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NONHOLONOMIC SIMPLE $\mathcal{D}$-MODULES FROM SIMPLE DERIVATIONS

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Abstract. We give new examples of affine sufaces whose rings of coordinates are d-simple and use these examples to construct simple nonholonomic $\mathcal{D}$-modules over these surfaces.

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1. Introduction. The d-simplicity of commutative rings was the subject of several papers in the 1970s and early 1980s, at least partly because of its applications to the construction of simple noncommutative noetherian rings [**12**, Proposition 1.14]. Although little was published on it from the mid 1980s to the mid 1990s, the subject has known something of a revival in recent years, fuelled perhaps by the construction of new examples of derivations with respect to which the ring of polynomials is d-simple [**5**], [**2**], [**16**], [**13**], and also by the application of these derivations to the construction of new families of simple modules over rings of differential operators [**4**], [**9**].

Despite these advances, some aspects of the theory have progressed very little since the 1980s. One of these is the construction of new examples of d-simple rings. The only examples known up to now were the ones already given in J. Archer's PhD thesis [**1**]; namely, coordinate rings of affine spaces, tori, quadrics, and products of these varieties with affine space.

This is precisely the question that we tackle in this paper. As an application of the theorems proved in Section 3, we give several new examples of smooth surfaces whose coordinate rings are d-simple. Two of these lead to new families of nonholonomic simple $\mathcal{D}$-modules over surfaces. One of these families is particularly interesting because all the previous examples of simple nonholonomic $\mathcal{D}$-modules required the affine surface to have trivial Picard group. However, the surface of Example 4.1 is the product of an elliptic curve E with an affine line, so its Picard group, which is isomorphic to Pic(E), must be nonzero. As a bonus we construct, in Example 4.4, an irreducible nonholonomic $\mathcal{D}$-module over an explicit surface of $\mathbb{C}^3$, taking a singular derivation as our starting point.

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2. Preliminaries. We begin by fixing some notation. Throughout the paper, K will be an algebraically closed field of characteristic zero. The coordinate ring of an affine variety X will be denoted by $\mathcal{O}(X)$. If x is an indeterminate over K then ∂_x will stand for the partial differential operator $\partial/\partial x$.

An ideal I of a commutative K-algebra A is *stable* under a derivation d of A if $d(I) \subseteq I$. If I is generated by f, then we also say that f is stable under d. The algebra A is d*-simple* if there exists a derivation d of A with no stable ideals apart from $\{0\}$ and A. In this case, d is called a *simple* derivation of A.

Since we will be discussing modules over a ring of differential operators in section 4, we review some basic facts about these rings before we proceed. Let X be an irreducible, smooth, affine variety over a field K of characteristic zero. The *ring of differential operators* $\mathcal{D}(X)$ is the K-subalgebra of $\mathrm{End}_K\mathcal{O}(X)$ generated by $\mathcal{O}(X)$ and its module of K-derivations $\mathrm{Der}_K(X)$. The ring $\mathcal{D}(X)$ admits a filtration, defined by

$$\mathcal{C}_0 = \mathcal{O}(X), \quad \mathcal{C}_1 = \mathcal{O}(X) + \mathrm{Der}_K(X) \text{ and } \mathcal{C}_k = \mathcal{C}_1^k \text{ if } k > 0.$$

An operator $d \in \mathcal{D}(X)$ has *order* k if $d \in \mathcal{C}_k \backslash \mathcal{C}_{k-1}$. It follows from [**14**, Proposition 15.4.5] that the graded ring associated with this filtration is isomorphic to the symmetric algebra on $\mathrm{Der}_K(X)$. We denote this algebra by $\mathrm{S}(X)$. Let $\mathrm{S}^k(X)$ be the k-th homogenous component of the symmetric algebra. The *symbol map* of order k, denoted by σ_k, is the composition

$$\sigma_k : \mathcal{C}_k \to \mathcal{C}_k/\mathcal{C}_{k-1} \xrightarrow{\sim} \mathrm{S}^k(X).$$

If $d \in \mathcal{C}_k \setminus \mathcal{C}_{k-1}$ then its *principal symbol* is $\sigma(d) = \sigma_k(d)$. Given an ideal I of $\mathcal{D}(X)$, denote by $\sigma(I)$ the ideal of $\mathrm{S}(X)$ generated the principal symbols $\sigma(d)$ of all $d \in I$.

