Canad. Math. Bull. Vol. **33** (2), 1990

COEFFICIENT BOUNDS IN THE LORENTZ REPRESENTATION OF A POLYNOMIAL

BY

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ABSTRACT. Each polynomial $P(x)$ has a "Lorentz representation", of the form $P(x) = \sum_{j=0}^{n} c_j x^j (1-x)^{n-j}$. This representation becomes unique if we insist that n equals the degree of P. Motivated partly by questions involving polynomials with integer coefficients, we investigate the relationship between $\|P\|_{L_\infty[0,1]}$ and $|c_j|, j = 0, 1, \ldots n$.

1. **Introduction and Statement of Results.** A *Lorentz representation* of a polynomial $P(x)$, is a representation of the form

$$P(x) = \sum_{j=0}^{n} c_j x^j (1-x)^{n-j}. \tag{1.1}$$

While it is not unique in general – for example

$$1 = \sum_{j=0}^{n} \binom{n}{j} x^j (1-x)^{n-j}, \text{ any } n \geqq 0,$$

– it becomes unique if we insist in (1.1) that n equals the degree of P.

One of the interesting features of the representation is that every polynomial P, positive in (0, 1), possesses a representation (1.1) with all $c_j \geqq 0$. Further, every polynomial P with integer coefficients has a representation (1.1), with all c_j integers, and in which n equals the degree of P. The representation (1.1) has been found useful in various contexts of approximation theory [2, 4, 6], and has helped indirectly to inspire others [10, 11].

In investigating extremal polynomials with integer coefficients [8], the problem of estimating the relationship between $\|P\|_{L_\infty[0,1]}$ and $|c_j|$, $j = 0, 1, \ldots n$, arose. In this paper, we present some sharp and near-sharp inequalities along these lines. The proofs involve maximum principles, conformal maps, and classical arguments of Markov.

Received September 26, 1988.

Research completed while the second author was visiting the CSIR.

AMS(MOS) Classification: Primary 26C05; Secondary 41A17.

Key words. Polynomials, Coefficient Bounds, Lorentz Representation, Maximum principle, Chebyshev Polynomials.

One feature of the results is that they depend mainly on the number n of terms in (1.1), not on the actual degree of P. The proofs are presented in Section 2. Given $\rho > 0$, we define

$$\lambda(\rho) := 4\rho^{1/2}/(1+\rho^{1/2})^2, \tag{1.2}$$

and if also $0 < p < \infty, n \geqq 0$, we set

$$I(p,\rho,n) := \left\{\frac{4}{\pi}\int_0^{\pi/4} |1-\lambda(\rho)\sin^2 y|^{pn} dy\right\}^{1/p}. \tag{1.3}$$

THEOREM 1.1. *Let $P(x)$ have the representation* (1.1), *and let $\rho > 0$. Then*

$$\left\{\sum_{j=0}^{n} |c_j|^2\rho^{2j}\right\}^{1/2} \leqq (1+\rho^{1/2})^{2n} I(2,\rho,n)\|P\|_{L_\infty[0,1]}. \tag{1.4}$$

Here

$$0 < I(2,\rho,n) \leqq 1, \tag{1.5}$$

with strict inequality unless $n = 0$. Further,

$$I(2,\rho,n) \leqq \{2\sqrt{2}[\pi\lambda(\rho)]^{-1/2}\Gamma(2n+1)/\Gamma(2n+3/2)\}^{1/2}, \tag{1.6}$$

while

$$\lim_{n\to\infty} I(2,\rho,n)n^{1/4} = \{2/[\pi\lambda(\rho)]\}^{1/4}. \tag{1.7}$$

We note that Theorem 1.1 is "nearly sharp". To be more precise, let $T_n(x)$ denote the usual Chebyshev polynomial of degree n. From the expressions given in [3, p.34], one readily derives the representation

$$T_n(2x-1) = \sum_{j=0}^{n} d_{n,j}x^j(1-x)^{n-j}(-1)^{n-j}, \tag{1.8}$$

where

$$d_{n,j} := \sum_{k=0}^{\min\{j,n-j\}} \binom{n}{2k}\binom{n-2k}{j-k}4^k,\ j = 0,\ 1,\ 2,\ \ldots n. \tag{1.9}$$

We shall see in Section 2 that for each fixed $\rho > 0$,

$$\left\{\sum_{j=0}^{n} |d_{n,j}|^2\rho^{2j}\right\}^{1/2} \Big/ [(1+\rho^{1/2})^{2n} I(2,\rho,n)\|T_n(2x-1)\|_{L_\infty[0,1]}] \longrightarrow \frac{1}{2},\ n\longrightarrow\infty. \tag{1.10}$$

https://doi.org/10.4153/CMB-1990-033-1 Published online by Cambridge University Press

Thus for n large enough, (1.4) cannot be improved by much more than 1/2. This "gap" of 1/2 also arises in classical majorization of polynomials on intervals, for technical reasons.

