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ON NONABELIAN TENSOR ANALOGUES OF 2-ENGEL CONDITIONS

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Abstract. Tensor analogues of right 2-Engel elements in groups were introduced by D. P. Biddle and L.-C. Kappe. We investigate the properties of right 2-Engel tensor elements and introduce the concept of $2_\otimes$-Engel margin. With the help of these results we describe the structure of $2_\otimes$-Engel groups. In particular, we prove a tensor version of Levi's theorem for 2-Engel groups and determine tensor squares of two-generator $2_\otimes$-Engel p-groups.

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1. Introduction. For any group G, *the nonabelian tensor square* $G \otimes G$ is a group generated by the symbols $g \otimes h$, subject to the relations

$$gg' \otimes h = (g^{g'} \otimes h^{g'})(g' \otimes h) \text{ and } g \otimes hh' = (g \otimes h')(g^{h'} \otimes h^{h'}),$$

where $g, g', h, h' \in G$ and $g^h = h^{-1}gh$. The more general concept of nonabelian tensor product of groups acting on each other in certain compatible way was introduced by R. Brown and J.-L. Loday in [**5**], following the ideas of R. K. Dennis [**6**]. This construction has its origins in algebraic K-theory as well as in homotopy theory, yet it has become interesting from a purely group-theoretical point of view since the paper of R. Brown, D. L. Johnson and E. F. Robertson [**4**]. Since then, many authors have been concerned with explicit computations of nonabelian tensor squares; see the paper of L.-C. Kappe [**9**] for a comprehensive survey of these results.

The main topic of [**3**] is consideration of tensor analogues of the center and centralizers in groups. More precisely, for a given group G the subgroup $Z^\otimes(G)$ consisting of all $a \in G$ with $a \otimes x = 1_\otimes$ for every $x \in G$ is called *the tensor center*. This concept was introduced by G. J. Ellis [**7**]. Moreover, for a group G and a non-empty subset X, the subgroup $C_G^\otimes(X) = \{a \in G : a \otimes x = 1_\otimes \text{ for all } x \in X\}$ is said to be *the tensor annihilator* of X in G. Also, tensor analogues of right n-Engel elements have been defined. Recall that *the set of right n-Engel elements* of a group G is defined by $R_n(G) = \{a \in G : [a, {}_n x] = 1 \text{ for all } x \in G\}$. Here $[a, {}_n x]$ stands for the commutator $[\cdots[[a, x], x], \cdots]$ with n copies of x. It is well-known that $R_1(G) = Z(G)$ and that $R_2(G)$ is a subgroup of G [**13**]. In contrast with this, it was shown that for $n \geq 3$ the set $R_n(G)$ is not necessarily a subgroup [**14**]. *The set of right $n_\otimes$-Engel elements* of a group G is

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then defined as

$$R_n^{\otimes}(G) = \{a \in G : [a, {}_{n-1}x] \otimes x = 1_{\otimes} \text{ for all } x \in G\}.$$

One of the results of [**3**] shows that $R_2^{\otimes}(G)$ is always a characteristic subgroup of G containing $Z(G)$ and contained in $R_2(G)$. It is also shown by an example that these inclusions may be proper.

The purpose of this paper is to further investigate tensor analogues of 2-Engel structure in groups. In the first part of the paper we determine some further information about $R_2^{\otimes}(G)$ and provide some new characterizations of this subgroup. In particular, we define the tensor analogue of 2-Engel margin and show that there is a striking resemblance between the results about 2-Engel margin and the results about its tensor analogue. We use these results to obtain the structure of $2_{\otimes}$-Engel groups. Here the group G is said to be *$n_{\otimes}$-Engel* when $[x, {}_{n-1}y] \otimes y = 1_{\otimes}$ for any $x, y \in G$. It is straightforward to see that every $2_{\otimes}$-Engel group is also 2-Engel. A well-known result of F. W. Levi (see [**15**, pp. 45–46]) states that every 2-Engel group G is metabelian and nilpotent of class ≤ 3 and the exponent of $\gamma_3(G)$ divides 3. Therefore it is hardly surprising that the following result is obtained: if G is a $2_{\otimes}$-Engel group, then $G \otimes G$ is abelian, $\gamma_3(G) \leq Z^{\otimes}(G)$ and $([x, y] \otimes z)^3 = 1_{\otimes}$ for every $x, y, z \in G$. As a consequence, we obtain several characterizations of $2_{\otimes}$-Engel groups, once again indicating the strong correspondence between 2-Engel groups and $2_{\otimes}$-Engel groups.

Let $\mathfrak{G}$ be a group-theoretic property. A group G is said to have *a finite covering by $\mathfrak{G}$-subgroups* if G equals, as a set, to the union of finite family of $\mathfrak{G}$-subgroups. The finite coverings of groups by their 2-Engel subgroups were studied by L.-C. Kappe [**10**]. It is proved in that paper that a group G has a finite covering by 2-Engel subgroups if and only if $|G : R_2(G)| < \infty$. The situation is similar in the context of $2_{\otimes}$-Engel groups. We prove that a group G can be covered by a finite family of $2_{\otimes}$-Engel subgroups if and only if $|G : R_2^{\otimes}(G)| < \infty$. Another result of [**10**] in this direction is that G has a finite covering by 2-Engel normal subgroups if and only if G is 3-Engel and $|G : R_2(G)| < \infty$. It is to be expected that there is a tensor analogue of this result, but we leave it for future consideration. It is not difficult to see that if G has a finite covering by $2_{\otimes}$-Engel normal subgroups, then G is $3_{\otimes}$-Engel and $|G : R_2^{\otimes}(G)| < \infty$. For the reverse conclusion one would probably need the characterization of $3_{\otimes}$-Engel groups by their normal closures analogous to [**12**].