The algebra $\mathrm{S}(X)$ has an additional structure of Lie algebra. Let f_1 and f_2 be homogeneous elements of $\mathrm{S}(X)$ of degrees r_1 and r_2. There exist $a_1, a_2 \in \mathcal{D}(X)$ of orders r_1 and r_2 respectively, such that $\sigma_{r_i}(a_i) = f_i$. The *Poisson bracket* of f_1 and f_2 is defined by

$$\{f_1, f_2\} = \sigma_{r_1+r_2-1}([a_1, a_2]),$$

where $[a_1, a_2]$ denotes the commutator in $\mathcal{D}(X)$. This is easily extended, by linearity, to all of $\mathrm{S}(X)$. An ideal of $\mathrm{S}(X)$ is *involutive* if it is closed under the Poisson bracket. Note that if I is a left ideal of $\mathcal{D}(X)$ then $\sigma(I)$ is an involutive ideal of $\mathrm{S}(X)$.

We finish this section with the problem of constructing simple $\mathcal{D}(X)$-modules over a smooth variety X. The key assumption will be that $\mathcal{O}(X)$ is d-simple with respect to a derivation d. Our first result appeared as Theorem 2.1 of [**4**]. However, as has been pointed out by D. Levcovitz, the proof given there works only when $\mathcal{O}(X)$ is a unique factorization domain. Since we need the same result in a more general setting, we give a new proof below.

THEOREM 2.1. *Let X be an irreducible, smooth, affine variety over K. Suppose that there exists a derivation d of $\mathcal{O}(X)$ with respect to which this ring is d-simple. Let $\mathfrak{S} \subseteq \mathcal{O}(X) \setminus \{0\}$ be a multiplicative set and put $M = \mathcal{D}(X)/\mathcal{D}(X)(d+f)$, where $f \in \mathcal{O}(X)$. Suppose that $d(\mathfrak{S}) \subset \mathfrak{S}$.*

(1) *If N is a nonzero submodule of M then $N_{\mathfrak{S}}$ is a nonzero submodule of $M_{\mathfrak{S}}$.*
(2) *If $M_{\mathfrak{S}}$ is a simple $\mathcal{D}(X)_{\mathfrak{S}}$-module then M is a simple $\mathcal{D}(X)$-module.*

Proof. Suppose that J is a left ideal of $\mathcal{D}(X)$ which contains $\mathcal{D}(X)(d+f)$ *properly*. To prove (1) it is enough to show that

$$\mathcal{D}(X)_{\mathfrak{S}}(d+f) \subsetneq J_{\mathfrak{S}}.$$

Assume, by contradiction, that these ideals are equal after the localisation has been performed. Thus, given a nonzero element a of J there exists $s \in \mathfrak{S}$ such that $sa \in \mathcal{D}(X)(d+f)$. Taking symbols, we have that $s\sigma(a)$ belongs to the ideal of $S(X)$ generated by $\sigma(d)$. Since $S(X)$ is noetherian and commutative, the set

$$\mathfrak{S}_0 = \{s \in \mathfrak{S} : s\sigma(a) \in S(X)\sigma(d) \text{ for all } \ a \in J\}$$

must be nonempty. Now, if $s \in \mathfrak{S}_0$, then, using the involutivity of S$(X)\sigma(d)$, we have that

$$\{s\sigma(a), \sigma(d)\} = \sigma(a)\{s, \sigma(d)\} + s\{\sigma(a), \sigma(d)\} \in S(X)\sigma(d).$$

