tau_q(Y_1|Y_2)^2 \\
&= \frac{1}{\overline{F}_{Y_2}(y_q)} \int_{-\infty}^{\infty}\int_{y_q}^{\infty} y_1^2 f_{\mathbf{Y}}(y_1, y_2)\,dy_2\,dy_1 - \tau_q(Y_1|Y_2)^2 \\
&= \frac{1}{\overline{F}_Z(z_{2,q})} \int_{-\infty}^{\infty}\int_{y_q}^{\infty} y_1^2 \frac{c_2}{\sqrt{|\Sigma|}} g_2\left(\tfrac{1}{2}(\mathbf{y} - \boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{y} - \boldsymbol{\mu})\right) dy_2\,dy_1 - \tau_q(Y_1|Y_2)^2 \\
&= \frac{1}{\overline{F}_Z(z_{2,q})} \times I - \tau_q(Y_1|Y_2)^2.
\end{aligned}$$

As in the bivariate case,

$$|\Sigma| = \begin{vmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{vmatrix} = \left(1 - \rho_{12}^2\right)\sigma_1^2\sigma_2^2,$$

the quadratic form can be represented as

$$
\begin{aligned}
&(\mathbf{y}-\boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{y}-\boldsymbol{\mu}) \\
&= \frac{1}{\left(1-\rho_{12}^2\right)}\left\{\left[\left(\frac{y_1-\mu_1}{\sigma_1}\right)-\rho_{12}-\left(\frac{y_2-\mu_2}{\sigma_2}\right)\right]^2+\left(1+\rho_{12}^2\right)\left(\frac{y_2-\mu_2}{\sigma_2}\right)^2\right\}.
\end{aligned}
$$

Using the transformations $z_1=\dfrac{y_1-\mu_1}{\sigma_1},\ z_2=\dfrac{y_2-\mu_2}{\sigma_2}$ results in

$$
I=\frac{c_2}{\sqrt{1-\rho_{12}^2}}\int_{z_{2,q}}^{\infty}\int_{-\infty}^{\infty}(\mu_1+\sigma_1 z_1)^2\, g_2\left(\frac{1}{2}\frac{(z_1-\rho_{12}z_2)^2}{\left(1-\rho_{12}^2\right)}+\frac{1}{2}z_2^2\right)dz_1dz_2 \tag{7.1}
$$

$$
=\frac{c_2}{\sqrt{1-\rho_{12}^2}}\int_{z_{2,q}}^{\infty}\int_{-\infty}^{\infty}\sigma_1^2 z_1^2\, g_2\left(\frac{1}{2}\frac{(z_1-\rho_{12}z_2)^2}{\left(1-\rho_{12}^2\right)}+\frac{1}{2}z_2^2\right)dz_1dz_2 \tag{7.2}
$$

$$
+\frac{c_2}{\sqrt{1-\rho_{12}^2}}\int_{z_{2,q}}^{\infty}\int_{-\infty}^{\infty}2\mu_1\sigma_1 z_1\, g_2\left(\frac{1}{2}\frac{(z_1-\rho_{12}z_2)^2}{\left(1-\rho_{12}^2\right)}+\frac{1}{2}z_2^2\right)dz_1dz_2 \tag{7.3}
$$

$$
+\frac{c_2}{\sqrt{1-\rho_{12}^2}}\int_{z_{2,q}}^{\infty}\int_{-\infty}^{\infty}\mu_1^2\, g_2\left(\frac{1}{2}\frac{(z_1-\rho_{12}z_2)^2}{\left(1-\rho_{12}^2\right)}+\frac{1}{2}z_2^2\right)dz_1dz_2 \tag{7.4}
$$

$$
=I_1+I_2+I_3.
$$

First, applying the property of marginal distributions of elliptical family, the integral (7.4) is simply

$$
\begin{aligned}
I_3&=\frac{c_2}{\sqrt{1-\rho_{12}^2}}\int_{z_{2,q}}^{\infty}\int_{-\infty}^{\infty}\mu_1^2\, g_2\left(\frac{1}{2}\frac{(z_1-\rho_{12}z_2)^2}{\left(1-\rho_{12}^2\right)}+\frac{1}{2}z_2^2\right)dz_1dz_2 \\
&=\mu_1^2 c_1\int_{-\infty}^{\infty}g_1\left(\tfrac{1}{2}z_2^2\right)dz_2=\mu_1^2\,\overline{F}_Z\left(z_{2,q}\right).
\end{aligned} \tag{7.5}
$$

As for I_2, we have (see 7.3)

$$
I_2=2\mu_1\sigma_1\frac{c_2}{\sqrt{1-\rho_{12}^2}}\int_{z_{2,q}}^{\infty}\int_{-\infty}^{\infty}z_1 g_2\left(\frac{1}{2}\frac{(z_1-\rho_{12}z_2)^2}{\left(1-\rho_{12}^2\right)}+\frac{1}{2}z_2^2\right)dz_1dz_2
$$

and after the transformation $t=\dfrac{z_1-\rho_{12}z_2}{\sqrt{1-\rho_{12}^2}}$ we get