Since every $2_{\otimes}$-Engel group has an abelian tensor square, there is a good chance to compute tensor squares of $2_{\otimes}$-Engel groups explicitly. We reduce these computations to consideration of tensor squares of groups of class ≤ 2. With the help of this we compute tensor squares of two-generator $2_{\otimes}$-Engel p-groups, using the results of [**1**] and [**11**]. It is worth mentioning that there is a minor error in the classification of two-generator p-groups of class 2 given by [**1**], so we give the correct result here. We also compute the kernel of the commutator map $\kappa : G \otimes G \to G'$ given by $g \otimes h \mapsto [g, h]$ for any nonabelian two-generator $2_{\otimes}$-Engel p-group G. The group $\ker \kappa$ is of interest as it is isomorphic to the third homotopy group of the space $SK(G, 1)$ [**5**]. In addition, we compute the Schur multiplier of G.

2. Preliminary results. In this section we summarize without proofs some basic results regarding computations in tensor squares and the results concerning 2-Engel groups which will be used throughout the paper without any further reference. The first lemma gives the right action version of [**5**, Proposition 3].

LEMMA 1 ([**5**]). *Let $g, g', h, h' \in G$. The following relations hold in $G \otimes G$:*
(a) $(g^{-1} \otimes h)^g = (g \otimes h)^{-1} = (g \otimes h^{-1})^h$.
(b) $(g' \otimes h')^{g \otimes h} = (g' \otimes h')^{[g,h]}$.
(c) $[g, h] \otimes g' = (g \otimes h)^{-1}(g \otimes h)^{g'}$.
(d) $g' \otimes [g, h] = (g \otimes h)^{-g'}(g \otimes h)$.
(e) $[g, h] \otimes [g', h'] = [g \otimes h, g' \otimes h']$.

Note here that G acts on $G \otimes G$ by $(g \otimes h)^{g'} = g^{g'} \otimes h^{g'}$. The next result is crucial in studying the analogy between commutators and tensors.

PROPOSITION 1 ([**4**]). *For a given group G there exists a homomorphism $\kappa : G \otimes G \to G'$ such that $\kappa : g \otimes h \mapsto [g, h]$. Moreover,* $\ker \kappa \leq Z(G \otimes G)$ *and G acts trivially on* $\ker \kappa$.

An element a of a group G is called *a right* 2*-Engel element* of G if $[a, x, x] = 1$ for each $x \in G$. In a similar fashion, an element a is said to be *a left* 2*-Engel element* of G if $[x, a, a] = 1$ for each $x \in G$. The sets of right 2-Engel elements and left 2-Engel elements of G are denoted by $R_2(G)$ and $L_2(G)$, respectively. For the properties of right 2-Engel elements we refer to [**15**, Theorem 7.13] and [**16**, Lemma 2.2, Theorem 2.3]. We list here some of them, especially those which turn out to have tensor analogues.

PROPOSITION 2 ([**15**], [**16**]). *Let G be a group, $a \in R_2(G)$ and $x, y, z \in G$.*
(a) *a is also a left* 2*-Engel element and a^G is abelian.*
(b) $[a, x]^{rs} = [a^r, x^s]$ *for all $r, s \in \mathbb{Z}$.*
(c) $[a, x, y] = [a, y, x]^{-1}$.
(d) $[a, [x, y]] = [a, x, y]^2$.
(e) $a^2 \in Z_3(G)$.
(f) $[a, [x, y], z] = 1$.

Here a^G denotes the normal closure of a in G. This result is the main ingredient of the proof of Levi's theorem [**15**, pp. 45–46] that every 2-Engel group G is nilpotent of class ≤ 3 and the exponent of $\gamma_3(G)$ divides 3. We also list some characterizations of 2-Engel groups which will serve as a model for $2_\otimes$-Engel groups.

PROPOSITION 3 ([**15**]). *For a group G the following assertions are equivalent:*
(a) *G is a* 2*-Engel group.*
(b) *$C_G(x)$ is a normal subgroup of G for every $x \in G$.*
(c) $[x, [y, z]] = [x, y, z]^2$ *for any $x, y, z \in G$.*
(d) $[x, z, y]^{-1} = [x, y, z]$ *for any $x, y, z \in G$.*
(e) *x^G is abelian for every $x \in G$.*

3. Right $2_\otimes$-Engel elements of groups.

The main object of this section is the study of tensor analogues of right (left) 2-Engel elements of a given group. More precisely, for an arbitrary group G we define *the sets of right* (*left*) *$2_\otimes$-Engel elements* of G by $R_2^\otimes(G) = \{a \in G : [a, x] \otimes x = 1_\otimes$ for all $x \in G\}$ and $L_2^\otimes(G) = \{a \in G : [x, a] \otimes a = 1_\otimes$ for all $x \in G\}$, respectively. At the beginning we formulate some elementary properties of these two sets.