However, $\sigma(J)$ is involutive, so that $\{\sigma(a), \sigma(d)\} \in \sigma(J)$. Hence, $s\{\sigma(a), \sigma(d)\} \in S(X)\sigma(d)$, because we chose $s \in \mathfrak{S}_0$. Therefore,

$$d(s)\sigma(a) = \{\sigma(d), s\}\sigma(a) \in S(X)\sigma(d).$$

Since this holds for every $a \in J$, it follows that $d(s) \in \mathfrak{S}_0$. Hence, $d^j(s) \in \mathfrak{S}_0$ for all $j \geq 0$. However, the ideal of $\mathcal{O}(X)$ generated by $d^j(s)$, for $j \geq 0$, is stable under d. If $0 \neq s \in K$, then we are done; otherwise, since $\mathcal{O}(X)$ is d-simple, there exist $b_0, \dots, b_k \in \mathcal{O}(X)$ such that

$$b_0 s + b_1 d(s) + \cdots + b_k d^k(s) = 1.$$

Thus,

$$\sigma(a) = b_0 s\sigma(a) + b_1 d(s)\sigma(a) + \cdots + b_k d^k(s)\sigma(a) \in S(X)\sigma(d).$$

Now take $a \in J \setminus \mathcal{D}(X)(d+f)$ to be an element of smallest possible order. Since $\sigma(a) = \sigma(b)\sigma(d)$, for some $b \in \mathcal{D}(X)$, it follows that $a - b(d+f) \in J$ has smaller order than a. Thus $a - b(d+f) \in \mathcal{D}(X)(d+f)$, and so $a \in \mathcal{D}(X)(d+f)$, a contradiction. This proves (1).

To prove (2) we must show that $\mathcal{D}(X)(d+f)$ is a maximal left ideal. Let J be as above; then $\mathcal{D}(X)/J$ is a homomorphic image of M. Since $M_{\mathfrak{S}}$ is simple and localisation is an exact functor, it follows from (1) that $J_{\mathfrak{S}} = \mathcal{D}(X)_{\mathfrak{S}}$. Thus $\mathfrak{S} \cap J \neq \emptyset$. In particular, $\mathcal{O}(X) \cap J$ is a nonzero ideal of $\mathcal{O}(X)$. But $d+f \in J$, and so if $a \in \mathcal{O}(X) \cap J$, then

$$[d+f, a] = d(a) \in \mathcal{O}(X) \cap J.$$

Thus $\mathcal{O}(X) \cap J$ is a nonzero d-ideal of $\mathcal{O}(X)$. Since $\mathcal{O}(X)$ is d-simple, we conclude that $1 \in J$. Hence $\mathcal{D}(X)(d+f)$ is a maximal left ideal, as we wanted to prove. □

As usual, we denote the nth complex Weyl algebra by A_n. In other words, A_n is the ring of differential operators over the complex nth affine space. We will often use the following result from [**9**, Theorem 2.1].

THEOREM 2.2. *Let a be a polynomial, and $d = \partial_x + a\partial_y$ a derivation of $\mathbb{C}[x, y]$. If $\mathbb{C}[x, y]$ is d-simple, then there exists $f \in \mathbb{C}[x, y]$ such that $A_2/A_2(d + f)$ is a simple nonholonomic left A_2-module.*

Recall that, if I is a left ideal of $\mathcal{D}(X)$, then the module $\mathcal{D}(X)/I$ is *nonholonomic* if the symbol ideal $\sigma(I)$ has dimension greater than $\dim(X)$. In particular, if I is cyclic and $\dim(X) > 1$ then $\mathcal{D}(X)/I$ is nonholonomic, because $\dim(\sigma(I)) = 2\dim(X) - 1$ in this case. For more details on holonomic and nonholonomic modules see [**3**, chapters 10 and 11].

3. Derivations on surfaces. Throughout this section we will assume that $n \geq 2$ is an integer, and that $g \in K[x, y]$. Let S be the affine surface with equation $z^n - g = 0$ in $\mathbb{A}^3(K)$.

PROPOSITION 3.1. *S is smooth if and only if the curve $g = 0$ is smooth in $\mathbb{A}^2(K)$. In particular, if S is smooth then g is squarefree.*

Proof. Let $f = z^n - g(x, y)$. If $p = (x_0, y_0, z_0) \in \mathbb{A}^3(K)$ is a singular point of S, then

$$\frac{\partial f}{\partial z}(p) = nz_0^{n-1} = 0,$$

so that $f(p) = \nabla f(p) = 0$ is equivalent to

$$z_0 = g(x_0, y_0) = \nabla g(x_0, y_0) = 0.$$

But such a (x_0, y_0) exists if and only if the curve $g = 0$ is not smooth in $\mathbb{A}^2(K)$. $\square$

The main result of this section is an application of the idea of lifting a holomorphic foliation by a finite projection. Since the surfaces we are dealing with are fairly special, we will be able to prove a result that is far sharper than [**15**, Theorem 1].

