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Jacobson Radicals of Skew Polynomial Rings of Derivation Type

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Abstract. We provide necessary and sufficient conditions for a skew polynomial ring of derivation type to be semiprimitive when the base ring has no nonzero nil ideals. This extends existing results on the Jacobson radical of skew polynomial rings of derivation type.

1 Introduction

Throughout this paper, R denotes an associative ring (not necessarily with unity), α is an automorphism of R, and δ is a derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + a\delta(b)$, for all $a, b \in R$. We denote by $R[x;\alpha]$ the skew polynomial ring of automorphism type whose elements are the polynomials $\sum_{i=0}^{n} r_i x^i$, $r_i \in R$, where the addition is defined as usual and the multiplication is subject to the relation $xa = \alpha(a)x$ for any $a \in R$. Also we denote by $R[x;\delta]$ the skew polynomial ring of derivation type whose elements are the polynomials $\sum_{i=0}^{n} r_i x^i$, $r_i \in R$, where the addition is defined as usual and the multiplication is subject to the relation $xa = ax + \delta(a)$ for any $a \in R$. The upper nil radical (*i.e.,* sum of all nil ideals), the Jacobson radical, and the set of all nilpotent elements of R are denoted by $\mathrm{Nil}^*(R)$, $J(R)$, and $\mathrm{Nil}(R)$, respectively.

Amitsur [1] showed that $J(R[x]) = (J(R[x]) \cap R)[x]$, and $J(R[x]) \cap R$ is a nil ideal of R. In particular, if R has no nonzero nil ideals, then $R[x]$ is semiprimitive. Subsequently, there has been a great deal of work examining the Jacobson radicals of more general ring extensions, such as skew polynomial rings (of automorphism type or derivation type). In [2], S. S. Bedi and J. Ram proved that $J(R[x;\alpha]) = I \cap J(R) + Ix + Ix^2 + \cdots + Ix^n + \cdots$, where $I = \{r \in R \mid rx \in J(R[x;\alpha])\}$. They also showed that even if R is commutative and reduced (*i.e.,* has no nonzero nilpotent element), then $R[x;\alpha]$ can be nonsemiprimitive. For skew polynomial rings $R[x;\delta]$ of derivation type, Ferrero, Kishimoto, and Motose showed that $J(R[x;\delta]) = (J(R[x;\delta]) \cap R)[x;\delta]$. But it is still unknown if $J(R[x;\delta]) \cap R$ must be nil. Although this is still open in general, there exist some partial answers for this problem. Jordan [7] proved that if R is a right Noetherian ring with unity then $J(R[x;\delta]) \cap R$ is nil. In [5], Ferrero et al. showed that $J(R[x;\delta]) \cap R$ is nil if R is commutative. Also it was shown in [3,10] that $J(R[x;\delta]) \cap R$ is nil if one assumes either that R is reduced or satisfies a polynomial identity or satisfies the ascending chain condition on right annihilators.

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A ring R is called strongly π-regular if R satisfies the descending chain condition on principal right ideals of the form $aR \supseteq a^2R \supseteq \cdots$, for every a in R. Strongly π-regular rings were introduced by Kaplansky [8] as a common generalization of algebraic algebras and artinian rings. Strong π-regularity has roles in module theory and ring theory. In this paper we show that $J(R[x;\delta]) \cap R$ is nil if and only if $J(R[x;\delta]) \cap R$ is a strongly π-regular ring if and only if $J(R[x;\delta]) \cap \mathrm{Nil}(R)$ is an ideal of R. As a corollary we extend some known results.

One drawback of Amitsur's Theorem is that it does not determine what $J(R[x]) \cap R$ really is, other than that it is a nil ideal. An interesting problem is to determine whether $J(R[x]) \cap R$ is indeed equal to $\mathrm{Nil}^*(R)$. In fact, it is equivalent to another famous problem in ring theory called Kothe's Conjecture. E. Snapper showed that for a commutative ring R, $J(R[x]) = \mathrm{Nil}(R[x]) = \mathrm{Nil}(R)[x]$. In this paper we also extend Snapper's Theorem and show that $J(R[x;\delta]) = \mathrm{Nil}(R[x;\delta]) = \mathrm{Nil}(R)[x;\delta] = \mathrm{Nil}^*(R)[x;\delta] = \mathrm{Nil}^*(R[x;\delta])$ if and only if $\mathrm{Nil}(R)$ is a δ-ideal of R and $\mathrm{Nil}(R[x;\delta]) = \mathrm{Nil}(R)[x;\delta]$. As a corollary we show that if $R[x;\delta]$ is NI ring, then

$$J\big(R[x;\delta]\big) = \mathrm{Nil}\big(R[x;\delta]\big) = \mathrm{Nil}(R)[x;\delta] = \mathrm{Nil}^*(R)[x;\delta] = \mathrm{Nil}^*\big(R[x;\delta]\big).$$

2 Main Results

In this section we prove the main results of the paper.