LEMMA 2. *Let G be any group.*
(a) *$R_2^\otimes(G) \subseteq R_2(G)$, $L_2^\otimes(G) \subseteq L_2(G)$.*
(b) *Every right $2_\otimes$-Engel element of G also belongs to $L_2^\otimes(G)$.*
(c) $L_2^\otimes(G) = \{a \in G : a^x \otimes a^y = a \otimes a$ *for all $x, y \in G\}$.*

Proof. Let $\kappa : G \otimes G \to G'$ be the commutator map. Let $a \in R_2^{\otimes}(G)$ and $x \in G$. Then we get $1 = \kappa([a, x] \otimes x) = [a, x, x]$, hence $a \in R_2(G)$. The inclusion $L_2^{\otimes}(G) \subseteq L_2(G)$ is proved in a similar way, therefore (a) is proved. To prove (b), pick $a \in R_2^{\otimes}(G)$ and $x \in G$. Then we have $1_{\otimes} = [a, ax] \otimes ax = [a, x] \otimes ax = ([a, x] \otimes a)^x = ([x, a] \otimes a)^{-[a,x]x}$, hence $[x, a] \otimes a = 1_{\otimes}$ and therefore $a \in L_2^{\otimes}(G)$. So we are left with the proof of (c). Let $S = \{a \in G : a^x \otimes a^y = a \otimes a$ for all $x, y \in G\}$. For $a \in S$ and $x \in G$ we have $[a, x] \otimes a = a^{-1}a^x \otimes a = (a^{-1} \otimes a)^{a^x}(a^x \otimes a) = 1_{\otimes}$, hence $a \in L_2^{\otimes}(G)$. Conversely, let $a \in L_2^{\otimes}(G)$ and $x, y \in G$. Then we obtain $a^x \otimes a^y = (a^{xy^{-1}} \otimes a)^y = (a[a, xy^{-1}] \otimes a)^y = (a \otimes a)^{[a,xy^{-1}]y}([a, xy^{-1}] \otimes a)^y$. Since G acts trivially on $\ker \kappa$, we have $(a \otimes a)^{[a,xy^{-1}]y} = a \otimes a$, whereas $[a, xy^{-1}] \otimes a = 1_{\otimes}$ by (b). This proves the assertion. □

The following theorem is already proved in [**3**].

Theorem 1 ([**3**]). *For any group G, the set of all right $2_{\otimes}$-Engel elements of G is a characteristic subgroup of G.*

The computations with tensors involving right $2_{\otimes}$-Engel elements are facilitated by the following result which has roots in corresponding rules for computation with 2-Engel elements [**15**, Theorem 7.13]. Before formulating the result, note that

$$Z_n^{\otimes}(G) = \{a \in G : [a, x_1, \dots, x_{n-1}] \otimes x_n = 1_{\otimes} \text{ for all } x_1, \dots, x_n \in G\}$$

is a characteristic subgroup of G contained in the n-th center $Z_n(G)$. This subgroup is called *the n-th tensor center* of G [**3**].

Proposition 4. *Let G be a group, $x, y, z \in G$ and $a \in R_2^{\otimes}(G)$.*
(a) $[a, x] \otimes y = ([a, y] \otimes x)^{-1}$.
(b) $[a, x] \in C_G^{\otimes}(x^G)$.
(c) $[a, x]^n \otimes y = ([a, x] \otimes y)^n$ *for any* $n \in \mathbb{Z}$.
(d) $a \otimes x^n = (a \otimes x)^n$ *for any* $n \in \mathbb{Z}$.
(e) $[a, x] \otimes [y, z] = 1_{\otimes}$.
(f) $[x, y] \otimes a = ([x, a] \otimes y)^2$ *and* $a \otimes [x, y] = ([a, x] \otimes y)^2$.
(g) $a^2 \in Z_3^{\otimes}(G)$.

Proof. The identities (a) and (b) are already proved in [**3**, Lemma 5.1 and Lemma 5.2]. To prove (c), it suffices to assume that $n > 0$. Now observe that $[a, x]^n \otimes y = ([a, x] \otimes y)([a, x]^{n-1} \otimes y)$; hence (c) follows by an induction on n.

Before we proceed, note first that (a) implies that the elements of the form $b \otimes z$, where $b \in a^G$ and $z \in G$, commute with each other. Expanding $a \otimes xy$ and $xy \otimes a$ using the tensor product rules, we have

$$a \otimes xy = (a \otimes x)(a \otimes y)([a, x] \otimes y) \tag{1}$$

and

$$xy \otimes a = (x \otimes a)(y \otimes a)([x, a] \otimes y). \tag{2}$$

The first equation yields

$$a \otimes [x, y] = a \otimes (yx)^{-1}(xy) = (a \otimes xy)(a \otimes yx)^{-1}([a, (yx)^{-1}] \otimes xy)$$

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by [**3**, Lemma 5.1]. Since xy is a conjugate of yx, we have $[a,(yx)^{-1}]\otimes xy = 1_\otimes$ by (b), hence $a\otimes[x,y]=([a,x]\otimes y)^2$. Similarly we prove $a\otimes[x,y]=([a,x]\otimes y)^2$. It is also clear that the equation (1) also implies (d).