THEOREM 3.2. *Let $d = a\partial_x + b\partial_y$ be a derivation of $K[x, y]$ with no stable ideal of height 1 and let $g \in K[x, y]$. If S is smooth then,*

$$nz^{n-1}d + d(g)\partial_z$$

induces a derivation Δ on $\mathcal{O}(S)$ that does not have any stable height 1 ideals. Moreover, the singularities of Δ are the zeros of the ideal $(z, g, d(g)) \cdot (z^n - g, a, b)$. In particular, Δ has a finite number of singularities.

Proof. A simple computation shows that

$$(nz^{n-1}d + d(g)\partial_z)(z^n - g) = 0,$$

so that $nz^{n-1}d + d(g)\partial_z$ induces a derivation Δ over $\mathcal{O}(S)$. Now, let $I \neq 0$ be a prime ideal of $\mathcal{O}(S)$ that is stable under Δ. It is convenient to split the proof into three parts.

FIRST PART: *If $I \cap K[x, y]$ is not stable under d then $(g, d(g)) \subseteq I$.*

Since $\mathcal{O}(S)$ is finite over $K[x, y]$, it follows that $I \cap K[x, y] \neq 0$ is a prime ideal of $K[x, y]$ of the same height as I. But I is stable under Δ, so that

$$nz^{n-1}d(I \cap K[x, y]) = \Delta(I \cap K[x, y]) \subseteq I.$$

Since I is prime, either $z \in I$ or $I \cap K[x, y]$ is stable under d. Also we are assuming that the latter does not occur, and so $z \in I$. Thus,

$$d(g) = \Delta(z) \in I \text{ and } g \in I,$$

which completes the proof of the first part.

SECOND PART: *No height one ideals of $\mathcal{O}(S)$ are stable under Δ.*

Suppose, now, that I is a height one prime ideal of $\mathcal{O}(S)$. Hence, $I \cap K[x, y]$ must be a height one prime ideal of $K[x, y]$. Thus, there exists an irreducible polynomial p which generates $I \cap K[x, y]$. But, by the first part,

$$(g, d(g)) \subseteq I \cap K[x, y] = (p).$$

Let $g = ph$, for some $h \in K[x, y]$. Since S is smooth, p cannot divide h. But

$$d(g) = hd(p) + d(h)p \in I \cap K[x, y] = (p),$$

implies that p divides $d(p)$, which contradicts the hypothesis on d. Therefore, I cannot have height one and be stable under Δ at the same time.

THIRD PART: We compute the singularities of Δ.

If $p \in K^3$ is a singularity of Δ, then either p belongs to the plane $z = 0$, or p is a singularity of d. Moreover, in the first case, p must also be a zero of g and $d(g)$. Thus, the singular set of Δ is equal to the zero set in K^3 of the ideal

$$I = (z, g, d(g)) \cdot (z^n - g, a, b).$$

Since this ideal is stable under Δ, it cannot have height one by the second part of the proof. In particular, the set of zeros of I in S is finite. □

COROLLARY 3.3. *If $K[x, y]$ is d-simple and $(g, d(g)) = K[x, y]$, then*
(1) *$\mathcal{O}(S)$ is Δ-simple;*
(2) *the module of Kähler differentials of S is free of rank two.*

Proof. The first part is an immediate consequence of the theorem. The second part follows from the first and from the following result of J. Archer [**1**, Theorem 2.5.18, p. 101].

> *Let S be a smooth surface in $\mathbb{A}^3(K)$. The module of Kähler differentials of S is free of rank two if and only if $\mathrm{Der}_K(S)$ contains a nonsingular derivation.* □

The second corollary combines the results above on derivations with the theorems of Section 2.