To fully describe the multiplication in skew polynomial rings of derivation type $R[x;\delta]$, we must iterate the rule $xa = ax + \delta(a)$, which leads us to the formula $x^i a = \sum_{j=0}^{i} \binom{i}{j} \delta^{i-j}(a) x^j$ for each positive integer i and $a \in R$. We use this formula in the proof of the next theorem.

Theorem 2.1 *Let R be a ring and δ a derivation of R. Then the following are equivalent:*

(i) *$J(R[x;\delta]) \cap R$ is nil;*
(ii) *$J(R[x;\delta]) \cap R$ is a strongly π-regular ring;*
(iii) *$J(R[x;\delta]) \cap \mathrm{Nil}(R)$ is an ideal of R.*

Proof (i)⇒(ii). This is obvious.

(ii)⇒(i). Assume that $J(R[x;\delta]) \cap R$ is a strongly π-regular ring and let $a \in J(R[x;\delta]) \cap R$. We show that a is nilpotent. Since by Dischinger's result in [4] the strongly π-regularity is left-right symmetric, the descending chain $(J(R[x;\delta]) \cap R)a \supseteq (J(R[x;\delta]) \cap R)a^2 \supseteq \cdots$ stabilizes. Thus there exists a positive integer n such that $a^n \in (J(R[x;\delta]) \cap R)a^{n+1}$. Then $xa^n \in J(R[x;\delta])$, and so there exists $g(x) = b_0 + b_1x + \cdots + b_tx^t \in R[x;\delta]$ such that

$$xa^n + g(x) = xa^n g(x). \tag{2.1}$$

If $a^n g(x) = 0$, then by multiplying equation (2.1) on the left by a^n we have $a^n x a^n = 0$, and so $a^{2n} = 0$. Now assume that $a^n g(x) \neq 0$, then there exists a largest integer $m \geq 0$ such that $a^n b_m \neq 0$. Multiplying equation (2.1) on the left by a^n gives us $a^n x a^n + a^n b_0 + a^n b_1 x + \cdots + a^n b_m x^m = a^n x a^n b_0 + \cdots + a^n x a^n b_m x^m$. On the other hand, since $a^n \in (J(R[x;\delta]) \cap R)a^{n+1}$, we have $a^{2n} b_m \neq 0$. Therefore the left-hand side of the

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equation has degree m, but the right-hand side has degree $m+1$. So $m = 0$ and hence $a^n xa^n + a^n b_0 = a^n xa^n b_0$. Then $a^{2n} = a^{2n} b_0$. Since $a^n \in (J(R[x;\delta]) \cap R)a^{n+1}$, there exists $b \in (J(R[x;\delta]) \cap R)$ such that $a^n = ba^{n+1}$, and so we can see that $a^n = b^i a^{n+i}$ for each integer $i \geq 1$. Thus $a^n = b^n a^{2n} = b^n a^{2n} b_0 = a^n b_0$, and so $a^n xa^n + a^n b_0 = a^n xa^n b_0 + a^n b_0 = a^n xa^n b_0$. Then $a^n = a^n b_0 = 0$, and the result follows.

(i)⇒(iii). If $J(R[x;\delta]) \cap R$ is nil, then $J(R[x;\delta]) \cap \mathrm{Nil}(R) = J(R[x;\delta]) \cap R$, and the result follows.

(iii)⇒(i). Assume that $J(R[x;\delta]) \cap \mathrm{Nil}(R)$ is an ideal of R and let $a \in J(R[x;\delta]) \cap R$. We show that a is nilpotent. Now $ax \in J(R[x;\delta])$, and so there exists $f(x) = a_0 + a_1 x + \cdots + a_t x^t \in R[x;\delta]$ such that

$$f(x) + ax = f(x)ax. \tag{2.2}$$

If the degree of $f(x)$ is zero, then it is easy to see that $a = 0$, and the result follows. Now assume that the degree of $f(x)$ is nonzero. Assume that each coefficient of $af(x)$ is nilpotent. Since $J(R[x;\delta]) \cap \mathrm{Nil}(R)$ is an ideal of R, then it is easy to see that each coefficient of $af(x)ax$ is also nilpotent. Multiplying equation (2.2) on the left by a gives us