We note that by comparing coefficients of x^n on both sides of (1.8), we see that

$$2^{2n-1} = \sum_{j=0}^{n} d_{n,j}. \tag{1.11}$$

For individual coefficients, we shall prove

THEOREM 1.2. *Let $P(x)$ have the representation* (1.1) *and let $0 < \mu \leqq 1/2$. Then for $0 \leqq j \leqq \mu n$, and for $n(1-\mu) \leqq j \leqq n$,*

$$\begin{aligned}|c_j| \leqq &\{\mu^{-\mu}(1-\mu)^{-1+\mu}\}^{2n}\|P\|_{L_\infty[0,1]} \\ &\times \min\left\{1, \frac{\sqrt{2}\Gamma(n+1)}{\Gamma(n+3/2)}(\pi\mu(1-\mu))^{-1/2}\right\}.\end{aligned} \tag{1.12}$$

Note that the 1 is the smaller term in the minimum if n is small and μ is close to zero. For large n, the minimum decreases like a constant multiple of $n^{-1/2}$.

A cursory examination of $\{d_{n,j}\}_{j=0}^{n}$ shows that (1.12) is sharp for $j = 0$ or $j = n$, if we let $\mu \longrightarrow 0+$. Further, a crude application of Stirling's formula shows that (1.12) is sharp for j close to $n/2$.

While Theorem 1.2 is not fully sharp, it is at least easily applicable. A sharp, but less convenient, inequality can be derived using a classical argument of Markov:

THEOREM 1.3. *Let $P(x)$ have the representation* (1.1), *and let $0 \leqq j \leqq n$. Let $d_{n,j}$ be defined by* (1.9). *Then,*

$$|c_j| \leqq d_{n,j}\|P\|_{L_\infty[0,1]}, \tag{1.13}$$

with equality if and only if $P(x)$ is a constant multiple of $T_n(2x-1)$.

A pleasant feature of Theorem 1.3 is that it is in a sense more elegant than its classical cousin [9, p. 56, Cor. 2] involving ordinary powers, since there the parity of the degree of P and of j (that is, whether they are even or odd) plays a role.

In the opposite direction to Theorems 1.1 to 1.3, we note the following fairly immediate (and sharp) consequence of (1.1):

$$\|P\|_{L_\infty[0,1]} \leqq \max_{0\leqq j\leqq n}\left\{|c_j| \Big/ \binom{n}{j}\right\}. \tag{1.14}$$

2. **Proofs.**

LEMMA 2.1. *Let $R(u)$ be a polynomial of degree at most n. Then for $u \in \mathbf{C}\setminus[0,\infty)$,*

$$|R(u)| \leqq |1+\sqrt{-u}|^{2n}\|R(s)/(1+s)^n\|_{L_\infty[0,\infty)}, \tag{2.1}$$

where the branch of the square root is the principal one.

PROOF. Note that for $u \in \mathbf{C}\setminus[0,\infty)$, we have $Re\sqrt{-u} > 0$, so $f(u) := R(u)/(1+\sqrt{-u})^{2n}$ is analytic in $\mathbf{C}\setminus[0,\infty)$, and has a finite limit at ∞, namely $(-1)^n c$, where c is the coefficient of u^n in $R(u)$. By the maximum modulus principle,

$$|f(u)| \leqq \|f\|_{L_\infty[0,\infty)},\ u \in \mathbf{C}\setminus[0,\infty).$$

But as $u \longrightarrow s \in [0,\infty)$, from the upper or lower half planes,

$$|f(u)| \longrightarrow |R(s)/(1+i\sqrt{s})^{2n}| = |R(s)|/(1+s)^n.$$

Therefore

$$\|f\|_{L_\infty[0,\infty)} = \|R(s)/(1+s)^n\|_{L_\infty[0,\infty)}.$$

Hence (2.1). □

We note that (2.1) is nearly sharp: Let

$$R_n(u) := T_n\left(\frac{u-1}{u+1}\right)(1+u)^n,\ n \geqq 1.$$

Then uniformly in compact subsets of $\mathbf{C}\setminus[0,\infty)$,

$$\lim_{n\to\infty} |R_n(u)|/\{|1+\sqrt{-u}|^{2n}\|R_n(s)/(1+s)^n\|_{L_\infty[0,\infty)}\} = \frac{1}{2}.$$

See the proof of (1.10) below.