COROLLARY 3.4. *Let $f \in K[x, y]$ and $n > 1$ be an integer. If*
(1) *$K[x, y]$ is d-simple,*
(2) $\deg(g) > 1$,
(3) $(g, d(g)) = K[x, y]$,
(4) $\deg(a) \neq \deg(b)$*, and*
(5) *$A_2(d + f)$ is a maximal left ideal of A_2,*
then $\mathcal{D}(S)/\mathcal{D}(S)(\Delta + nz^{n-1}f)$ is a simple nonholonomic $\mathcal{D}(S)$-module.

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Proof. The proof consists in reducing the problem to the A_2-module $A_2/A_2(d+f)$, using Theorem 2.1. However, to do this, we must introduce an adequate multiplicative set of $\mathcal{O}(S)$. Consider

$$\mathfrak{S} = \{z^j h : 0 \leq j \leq n-1 \text{ and } h \in K[x, y] \setminus K\} \cup \{z^{n-1}\}.$$

Since

$$(z^{n-1})^2 = z^n \cdot z^{n-2} = gz^{n-2} \in \mathfrak{S},$$

it is easy to check that $\mathfrak{S}$ is a multiplicative set of $\mathcal{O}(S)$. However, we must also show that it is stable under Δ in order to apply Theorem 2.1. Recall, first of all, that Δ is simple by Corollary 3.3; so there exists no $h \in K[x, y] \setminus K$ such that $\Delta(h) = 0$. Now, $\deg(g) > 1$ implies that

$$\Delta(z^{n-1}) = (n-1)z^{n-2}d(g) \in \mathfrak{S}.$$

Moreover,

$$\Delta(h) = nz^{n-1}d(h) \in \mathfrak{S},$$

even when $d(h) \in K \setminus \{0\}$. Thus, we need only show that $\Delta(z^j h) \in \mathfrak{S}$ for some $j \geq 1$ and $h \in K[x, y] \setminus K$. But, under these hypotheses,

$$\Delta(z^j h) = z^{j-1}(nz^n d(h) + jd(g)h) = z^{j-1}(ngd(h) + jd(g)h),$$

in $\mathcal{O}(S)$. Hence, the right hand side of this equation belongs to $\mathfrak{S}$ if and only if $ngd(h) + jd(g)h$ is not a constant. Since this constant cannot be zero, we may assume, without loss of generality, that

$$ngd(h) + jd(g)h = 1.$$

Thus,

$$d(h^n g^j) = h^{n-1}g^{j-1}(ngd(h) + jd(g)h) = h^{n-1}g^{j-1}.$$

However,

$$\deg(h^{n-1}g^{j-1}) = (n-1)\deg(h) + (j-1)\deg(g),$$

while

$$\deg(d(h^n g^j)) \geq n\deg(h) + j\deg(g) + \min\{\deg(a), \deg(b)\} - 1.$$

Comparing the last two equations, we conclude that

$$\deg(g) + \deg(h) \leq 1,$$

which is a contradiction because $\deg(gh) \geq 2$. Therefore, $d(\mathfrak{S}) \subseteq \mathfrak{S}$, as required by Theorem 2.1.

Now, let $M = \mathcal{D}(S)/\mathcal{D}(S)(\Delta + nz^{n-1}f)$. Since $\mathcal{O}(S)$ is Δ-simple by Corollary 3.3, it follows from Theorem 2.1 that M is simple if and only if $M_{\mathfrak{S}}$ is simple. Therefore, we need only prove that $M_{\mathfrak{S}}$ is a simple $\mathcal{D}(S)_{\mathfrak{S}}$-module.

Denote by L and L_0 the quotient fields of $\mathcal{O}(S)$ and $K[x, y]$, respectively. Since

$$\left(\frac{\Delta}{nz^{n-1}} + f\right)\bigg|_{D(L_0)} = d + f,$$

it follows that

$$N = \mathcal{D}(L_0)/\mathcal{D}(L_0)((n^{-1}z^{1-n})\Delta + f) \cong \mathcal{D}(L_0)/\mathcal{D}(L_0)(d + f).$$

But this last module is a localization of $A_2/A_2(d + f)$, which is simple by hypothesis. Therefore, N is a simple module contained in

$$\mathcal{D}(L)/\mathcal{D}(L)((n^{-1}z^{1-n})\Delta + f) \cong M_{\mathfrak{S}}.$$