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$$I_2 = 2\mu_1\sigma_1\sqrt{1-\rho_{12}^2}\int_{z_{2,q}}^{\infty}\int_{-\infty}^{\infty} c_2\left(t+\frac{\rho_{12}z_2}{\sqrt{1-\rho_{12}^2}}\right) g_2\left(\tfrac{1}{2}t^2+\tfrac{1}{2}z_2^2\right)dt dz_2.$$

Next, by noticing that the integral of the odd function

$$\int_{-\infty}^{\infty} t c_2 g_2\left(\tfrac{1}{2}\left(t^2+z_2^2\right)\right)dt = 0$$

and due to the property of elliptical marginals

$$\int_{-\infty}^{\infty} c_2 g_2\left(\tfrac{1}{2}\left(t^2+z_2^2\right)\right)dt = c_1 g_1\left(\tfrac{1}{2}z_2^2\right),$$

we rewrite I_2 as

$$\begin{aligned} I_2 &= 2\mu_1\sigma_1\rho_{12}\int_{z_{2,q}}^{\infty} z_2 c_1 g_1\left(\tfrac{1}{2}z_2^2\right)dz_2 = 2\mu_1\sigma_1\rho_{12}\int_{\frac{1}{2}z_{2,q}^2}^{\infty} c_1 g_1(u)\,du \\ &= 2\mu_1\sigma_1\rho_{12}\overline{G}\left(\tfrac{1}{2}z_{2,q}^2\right). \end{aligned} \tag{7.6}$$

Finally (see 7.2)

$$\begin{aligned} I_1 &= \sigma_1^2\int_{z_{2,q}}^{\infty}\int_{-\infty}^{\infty}\frac{z_1^2}{\sqrt{1-\rho_{12}^2}}\,c_2 g_2\left(\frac{1}{2}\frac{(z_1-\rho_{12}z_2)^2}{\left(1-\rho_{12}^2\right)}+\frac{1}{2}z_2^2\right)dz_1 dz_2 \\ &= \sigma_1^2\int_{z_{2,q}}^{\infty} I_1^*(z_2)\,dz_2, \end{aligned} \tag{7.7}$$

where the function $I_1^*(z_2)$ can be represented as

$$I_1^*(z_2) = \frac{1}{\sqrt{1-\rho_{12}^2}}\int_{-\infty}^{\infty} z_1^2 c_2 g_2\left(\frac{1}{2}\frac{(z_1-\rho_{12}z_2)^2}{\left(1-\rho_{12}^2\right)}+\frac{1}{2}z_2^2\right)dz_1.$$

Then, applying the familiar transformation $u = (z_1-\rho_{12}z_2)/\sqrt{1-\rho_{12}^2}$ results in

$$\begin{aligned} I_1^*(z_2) &= \int_{-\infty}^{\infty}\left(u\sqrt{1-\rho_{12}^2}+\rho_{12}z_2\right)^2 c_2 g_2\left(\tfrac{1}{2}u^2+\tfrac{1}{2}z_2^2\right)du \\ &= \left(1-\rho_{12}^2\right)\int_{-\infty}^{\infty}\left(u+\frac{\rho_{12}z_2}{\sqrt{1-\rho_{12}^2}}\right)^2 c_2 g_2\left(\tfrac{1}{2}u^2+\tfrac{1}{2}z_2^2\right)du \end{aligned}$$

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$$= \left(1-\rho_{12}^2\right)\int_{-\infty}^{\infty} u^2 c_2\, g_2\left(\tfrac{1}{2}u^2+\tfrac{1}{2}z_2^2\right)du$$
$$+\sqrt{1-\rho_{12}^2}\int_{-\infty}^{\infty} 2u\,\rho_{12}\, z_2\, c_2\, g_2\left(\tfrac{1}{2}u^2+\tfrac{1}{2}z_2^2\right)du$$
$$+\int_{-\infty}^{\infty}\left(\rho_{12}\, z_2\right)^2 c_2\, g_2\left(\tfrac{1}{2}u^2+\tfrac{1}{2}z_2^2\right)du.$$

As the integral of the odd function is again zero

$$\int_{-\infty}^{\infty} 2u\,\rho_{12}\, z_2\, c_2\, g_2\left(\tfrac{1}{2}u^2+\tfrac{1}{2}z_2^2\right)du = 0$$

and applying Lemma 1, considering the consistency property of the associated measure, gives

$$\begin{aligned} I_1^*(z_2) &= -\left(1-\rho_{12}^2\right)\int_{-\infty}^{\infty} u\, d\,\overline{G}_2\left(\tfrac{1}{2}u^2+\tfrac{1}{2}z_2^2\right) \\ &\quad + \left(\rho_{12}\, z_2\right)^2 c_1\, g_1\left(\tfrac{1}{2}z_2^2\right) \\ &= \left(1-\rho_{12}^2\right)\sigma_Z^2\int_{-\infty}^{\infty}\frac{1}{\sigma_Z^2}\,\overline{G}_2\left(\tfrac{1}{2}u^2+\tfrac{1}{2}z_2^2\right)du \\ &\quad + \left(\rho_{12}\, z_2\right)^2 c_1\, g_1\left(\tfrac{1}{2}z_2^2\right) \\ &= \left(1-\rho_{12}^2\right)\sigma_Z^2\, f_{Z^*}(z_2) + \left(\rho_{12}\, z_2\right)^2 c_1\, g_1\left(\tfrac{1}{2}z_2^2\right). \end{aligned}$$