It remains to prove that $[a,x]\otimes[y,z]=1_\otimes$ and $a^2\in Z_3^\otimes(G)$. Expanding the identity $[a,x]\otimes yz=([a,yz]\otimes x)^{-1}$, we obtain that $([a,x]\otimes z)([a,x]\otimes y)^z = ([a,z]\otimes x)^{-[a,y]^z}([a,y]\otimes[z^{-1},x^{-1}]x)^{-z}$. Since $[a,z,x]\otimes[a^z,y^z]=1_\otimes$, it follows that $[a,y]^z$ acts trivially on $[a,z]\otimes x$. Thus we obtain, after cancellation and relabelling, $1_\otimes = [a,y]\otimes[x,z]=([a,[x,z]]\otimes y)^{-1}=([a,x,z]^2\otimes y)^{-1}$, hence $[a^2,x,y]\otimes z = 1_\otimes$. □

The immediate consequence of Proposition 4 is the following characterization of $R_2^\otimes(G)$.

Corollary 1. *For any group G we have $R_2^\otimes(G)=\{a\in G:[a,x]\in C_G^\otimes(x^G)$ for all $x\in G\}$.*

It is known that $a\in R_2(G)$ implies that a^G is abelian. The following corollary gives the corresponding result for right $2_\otimes$-Engel elements.

Corollary 2. *Let $a\in R_2^\otimes(G)$. Then the normal closure $(a\otimes x)^{G\otimes G}$ is an abelian group for any $x\in G$.*

Proof. Let $a\in R_2^\otimes(G)$ and $\tau\in G\otimes G$. As usual, denote with κ the commutator map $G\otimes G\to G'$. Then we have $[(a\otimes x),(a\otimes x)^\tau]=[a\otimes x,(a\otimes x)^{\kappa(\tau)}]=[a,x]\otimes[a^{\kappa(\tau)},x^{\kappa(\tau)}]=1_\otimes$ by Proposition 4. It follows by conjugation that every two elements of $(a\otimes x)^{G\otimes G}$ commute, as required. □

Let $\phi(x_1,\ldots,x_n)$ be any word in the variables $x_1,\ldots,x_n$. For a group G *the associated marginal subgroup* $\phi^*(G)$ (also called *the ϕ-margin* of G) consists of all $a\in G$ such that $\phi(g_1,\ldots,ag_i,\ldots,g_n)=\phi(g_1,\ldots,g_i,\ldots,g_n)$ for every $g_i\in G$ and $1\le i\le n$. It is clear that $\phi^*(G)$ is always a characteristic subgroup of G. Margins were first introduced by P. Hall [**8**]. In particular, marginal subgroups for the 2-Engel word $\phi(x,y)=[x,y,y]$ were studied by T. K. Teague [**16**]. Let $E_1(G)=\{a\in G:[ax,y,y]=[x,y,y]$ for all $x,y\in G\}=R_2(G)$ and $E_2(G)=\{a\in G:[x,ay,ay]=[x,y,y]$ for all $x,y\in G\}$. Then *the 2-Engel margin* of G is $E(G)=E_1(G)\cap E_2(G)$. Now, the tensor analogues of these subgroups can be defined as

$$E_1^\otimes(G)=\{a\in G:[ax,y]\otimes y=[x,y]\otimes y \text{ for all } x,y\in G\},$$
$$E_2^\otimes(G)=\{a\in G:[x,ay]\otimes ay=[x,y]\otimes y \text{ for all } x,y\in G\},$$

and let $E^\otimes(G)=E_1^\otimes(G)\cap E_2^\otimes(G)$. It is not difficult to see that these sets are characteristic subgroups of G. Using Proposition 4, we also conclude that $E_1^\otimes(G)=R_2^\otimes(G)$.

In [**16**, Theorem 2.4] it is proved that $E(G)=\{a\in G:[x,a,y][x,y,a]=1$ for all $x,y\in G\}$. The following result is therefore hardly surprising.

Theorem 2. *For any group G we have*

$$E^\otimes(G)=\{a\in G:([x,a]\otimes y)([x,y]\otimes a)=1_\otimes \text{ for all } x,y\in G\}.$$

Proof. Let $S=\{a\in G:([x,a]\otimes y)([x,y]\otimes a)=1_\otimes$ for all $x,y\in G\}$, let $a\in S$ and $x,y\in G$. It is clear that $a\in R_2^\otimes(G)=E_1^\otimes(G)$. Using Proposition 4, we have that $[x,ay]\otimes ay=[x,y][x,a]^y\otimes ay=([x,y][x,a]^y\otimes y)([x,y][x,a]^y\otimes a)^y = ([x,y]\otimes y)^{[x,a]^y}([x,a]\otimes y)^y([x,y]\otimes a)^{[x,a]^y y}([x,a]^y\otimes a)^y=([x,y]\otimes y)^{[x,a]^y}([x,a]^y\otimes a)^y$. Observe that $([x,a]^y\otimes a)^y=(a^{-xy}a^y\otimes a)^y=(a\otimes a)^{-1}(a\otimes a)=1_\otimes$ by Lemma 2;

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hence we only have to prove that $[x, a]^y$ acts trivially on $[x, y] \otimes y$. To see this, we first note that $y^{[x,a]^y} = [y, [x, a]]y$, hence $([x, y] \otimes y)^{[x,a]^y} = [x, y] \otimes [y, [x, a]]y$. As $[x, a] \in R_2^{\otimes}(G)$, we get $[[x, a], y] \otimes [x, y] = ([[x, a], [x, y]] \otimes y)^{-1} = 1_{\otimes}$ by Proposition 4, thus the inclusion $S \subseteq E^{\otimes}(G)$ is proved. Conversely, every $a \in E^{\otimes}(G)$ also belongs to $R_2^{\otimes}(G)$. Reversing the above arguments, we obtain $a \in S$, as required. □

Let us mention an important consequence of this theorem.