But this implies that $M_{\mathfrak{S}}$ is simple, by the proof of [**4**, Theorem 2.2(2), p. 408]. This completes the proof. □

4. The examples. The following notation will be in force throughout the section. Given an integer $n \geq 2$, a polynomial $g \in K[x, y]$, and a derivation d of $K[x, y]$, the surface of $\mathbb{A}^3(K)$ with equation $z^n - g = 0$ will be denoted by S and Δ will stand for the derivation of $\mathcal{O}(S)$ induced by $nz^{n-1}d - d(g)\partial_z$. We begin with the example of a simple nonholonomic module mentioned at the introduction.

EXAMPLE 4.1. Let $g(x) \in K[x]$ be a squarefree polynomial of degree 3. Then,
(1) $\mathrm{Pic}(S) \neq 0$, and
(2) there exist nonholonomic simple modules over $\mathcal{D}(S)$.

Proof. Let E be the curve of $\mathbb{A}^3(K)$ with equations $y = z^2 - g = 0$. Then, $S \cong E \times \mathbb{A}^1$. Thus,

$$\mathrm{Pic}(S) \cong \mathrm{Pic}(E) \neq 0,$$

since E is an elliptic curve. This proves (1). In order to prove (2), choose a derivation of $K[x, y]$ of the form $d = \partial_x + h(x, y)\partial_y$, with respect to which $K[x, y]$ is d-simple; see [**1**], [**4**], [**16**] or Example 4.2. Then, by Theorem 2.2, there exists $f(x, y) \in K[x, y]$ such that $A_2(d + f)$ is a maximal left ideal of A_2, the Weyl algebra over $K[x, y]$. Now consider the module

$$M = \mathcal{D}(S)/\mathcal{D}(S)(\Delta + nz^{n-1}f).$$

Since g is squarefree in one variable, it follows that

$$(g, d(g)) = (g, dg/dx) = (1).$$

Therefore, M is simple by Corollary 3.4. □

A similar construction can be made by taking the simple derivation defined in [**16**] as a starting point. Instead of that, we give a more general example in the same vein.

EXAMPLE 4.2. Let $g_0, g_1 \in K[y]$ be nonzero polynomials with $\deg(g_1) \geq \deg(g_0) > 1$ and no common roots. If $g = xg_1 + g_0$ and $d = \partial_x + g\partial_y$, then
(1) $K[x, y]$ is d-simple;
(2) $\mathcal{O}(S)$ is Δ-simple, and

(3) there exists $f \in K[x, y]$ such that $\mathcal{D}(S)/\mathcal{D}(S)(\Delta + nz^{n-1}f)$ is a simple nonholonomic $\mathcal{D}(S)$-module.

Proof. Throughout the proof we denote the derivative of a polynomial $q \in K[y]$ with respect to y by q'. We have that

$$(g, d(g)) = (g, g_1 + xgg_1' + gg_0') = (g, g_1) = (g_1, g_0) = (1),$$

because these polynomials in one variable have no common roots. Thus, (2) follows from (1) and Corollary 3.3, whilst (3) follows from (1), Theorem 2.2 and Corollary 3.4. Hence, it is enough to prove (1). The proof follows the approach introduced by D. Jordan in [**10**].

We proceed by contradiction. Suppose that d has a stable nonconstant irreducible polynomial $f \in K[x, y]$, and write

$$f = a_n(y)x^n + \cdots + a_1(y)x + a_0(y),$$

where $a_n, \ldots, a_0 \in K[y]$. Thus, $d(f) = hf$, for some $h \in K[x, y]$. If $n = 0$ then

$$g\frac{\partial f}{\partial y} = hf;$$

which implies that f divides g. But this is a contradiction since g is an irreducible polynomial. Therefore, $n \geq 1$. Thus,

$$d\left(\frac{f}{a_n}\right) = \frac{f}{a_n}\left(h - \frac{d(a_n)}{a_n}\right). \tag{4.1}$$

In other words, $\widehat{f} = f/a_n \in K(y)[x]$ is stable under d. Let

$$\widehat{f} = x^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0,$$

where $b_j = a_j/a_n \in K(y)$. The term of degree n of $d(\widehat{f})$ as a polynomial in x is

$$g_1b_{n-1}'x^n,$$

which has degree n in x. Since $\deg_x(\widehat{f}) = n$, it follows from (4.1) that $h - d(a_n)/a_n$ must have degree zero as a polynomial in x. In particular, $h \in K[y]$.