$$af(x) + a^2 x = af(x)ax. \tag{2.3}$$

The coefficient of x in the left-hand side of equation (2.3) is $aa_1 + a^2$ and the coefficient of x in the right-hand side of equation (2.3) is $aa_0 a + aa_1\delta(a) + aa_2\delta^2(a) + \cdots + aa_t\delta^t(a)$. So we have $aa_1 + a^2 = aa_0 a + aa_1\delta(a) + aa_2\delta^2(a) + \cdots + aa_t\delta^t(a)$ and then $a^2 \in \mathrm{Nil}(R)$, since $J(R[x;\delta]) \cap \mathrm{Nil}(R)$ is an ideal of R. Now assume that there exists a largest integer $0 \leq m \leq t$ such that aa_m is not nilpotent. The coefficient of x^{t+1} in the left-hand side of equation (2.3) is zero, and the coefficient of x^{t+1} in the right-hand side of equation (2.3) is $aa_t a$. Thus $aa_t a = 0$, so aa_t is nilpotent and hence $m \neq t$. By equation (2.3), $aa_0 = 0$ and so $m \neq 0$. Thus $t > m > 0$. The coefficient of x^{m+1} in left hand side of equation (2.3) is aa_{m+1}, and the coefficient of x^{m+1} in the right-hand side of equation (2.3) is $aa_m a + \binom{m+1}{m} aa_{m+1}\delta(a) + \cdots + \binom{t}{m} aa_t \delta^{t-m}(a)$. Thus $aa_m a \in J(R[x;\delta]) \cap \mathrm{Nil}(R)$ and so $aa_m aa_m \in J(R[x;\delta]) \cap \mathrm{Nil}(R)$, which is a contradiction, and the result follows. ■

By Theorem 2.1 and [5, Theorem 3.2] we have the following corollary.

Corollary 2.2 *Let R be a ring with no nonzero nil ideals and δ a derivation of R. Then the following are equivalent:*

(i) *$R[x;\delta]$ is semiprimitive;*
(ii) *$J(R[x;\delta]) \cap R$ is a strongly π-regular ring;*
(iii) *$J(R[x;\delta]) \cap \mathrm{Nil}(R)$ is an ideal of R.*

In a commutative ring, the set of nilpotent elements form an ideal. This property is also possessed by certain noncommutative rings, which are known as NI rings. Note that the class of NI rings contain a large class of noncommutative rings such as 2-primal rings, (PS I) rings, (S I) rings, one-sided duo rings, reversible rings, symmetric rings, and reduced rings (for more details see [9]). By Theorem 2.1 we have the following corollary, which is a generalization of [5, Theorem 3.3].

Corollary 2.3 *Let R be a NI ring and δ a derivation of R. Then $J(R[x;\delta]) \cap R$ is nil.*

A ring is called *locally finite* if every finite subset in it generates a finite semigroup multiplicatively. Finite rings are clearly locally finite, and an algebraic closure of a finite field is locally finite but not finite. By [6, Proposition 2.1 and Lemma 2.4], subrings (not necessarily with identity) of locally finite rings are locally finite and every locally finite ring is strongly π-regular. So we have the following corollary.

Corollary 2.4 *Let R be a locally finite ring and δ a derivation of R. Then $J(R[x;\delta]) \cap R$ is nil.*

It is known that each two-sided ideal of a strongly π-regular ring R is strongly π-regular. So we have the following corollary.

Corollary 2.5 *Let R be a strongly π-regular ring and δ a derivation of R. Then $J(R[x;\delta]) \cap R$ is nil.*

Note that each algebraic algebra is strongly π-regular (for more details see [8]), and so if R is an algebraic algebra, then $J(R[x;\delta]) \cap R$ is nil.

Let L be a Lie algebra acting on R as derivations and $U(L)$ be the universal enveloping algebra of L. The construction of $R[x;\delta]$ using a single derivation can be extended to construct the smash product $R\#U(L)$ (for more details on $R\#U(L)$, see [3]).