COROLLARY 3. *Let G be a group, $x, y \in G$ and $a \in E^{\otimes}(G)$. Then* $([a, x] \otimes y)^3 = [a^3, x] \otimes y = 1_{\otimes}$.

Proof. For $a \in E^{\otimes}(G)$ we get $1_{\otimes} = ([x, y] \otimes a)([x, a] \otimes y) = ([x, a] \otimes y)^3$ by Proposition 4, hence also $[a^3, x] \otimes y = 1_{\otimes}$. □

It is proved in [**16**] that $Z_2(G) \leq E(G) \leq Z_3(G)$ for any group G. Similar arguments show the following.

PROPOSITION 5. *For any group G we have* $Z_2^{\otimes}(G) \leq E^{\otimes}(G) \leq Z_3^{\otimes}(G)$.

Proof. It is clear that $Z_2^{\otimes}(G) \leq E^{\otimes}(G)$. Now, if $a \in E^{\otimes}(G)$, then $a^3 \in Z_2^{\otimes}(G) \leq Z_3^{\otimes}(G)$. On the other hand, we have $a^2 \in Z_3^{\otimes}(G)$ by Proposition 4, hence $a \in Z_3^{\otimes}(G)$. □

4. $2_{\otimes}$-Engel groups. A group G is said to be $2_{\otimes}$-*Engel* when $[x, y] \otimes y = 1_{\otimes}$ for any $x, y \in G$. It is worth noting that G is $2_{\otimes}$-Engel precisely when $R_2^{\otimes}(G) = G$, which is equivalent to $L_2^{\otimes}(G) = G$ and is also equivalent to $E^{\otimes}(G) = G$. Using the commutator map argument, it becomes clear that every $2_{\otimes}$-Engel group is also 2-Engel. The structure of $2_{\otimes}$-Engel groups is described in the next result which corresponds to the well-known Levi's theorem about 2-Engel groups [**15**, pp. 45–46]:

THEOREM 3. *Let G be a $2_{\otimes}$-Engel group. Then we have:*
(a) *$G \otimes G$ is abelian group;*
(b) *$\gamma_3(G) \leq Z^{\otimes}(G)$;*
(c) $([x, y] \otimes z)^3 = 1_{\otimes}$ *for any $x, y, z \in G$.*

Proof. It follows directly from Proposition 4 that $G \otimes G$ is abelian. From the same proposition we obtain $([x, y, z] \otimes v)^2 = [x, y, z]^2 \otimes v = [x, [y, z]] \otimes v = ([x, v] \otimes [y, z])^{-1} = 1_{\otimes}$. Furthermore, since $E^{\otimes}(G) = G$, we get (b) and (c) by Corollary 3. □

In contrast with this result, there exists a 2-Engel group G such that $\mathrm{cl}\,(G \otimes G) = 2$ [**2**]. The following is a tensor analogue of Proposition 3.

COROLLARY 4. *The following statements for a group G are equivalent.*
(a) *G is $2_{\otimes}$-Engel.*
(b) $[x, y] \otimes z = ([x, z] \otimes y)^{-1}$ *for any $x, y, z \in G$.*
(c) $x \otimes [y, z] = ([x, y] \otimes z)^2$ *for any $x, y, z \in G$.*
(d) $x^y \otimes x^z = x \otimes x$ *for any $x, y, z \in G$.*
Additionally, if G is a $2_{\otimes}$-Engel group, then $C_G^{\otimes}(g) \lhd G$ for any $g \in G$.

Proof. By Proposition 4, (a), (b) and (c) are equivalent. The equivalence between (a) and (d) is established in Lemma 2, (c). Now let G be a $2_{\otimes}$-Engel group, let $g, y \in G$ and let $x \in C_G^{\otimes}(g) \leq C_G(g)$. Then we have $x^y \otimes g = x[x, y] \otimes g = [x, y] \otimes g = ([x, g] \otimes y)^{-1} = 1_{\otimes}$, thus $x^y \in C_G^{\otimes}(g)$. This proves the corollary. □

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It is evident that the condition "$C_G^{\otimes}(g) \lhd G$ for any $g \in G$" may fail to imply that G is $2_{\otimes}$-Engel, as $C_G^{\otimes}(g)$ does not necessarily contain g.

Turning our attention to finite coverings by $2_{\otimes}$-Engel subgroups, we mention here a related result of L.-C. Kappe [**10**] which states that a group G has a finite covering by 2-Engel subgroups if and only if $|G : R_2(G)| < \infty$. Our proof of the tensor analogue follows the lines of Kappe's proof.

THEOREM 4. *A group G has a finite covering by $2_{\otimes}$-Engel subgroups if and only if $|G : R_2^{\otimes}(G)| < \infty$.*

Proof. Suppose that $G = \bigcup_{i=1}^{n} H_i$, where H_i are $2_{\otimes}$-Engel subgroups of G. The standard reduction step, due to B. H. Neumann (see [**10**]), shows that we may assume that $|G : H_i| < \infty$ for every i. Hence the subgroup $D = \bigcap_{i=1}^{n} H_i$ has a finite index in G. It is clear that $D \le R_2^{\otimes}(G)$; hence $|G : R_2^{\otimes}(G)| < \infty$.