However, the term of degree n of $d(f)$, as a polynomial in x, is

$$g_1a_n'x^{n+1},$$

whilst hf has degree n in x. Since $g_1 \neq 0$, it follows that $a_n' = 0$. Therefore, we may assume, without loss of generality, that $a_n = 1$.

Equating the coefficients of the terms of degree j in x on both sides of $d(f) = hf$, we obtain

$$(j+1)a_{j+1} + g_1a_{j-1}' + g_0a_j' = ha_j. \tag{4.2}$$

For $j = n$, this implies that $h = g_1a_{n-1}'$. Taking this into (4.2) we have that

$$(j+1)a_{j+1} + g_1a_{j-1}' + g_0a_j' = g_1a_{n-1}'a_j. \tag{4.3}$$

Suppose first that $a_{n-1} \in K$. Hence,

$$h = g_1 a'_{n-1} = 0.$$

Taking this into (4.3) with $j = n-1$, we get

$$n + g_1 a'_{n-2} = 0,$$

which implies that $n = 0$, a contradiction. Thus, we may assume from now on that $a'_{n-1} \neq 0$.

We will now prove, by induction on k, the equality

$$\deg(a_k) = (n-k)\deg(a_{n-1}), \tag{$E(k)$}$$

for all $-1 \leq k \leq n-1$. Since $E(n)$ and $E(n-1)$ are obviously true, we show that $E(j-1)$ holds whenever $E(k)$ is true for all $j \leq k \leq n-1$.

From $E(j)$ and $E(j+1)$ we get that $\deg(a_{j+1}) \leq \deg(a_j)$, and since $\deg(g_1) \geq \deg(g_0) > 1$, it follows that

$$\deg(a_{j+1}) \leq \deg(g_0 a'_j) < \deg(g_1 a'_{n-1} a_j).$$

Hence, $\deg(g_1 a'_{j-1}) = \deg(g_1 a'_{n-1} a_j)$, so that

$$\deg(a'_{j-1}) = \deg(a'_{n-1}) + \deg(a_j).$$

Thus, by the induction hypothesis,

$$\deg(a_{j-1}) = \deg(a_{n-1}) + \deg(a_j) = \deg(a_{n-1}) + (n-j)\deg(a_{n-1}),$$

from which $E(j-1)$ is an immediate consequence. However, $E(-1)$ gives

$$\deg(a_{-1}) = (n+1)\deg(a_{n-1}),$$

which is a contradiction, since f is a polynomial. Therefore, d does not have a stable polynomial with $a'_{n-1} \neq 0$, and the proof is complete. □

Example 4.3. Let $\beta \in \mathbb{C}[x, y]$ be a generic homogeneous polynomial of degree $k \geq 3$, and let g be a linear factor of β. If $b \in \mathbb{C} \setminus \{0\}$ and $d = ((x+y)\beta + b)\partial_x + \beta\partial_y$ then $\mathcal{O}(S)$ is a Δ-simple ring.

Proof. Let $\lambda = x + y$. By [**5**, Corollary 4.3, p. 460] the polynomial ring $\mathbb{C}[x, y]$ is d-simple. Since β is generic, we may assume that $g = \alpha_1 x + \alpha_2 y$, where $\alpha_1, \alpha_2 \in \mathbb{C}$ and $\alpha_1 \neq 0$. Thus,

$$(g, d(g)) = (g, \alpha_1(\lambda\beta + b) + \beta\alpha_2).$$

Since g divides β, it follows that

$$(g, d(g)) = (g, \alpha_1 b) = (1),$$

because $\alpha_1 b \in \mathbb{C} \setminus \{0\}$. The result now follows from Corollary 3.3. □

Finally, we give an example where the singularity set of the derivation is nonempty. Compare this with [**7**].