Corollary 2.6 *Let R be an algebra with no nonzero nil ideals and L a Lie algebra acting on R as derivations. Then $J(R\#U(L)) = 0$ in all of the following cases:*

(i) *R is NI;*
(ii) *R is locally finite;*
(iii) *R is strongly π-regular:*
(iv) *R is algebraic algebra.*

Proof Assume on the contrary that $J(R\#U(L)) \neq 0$. Then by [3, Proposition 3.7], $J(R\#U(L)) \cap R \neq 0$. Now let $0 \neq x \in L$, and let δ be a derivation of R corresponding to x. Thus, by [3, Lemma 3.8], $J(R\#U(L)) \cap R[x;\delta] \subseteq J(R[x;\delta])$. But in all cases we have $J(R[x;\delta]) = 0$, and so $0 \neq J(R\#U(L)) \cap R \subseteq J(R\#U(L)) \cap R[x;\delta] \subseteq J(R[x;\delta]) = 0$ which is contradiction. Then $J(R\#U(L)) = 0$, and the result follows. ∎

Proposition 2.7 *Let R be a ring and δ a derivation of R. Then*

$$J(R[x;\delta]) = \mathrm{Nil}(R[x;\delta]) = \mathrm{Nil}(R)[x;\delta]$$

if and only if $\mathrm{Nil}(R)$ *is a δ-ideal of R and* $\mathrm{Nil}(R[x;\delta]) = \mathrm{Nil}(R)[x;\delta]$.

Proof If $J(R[x;\delta]) = \mathrm{Nil}(R[x;\delta]) = \mathrm{Nil}(R)[x;\delta]$, then by [5, Theorem 3.2], $J(R[x;\delta]) \cap R = \mathrm{Nil}(R)$, and the result follows. Now assume that $\mathrm{Nil}(R)$ is a δ-ideal of R and $\mathrm{Nil}(R[x;\delta]) = \mathrm{Nil}(R)[x;\delta]$. Since $\mathrm{Nil}(R)$ is an ideal of R, by Theorem 2.1 $J(R[x;\delta]) \cap R$ is nil and so by [5, Theorem 3.2], $J(R[x;\delta]) \subseteq \mathrm{Nil}(R)[x;\delta] =$

$\mathrm{Nil}(R[x;\delta])$. Since $\mathrm{Nil}(R[x;\delta]) = \mathrm{Nil}(R)[x;\delta]$ and $\mathrm{Nil}(R)$ is δ-ideal, $\mathrm{Nil}(R)[x;\delta]$ is a nil ideal, and so $\mathrm{Nil}(R)[x;\delta] \subseteq J(R[x;\delta])$. ■

Proposition 2.8 *Let R be a ring and δ a derivation of R. If $R[x;\delta]$ is NI, then* $J(R[x;\delta]) = \mathrm{Nil}(R[x;\delta]) = \mathrm{Nil}(R)[x;\delta] = \mathrm{Nil}^*(R)[x;\delta] = \mathrm{Nil}^*(R[x;\delta])$.

Proof Assume that $R[x;\delta]$ is NI. It is easy to see that $\mathrm{Nil}(R)$ is an ideal of R. Let $a \in \mathrm{Nil}(R)$. Since $\mathrm{Nil}(R[x;\delta])$ is ideal, $ax+\delta(a) = xa \in \mathrm{Nil}(R[x;\delta])$ and so $\delta(a) \in \mathrm{Nil}(R)$. Thus $\mathrm{Nil}(R)$ is a δ-ideal of R. Also $(R[x;\delta])/(\mathrm{Nil}(R)[x;\delta]) \cong R/(\mathrm{Nil}(R))[x;\delta]$, and $R/(\mathrm{Nil}(R))$ is a reduced ring. Then $(R[x;\delta])/(\mathrm{Nil}(R)[x;\delta])$ is reduced, and so $\mathrm{Nil}(R[x;\delta]) \subseteq \mathrm{Nil}(R)[x;\delta]$. Since $\mathrm{Nil}(R[x;\delta])$ is ideal, $\mathrm{Nil}(R[x;\delta]) \supseteq \mathrm{Nil}(R)[x;\delta]$ and so $\mathrm{Nil}(R[x;\delta]) = \mathrm{Nil}(R)[x;\delta]$. Thus by Proposition 2.7, $J(R[x;\delta]) = \mathrm{Nil}(R[x;\delta]) = \mathrm{Nil}(R)[x;\delta]$. Since $\mathrm{Nil}(R[x;\delta])$ is ideal, $\mathrm{Nil}^*(R[x;\delta]) = \mathrm{Nil}(R[x;\delta])$ and $\mathrm{Nil}(R) = \mathrm{Nil}^*(R)$ and the result follows. ■

The following corollary generalizes Snapper's Theorem.

Corollary 2.9 *Let $R[x]$ be a NI ring. Then* $J(R[x]) = \mathrm{Nil}(R[x]) = \mathrm{Nil}(R)[x] = \mathrm{Nil}^*(R)[x] = \mathrm{Nil}^*(R[x])$.

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