Afterwards, substituting the results in (7.7), we obtain

$$\begin{aligned} I_1 &= \left(1-\rho_{12}^2\right)\sigma_1^2\sigma_Z^2\int_{z_{2,q}}^{\infty} f_{Z^*}(z_2)\,dz_2 + \sigma_1^2\rho_{12}^2\int_{z_{2,q}}^{\infty} z_2^2\, c_1\, g_1\left(\tfrac{1}{2}z_2^2\right)dz_2 \\ &= \left(1-\rho_{12}^2\right)\sigma_1^2\sigma_Z^2\,\overline{F}_{Z^*}\left(z_{2,q}\right) + \rho_{12}^2\, z_{2,q}\,\sigma_1^2\,\overline{G}_Z\left(\tfrac{1}{2}z_{2,q}^2\right) + \rho_{12}^2\sigma_1^2\sigma_Z^2\,\overline{F}_{Z^*}\left(z_{2,q}\right) \\ &= Var(Y_1)\left[\overline{F}_{Z^*}\left(z_{2,q}\right) + \rho_{12}^2\, f_{Z^*}\left(z_{2,q}\right)\cdot z_{2,q}\right]. \end{aligned} \tag{7.8}$$

Consequently, combining (7.5), (7.6) and (7.8) we get

$$\begin{aligned} &E\left(Y_1^2\,\middle|\,Y_2 > y_q\right) \\ &= Var(Y_1)\left[\frac{\overline{F}_{Z^*}\left(z_{2,q}\right)}{\overline{F}_Z\left(z_{2,q}\right)} + \rho_{12}^2\,\frac{f_{Z^*}\left(z_{2,q}\right)}{\overline{F}_Z\left(z_{2,q}\right)}\cdot z_{2,q}\right] + \mu_1^2 + 2\mu_1\sigma_1\rho_{12}\,\frac{\overline{G}\left(\tfrac{1}{2}z_{2,q}^2\right)}{\overline{F}_Z\left(z_{2,q}\right)}. \end{aligned}$$

Recall that due to Landsman and Valdez (2003) the expression for tail conditional expectation is

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$$TCE_q(Y_1 \,|\, Y_2) = \mu_1 + \frac{\overline{G}\left(\frac{1}{2} z_{2,q}^2\right)}{\overline{F}_Z\left(z_{2,q}\right)} \cdot \sigma_1 \rho_{12}$$

and after all

$$\begin{aligned}
&E\left(\left(Y_1 - \tau_q(Y_1 \,|\, Y_2)\right)^2 \,\middle|\, Y_2 > y_q\right) \\
&= Var(Y_1)\left[\frac{\overline{F}_{Z^*}\left(z_{2,q}\right)}{\overline{F}_Z\left(z_{2,q}\right)} + \rho_{12}^2 \frac{f_{Z^*}\left(z_{2,q}\right)}{\overline{F}_Z\left(z_{2,q}\right)} \cdot z_{2,q}\right] + \mu_1^2 + 2\mu_1 \sigma_1 \rho_{12} \frac{\overline{G}\left(\frac{1}{2} z_{2,q}^2\right)}{\overline{F}_Z\left(z_{2,q}\right)} \\
&- \left(\mu_1 + \frac{\frac{1}{\sigma_2}\overline{G}\left(\frac{1}{2} z_{2,q}^2\right)}{\overline{F}_Z\left(z_{2,q}\right)} \cdot \sigma_1 \sigma_2 \rho_{12}\right)^2 \\
&= Var(Y_1)\left[\frac{\overline{F}_{Z^*}\left(z_{2,q}\right)}{\overline{F}_Z\left(z_{2,q}\right)} + \rho_{12}^2 \frac{f_{Z^*}\left(z_{2,q}\right)}{\overline{F}_Z\left(z_{2,q}\right)} \cdot z_{2,q} - \sigma_Z^2 \rho_{12}^2 \left(\frac{f_{Z^*}\left(z_{2,q}\right)}{\overline{F}_Z\left(z_{2,q}\right)}\right)^2\right] \\
&= Var(Y_1)\left[\frac{\overline{F}_{Z^*}\left(z_{2,q}\right)}{\overline{F}_Z\left(z_{2,q}\right)} + \frac{f_{Z^*}\left(z_{2,q}\right)}{\overline{F}_Z\left(z_{2,q}\right)}\left(z_{2,q} - \frac{f_{Z^*}\left(z_{2,q}\right)}{\overline{F}_Z\left(z_{2,q}\right)} \sigma_Z^2\right) \cdot \rho_{12}^2\right].
\end{aligned}$$

■

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