Assume now $|G : R_2^{\otimes}(G)| < \infty$. Let $\{g_1, \dots, g_n\}$ be a transversal of $R_2^{\otimes}(G)$ in G and let $H_i = \langle g_i \rangle R_2^{\otimes}(G)$. We have $G = \bigcup_{i=1}^{n} H_i$, hence it suffices to prove that each H_i is $2_{\otimes}$-Engel. Let $y = g^i a$ and $x = g^j b$ be arbitrary elements of $\langle g \rangle R_2^{\otimes}(G)$, where $i, j \in \mathbb{Z}$ and $a, b \in R_2^{\otimes}(G)$. Since $R_2^{\otimes}(G) = E_1^{\otimes}(G)$, we obtain, using Proposition 4, $[x, y] \otimes y = [g^j, g^i a] \otimes g^i a = [g^j, a] \otimes g^i a = ([g^j, a] \otimes a)([g^j, a] \otimes g^i)^a = (([g, a] \otimes g)^a)^{ij} = 1_{\otimes}$, as required. □

REMARK. Suppose that a group G has a finite covering by $2_{\otimes}$-Engel normal subgroups $N_1, \dots, N_n$. Again we may assume that $|G : N_i| < \infty$ and by Theorem 4 we also have $|G : R_2^{\otimes}(G)| < \infty$. Since for every $x \in G$ we have $x^G \le N_i$ for some i, we conclude that every normal closure of an element of G is $2_{\otimes}$-Engel. In particular, we have $1_{\otimes} = [x^{-y}, x] \otimes x = ([y, x, x] \otimes x)^{x^{-1}}$, hence G is $3_{\otimes}$-Engel. In view of [**10**] it is likely that a $3_{\otimes}$-Engel group G with $|G : R_2^{\otimes}(G)| < \infty$ has a finite normal covering by $2_{\otimes}$-Engel subgroups, but we have not been able to (dis)prove this, since there are no known tensor analogues of results regarding 3-Engel groups [**12**].

5. Tensor squares of $2_{\otimes}$-Engel groups. We have proved in the previous section that $2_{\otimes}$-Engel groups have abelian tensor squares. Moreover, if G is a $2_{\otimes}$-Engel group, then $\gamma_3(G) \le Z^{\otimes}(G)$ by Theorem 3. Using a result of G. J. Ellis [**7**], we see that $G \otimes G \cong G/\gamma_3(G) \otimes G/\gamma_3(G)$, hence the calculations of tensor squares reduce to the calculations of tensor squares of class 2 groups (of course, the situation becomes even better when G is abelian).

Let G be a nonabelian two-generator $2_{\otimes}$-Engel p-group. The group $G/\gamma_3(G)$ is a two-generator $2_{\otimes}$-Engel p-group of class 2. From [**1**] and [**11**] we obtain the complete classification of two-generator p-groups of class 2, hence we only have to check which of these groups are $2_{\otimes}$-Engel. The following lemma provides a useful criterion for this task.

LEMMA 3. *Let G be a two-generator group of class two. Then G is $2_{\otimes}$-Engel if and only if $G \otimes G \cong G^{ab} \otimes G^{ab}$.*

Proof. Let $G = \langle a, b \rangle$ be a group of class two and let $x, y \in G$. Then $x = a^i b^j [a, b]^k$ and $y = a^{i'} b^{j'} [a, b]^{k'}$ for some $i, i', j, j', k, k' \in \mathbb{Z}$. By means of linear expansion we obtain $[x, y] = [a, b]^{ij' - i'j}$, hence $[x, y] \otimes y = (a \otimes [a, b])^{j' - i i' j' + i'^2 j} (b \otimes [a, b])^{-i' - i j'^2 + i' j j'}$. Therefore G is $2_{\otimes}$-Engel if and only if $a \otimes [a, b] = b \otimes [a, b] = 1_{\otimes}$, which is equivalent to

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$x \otimes [y, z] = 1_\otimes$ for all $x, y, z \in G$. By [**9**, Theorem 3], G is $2_\otimes$-Engel if and only if $G \otimes G \cong G^{ab} \otimes G^{ab}$. □

The recipe for computing tensor squares of two-generator $2_\otimes$-Engel p-groups therefore consists of looking for those two-generator p-groups G of class two which satisfy the condition $G \otimes G \cong G^{ab} \otimes G^{ab}$. Note also that if $G^{ab} \cong \mathbb{Z}_{a_1} \times \cdots \times \mathbb{Z}_{a_r}$, then $G^{ab} \otimes G^{ab}$ is isomorphic to the direct product of all $\mathbb{Z}_{\gcd(a_i,a_j)}$, where $i, j = 1, \ldots, r$.

First assume p is odd. Then we have the following cases [**1**].

(*Case* 1.) $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, where $[a, b] = c$, $[a, c] = [b, c] = 1$, $|a| = p^\alpha$, $|b| = p^\beta$, $|c| = p^\gamma$ and $\alpha \geq \beta \geq \gamma \geq 1$. Here we have $G \otimes G \cong \mathbb{Z}_{p^\alpha} \times \mathbb{Z}_{p^\beta}^3 \times \mathbb{Z}_{p^\gamma}^2$, hence $G \otimes G \not\cong G^{ab} \otimes G^{ab}$.