EXAMPLE 4.4. Suppose that $n \geq 5$ is prime, and let

$$g = xy^{n-1} + y + x^{n-1}.$$

If d is a derivation of $\mathbb{C}[x, y]$ without any stable height one ideals, but whose singular set is nonempty, then there exists $f \in \mathcal{O}(S)$ such that

$$M = \mathcal{D}(S)/\mathcal{D}(S)(\Delta + nz^{n-1}f)$$

is a simple nonholonomic $\mathcal{D}(S)$-module.

Proof. An easy computation shows that S is a smooth surface. Moreover, we know from Theorem 3.2 that Δ is a derivation of S, without stable height one ideals, whose singular set is finite. However, since the singular set of d is nonempty, then so is that of Δ. Finally, the result follows from [**6**, Theorem 3.5, p. 350] because by [**17**, Theorem 4.1, p. 312] the Picard group of the surface S is zero. We need to know that n is prime in order to apply this last result; see [**17**, Equation (4.7), p. 313]. □

The best known example of a derivation satisfying the conditions of Example 4.4 is

$$(y^{k-1}x - 1)\partial_x + (y^k - x^{k-1})\partial_y,$$

which was originally proposed by Jouanolou in [**11**, p. 157]. One can also produce such examples using a computer, as shown in [**8**].

REFERENCES

1. J. Archer, *Derivations on commutative rings and projective modules over skew polynomial rings*, PhD thesis, Leeds University (1981).

2. P. Brumatti, Y. Lequain and D. Levcovitz, Differential simplicity in polynomial rings and algebraic independence of power series, *J. London Math. Soc. (2)* **68** (2003), 615–630.

3. S. C. Coutinho, A primer of algebraic $\mathcal{D}$-modules, London Mathematical Society Student Texts 33 (Cambridge University Press, 1995).

4. S. C. Coutinho, d-simple rings and simple $\mathcal{D}$-modules, *Math. Proc. Cambridge Phil. Soc.* **125** (1999), 405–415.

5. S. C. Coutinho, On the differential simplicity of polynomial rings, *J. Algebra* **264** (2003), 442–468.

6. S. C. Coutinho, Indecomposable non-holonomic $\mathcal{D}$-modules in dimension 2, *Proc. Edinburgh Math. Soc. (2)* **46** (2003), 341–355.

7. S. C. Coutinho, Non-holonomic irreducible $\mathcal{D}$-modules over complete intersections, *Proc. Amer. Math. Soc.* **131** (2003), 83–86.

8. S. C. Coutinho and L. Menasché Schechter, Algebraic solutions of holomorphic foliations: an algorithmic approach, *J. Symbolic Computation* **41**(5) (2006), 603–618.

9. A. M. Doering, Y. Lequain and C. C. Ripoll, Differential simplicity and cyclic maximal left ideals of the Weyl algebra $A_2(K)$, *Glasgow Math. J.* **48** (2006), 269–274.

10. D. Jordan, Differentiably simple rings with no invertible derivatives, *Quart. J. Math. Oxford Ser. 2*, **32** (1981), 417–424.

11. J. P. Jouanolou, *Equations de Pfaff algébriques* Lecture Notes in Mathematics No. 708 (Springer-Verlag, 1979).

12. K. R. Goodearl and R. B. Warfield, *An introduction to noncommutative noetherian rings*, London Mathematical Society Texts 16 (Cambridge University Press, 1989).

13. A. Maciejewski, J. Moulin-Ollagnier and A. Nowicki, Simple quadratic derivations in two variables, *Comm. Algebra* **29** (2001), 619–638.

14. J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings* (John Wiley & Sons, Chichester, 1987); revised edition by the American Mathematical Society, Providence, 2001.

15. L. G. Mendes, Algebraic foliations without algebraic solutions, *An. Acad. bras. Ci.* **69** (1997), 11–13.

16. A. Nowicki, An example of a simple derivation in two variables, preprint.

17. T. Shioda, On the Picard number of a complex projective variety, *Ann. Scient. Éc. Norm.* Sup., 4^{e} série, **14** (1981), 303–321.