(*Case* 2.) $G \cong \langle a\rangle \rtimes \langle b\rangle$, where $[a, b] = a^{p^{\alpha-\gamma}}$, $|a| = p^\alpha$, $|b| = p^\beta$, $|[a, b]| = p^\gamma$ and $\beta \geq \gamma \geq 1$, $\alpha \geq 2\gamma$; by a closer inspection of the proof of [**1**, Theorem 2.4] it becomes clear that the extra condition $\alpha \geq \beta$ given there is irrelevant. By [**1**, Theorem 4.2] we have $G \otimes G \cong \langle a \otimes a\rangle \times \langle b \otimes b\rangle \times \langle (b \otimes a)(a \otimes b)\rangle \times \langle b \otimes a\rangle$, where $|a \otimes a| = p^{\alpha-\gamma}$, $|b \otimes b| = p^\beta$, $|(b \otimes a)(a \otimes b)| = p^{\min\{\alpha-\gamma,\beta\}}$ and $|b \otimes a| = n$, where $n = \gcd(p^\alpha, \sum_{k=0}^{p^\beta-1}(p^\alpha - p^{\alpha-\gamma} + 1)^k)$. Applying [**1**, Lemma 4.1], we immediately obtain $n = p^{\min\{\alpha,\beta\}}$, hence $G \otimes G$ is isomorphic to $\mathbb{Z}_{p^\beta} \times \mathbb{Z}_{p^{\alpha-\gamma}} \times \mathbb{Z}_{p^{\min\{\alpha,\beta\}}} \times \mathbb{Z}_{p^{\min\{\alpha-\gamma,\beta\}}}$. Since $G^{ab} \cong \mathbb{Z}_{p^{\alpha-\gamma}} \times \mathbb{Z}_{p^\beta}$, we get $G^{ab} \otimes G^{ab} \cong \mathbb{Z}_{p^\beta} \times \mathbb{Z}_{p^{\alpha-\gamma}} \times \mathbb{Z}^2_{p^{\min\{\alpha-\gamma,\beta\}}}$. This yields that G is $2_\otimes$-Engel if and only if $\min\{\alpha - \gamma, \beta\} = \min\{\alpha, \beta\}$ which is equivalent to $\alpha \geq \beta + \gamma$.

(*Case* 3.) $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, where $[a, b] = a^{p^{\alpha-\gamma}} c$, $[c, b] = a^{-p^{2(\alpha-\gamma)}} c^{-p^{\alpha-\gamma}}$, $|a| = p^\alpha$, $|b| = p^\beta$, $|[a, b]| = p^\gamma$, $|c| = p^\sigma$, $\alpha \geq \beta \geq \gamma > \sigma \geq 1$ and $\alpha + \sigma \geq 2\gamma$. Let $\delta = \min\{\alpha - \gamma, \beta\}$ and $\tau = \min\{\alpha - \gamma, \sigma\}$. Then we have $G \otimes G \cong \mathbb{Z}_{p^{\alpha-\gamma}} \times \mathbb{Z}^3_{p^\delta} \times \mathbb{Z}^2_{p^\tau}$, hence it is not isomorphic to $G^{ab} \otimes G^{ab}$.

For $p = 2$ the situation is more complicated [**11**].

(*Case* 4.) $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, where $[a, b] = c$, $[a, c] = [b, c] = 1$, $|a| = 2^\alpha$, $|b| = 2^\beta$, $|c| = 2^\gamma$ and $\alpha \geq \beta \geq \gamma \geq 1$. Here we have

$$G \otimes G \cong \begin{cases} \mathbb{Z}_{2^\alpha} \times \mathbb{Z}^3_{2^\beta} \times \mathbb{Z}^2_{2^\gamma}, & : \beta > \gamma, \\ \mathbb{Z}_{2^\alpha} \times \mathbb{Z}^2_{2^\gamma} \times \mathbb{Z}_{2^{\gamma+1}} \times \mathbb{Z}_{2^{\gamma-1}} \times \mathbb{Z}_{2^{\min\{\alpha-1,\gamma\}}} & : \beta = \gamma. \end{cases}$$

It follows from here that $G \otimes G \not\cong G^{ab} \otimes G^{ab}$.

(*Case* 5.) $G \cong \langle a\rangle \rtimes \langle b\rangle$, where $[a, b] = a^{2^{\alpha-\gamma}}$ $|a| = 2^\alpha$, $|b| = 2^\beta$, $|[a, b]| = 2^\gamma$ and $\alpha, \beta, \gamma \in \mathbb{N}, \alpha \geq 2\gamma, \beta \geq \gamma$ and $\alpha + \beta > 3$. In this particular case, $G \otimes G$ is isomorphic to $\mathbb{Z}_{2^\beta} \times \mathbb{Z}_{2^{\alpha-\gamma+1}} \times \mathbb{Z}_{2^{\min\{\alpha-\gamma,\beta\}}} \times \mathbb{Z}_{2^{\min\{\alpha,\beta\}}}$. It is straightforward to verify that $G \otimes G \not\cong G^{ab} \otimes G^{ab}$.

(*Case* 6.) $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, where $[a, b] = a^{2^{\alpha-\gamma}} c$, $[c, b] = a^{-2^{2(\alpha-\gamma)}} c^{-2^{\alpha-\gamma}}$, $|a| = 2^\alpha$, $|b| = 2^\beta$, $|[a, b]| = 2^\gamma$, $|c| = 2^\sigma$ with $\alpha, \beta, \gamma, \sigma \in \mathbb{N}$, $\alpha + \sigma \geq 2\gamma$ and $\beta \geq \gamma > \sigma$. Let $\rho = \min\{\alpha - \gamma + \sigma, \beta\}$. Then we have

$$G \otimes G \cong \begin{cases} \mathbb{Z}^3_{2^\gamma} \times \mathbb{Z}_{2^{\gamma+1}} \times \mathbb{Z}^2_{2^{\gamma-1}} & : \alpha = \gamma + 1,\ \beta = \gamma, \\ \mathbb{Z}_{2^{\alpha-\gamma+\sigma+1}} \times \mathbb{Z}_{2^\beta} \times \mathbb{Z}_{2^{\min\{\alpha,\beta\}}} \times \mathbb{Z}_{2^\rho} \times \mathbb{Z}^2_{2^\sigma} & : \alpha \geq \gamma + 2 \text{ or } \beta \geq \gamma + 1. \end{cases}$$

It is clear that $G \otimes G$ is not isomorphic to $G^{ab} \otimes G^{ab}$.

We summarize our conclusions in the following theorem.

THEOREM 5. *Let G be a nonabelian two-generator $2_\otimes$-Engel p-group. Then $p \neq 2$ and $G/\gamma_3(G) \cong \langle a\rangle \rtimes \langle b\rangle$, where $[a, b] = a^{p^{\alpha-\gamma}}$, $|a| = p^\alpha$, $|b| = p^\beta$, $|[a, b]| = p^\gamma$ with $\alpha \geq$*

$\beta \geq \gamma \geq 1, \alpha \geq 2\gamma$ and $\alpha \geq \beta + \gamma$. We have $G \otimes G \cong \langle a \otimes a \rangle \times \langle b \otimes b \rangle \times \langle (b \otimes a)(a \otimes b) \rangle \times \langle b \otimes a \rangle \cong \mathbb{Z}_{p^\beta}^3 \times \mathbb{Z}_{p^{\alpha-\gamma}}$.

Our considerations also show the following.

COROLLARY 5. *Every $2_\otimes$-Engel 2-group is abelian.*

More generally, if G is a $2_\otimes$-Engel group without elements of order 3, then $G' \leq Z^\otimes(G)$ by Theorem 3. This, together with the result of Ellis [**7**], implies $G \otimes G \cong G^{ab} \otimes G^{ab}$.

Let G be a group. From a topological point of view, the third homotopy group $\pi_3 SK(G, 1)$ of the suspension of $K(G, 1)$ is of some interest. A combinatorial description of $\pi_n SK(G, 1)$ has been given by J. Wu [**17**]. Observing the formula $\pi_3 SK(G, 1) \cong \ker \kappa$ [**5**], one can use a different approach when $G \otimes G$ is explicitly computed. Applying Theorem 5, we describe $\pi_3 SK(G, 1)$ for any nonabelian two-generator $2_\otimes$-Engel p-group G. We also determine the Schur multiplier $H_2(G)$ of G.

COROLLARY 6. *Let G be a nonabelian two-generator $2_\otimes$-Engel p-group, let $\kappa : G \otimes G \to G'$ be the commutator map and let a, b, α, β, γ be as in Theorem 5. Then $\pi_3 SK(G, 1) \cong \ker \kappa \cong \langle a \otimes a \rangle \times \langle b \otimes b \rangle \times \langle (b \otimes a)(a \otimes b) \rangle \times \langle (b \otimes a)^{p^\gamma} \rangle \cong \mathbb{Z}_{p^\beta}^2 \times \mathbb{Z}_{p^{\alpha-\gamma}} \times \mathbb{Z}_{p^{\beta-\gamma}}$ and $H_2(G) \cong \mathbb{Z}_{p^{\beta-\gamma}}$.*

Proof. As $\kappa(a \otimes a) = \kappa(b \otimes b) = \kappa((b \otimes a)(a \otimes b)) = \kappa((b \otimes a)^{p^\gamma}) = 1$, Theorem 5 gives $\ker \kappa \cong \langle a \otimes a \rangle \times \langle b \otimes b \rangle \times \langle (b \otimes a)(a \otimes b) \rangle \times \langle (b \otimes a)^{p^\gamma} \rangle \cong \mathbb{Z}_{p^\beta}^2 \times \mathbb{Z}_{p^{\alpha-\gamma}} \times \mathbb{Z}_{p^{\beta-\gamma}}$, as required. To compute the Schur multiplier of G, note for instance that the exactness of rows and columns in commutative diagram (1) in [**4**] implies $H_2(G) \cong \ker \kappa / \Delta(G)$, where $\Delta(G) = \langle x \otimes x : x \in G \rangle$. Now, every $x \in \langle a, b \rangle$ can be written in the form $x = a^m b^n [a, b]^k$, where $m, n, k \in \mathbb{Z}$. Expanding $x \otimes x$ linearly, we obtain $x \otimes x = (a \otimes a)^{m^2} (b \otimes b)^{n^2} ((b \otimes a)(a \otimes b))^{mn}$. This yields $\Delta(G) \cong \langle a \otimes a \rangle \times \langle b \otimes b \rangle \times \langle (b \otimes a)(a \otimes b) \rangle \cong \mathbb{Z}_{p^\beta}^2 \times \mathbb{Z}_{p^{\alpha-\gamma}}$, hence the result. □

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