We can now prove:

THEOREM 2.2. *Let $p, \rho > 0$, and let $R(u)$ be a polynomial of degree at most n.*
(a) Then

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} |R(\rho e^{i\theta})|^p d\theta\right\}^{1/p} \leqq (1+\rho^{1/2})^{2n} I(p,\rho,n)\|R(s)/(1+s)^n\|_{L_\infty[0,\infty)}, \tag{2.2}$$

where $I(p,\rho,n)$ is defined by (1.3) *and* (1.2).
(b) If $\lambda(\rho)$ is defined by (1.2), *then*

$$0 < \lambda(\rho) \leqq 1, \tag{2.3}$$

with $\lambda(\rho) = 1$ if and only if $\rho = 1$, and

$$0 < I(p,\rho,n) \leqq 1, \tag{2.4}$$

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with $I(p,\rho,n)=1$ *if and only if* $n=0$, *while*

$$I(p,\rho,n) \leqq \{2\sqrt{2}[\pi\lambda(\rho)]^{-1/2}\Gamma(pn+1)/\Gamma(pn+3/2)\}^{1/p}. \tag{2.5}$$

(c) Further,

$$\lim_{n\to\infty} I(p,\rho,n)n^{1/(2p)} = \{2[\pi\lambda(\rho)p]^{-1/2}\}^{1/p}. \tag{2.6}$$

PROOF. (a) In view of Lemma 2.1, it suffices to estimate

$$J := \left\{\frac{1}{2\pi}\int_0^{2\pi} |1+\sqrt{-\rho e^{i\theta}}|^{2np}\,d\theta\right\}^{1/p}.$$

We see that

$$\begin{aligned}
J &= \left\{\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} |1+\sqrt{\rho e^{is}}|^{2np}\,ds\right\}^{1/p} \quad \left(s := \frac{\theta-\pi}{2}\right)\\
&= \left\{\frac{2}{\pi}\int_0^{\pi/2} [1+\rho+2\sqrt{\rho}\cos s]^{np}\,ds\right\}^{1/p}\\
&= \left\{\frac{2}{\pi}\int_0^{\pi/2} [(1+\rho^{1/2})^2-2\sqrt{\rho}(1-\cos s)]^{np}\,ds\right\}^{1/p}\\
&= (1+\rho^{1/2})^{2n}\left\{\frac{2}{\pi}\int_0^{\pi/2}\left[1-\lambda(\rho)\sin^2\left(\frac{s}{2}\right)\right]^{np} ds\right\}^{1/p} \quad \text{(by (1.2))}\\
&= (1+\rho^{1/2})^{2n} I(p,\rho,n),
\end{aligned}$$

by the substitution $y := s/2$, and by (1.3).

(b) The inequality (2.3) follows from

$$(1+\rho^{1/2})^2 - 4\rho^{1/2} = (1-\rho^{1/2})^2 \geqq 0,$$

while (2.4) is fairly obvious. Next, the substitution $t := 1-\lambda(\rho)\sin^2 y$ in (1.3) yields

$$\begin{aligned}
I(p,\rho,n)^p &= \frac{2}{\pi}\lambda(\rho)^{-1/2}\int_{1-\lambda(\rho)/2}^{1} t^{pn}(1-t)^{-1/2}(1-(1-t)/\lambda(\rho))^{-1/2}\,dt \qquad (2.7)\\
&\leqq \frac{2}{\pi}\sqrt{2}\lambda(\rho)^{-1/2}\int_0^1 t^{pn}(1-t)^{-1/2}\,dt\\
&= \frac{2}{\pi}\sqrt{2}\lambda(\rho)^{-1/2}\frac{\Gamma(pn+1)\Gamma(1/2)}{\Gamma(pn+3/2)}\\
&= 2\sqrt{2}[\pi\lambda(\rho)]^{-1/2}\frac{\Gamma(pn+1)}{\Gamma(pn+3/2)}.
\end{aligned}$$

Hence (2.5).

(c) We note that as $n \longrightarrow \infty$, on removing the $\sqrt{2}$, the inequality in the first line after (2.7) becomes essentially an equality, since for any $0 < \eta < 1$, the integral over $[0, 1-\eta]$ decreases geometrically to zero, as $n \longrightarrow \infty$, while $1 - (1-t)/\lambda(\rho) \longrightarrow 1$ as $t \longrightarrow 1-$. On applying Stirling's formula to the last right-hand side of (2.7), and omitting the $\sqrt{2}$ we obtain (2.6). □

We note that, as in Lemma 2.1, the polynomials

$$R_n(u) := T_n\left(\frac{u-1}{u+1}\right)(1+u)^n, \; n \geqq 1,$$

may be used to show that (2.2) is nearly sharp. We now turn to the

PROOF OF THEOREM 1.1. For the polynomial P with representation (1.1), let us define an associated polynomial

$$R(u) := \sum_{j=0}^{n} c_j u^j. \tag{2.8}$$

Consider the transformation

$$u := \frac{x}{1-x} \leftrightarrow x = \frac{u}{1+u} \tag{2.9}$$

for $x \in [0, 1]$ and $u \in [0, \infty)$. We see that $1 - x = (1+u)^{-1}$, so that

$$P(x) = (1-x)^n R\left(\frac{x}{1-x}\right) = R(u)/(1+u)^n. \tag{2.10}$$

Then by Theorem 2.2,

$$\begin{aligned}\left\{\sum_{j=0}^{n} |c_j|^2 \rho^{2j}\right\}^{1/2} &= \left\{\frac{1}{2\pi}\int_0^{2\pi} |R(\rho e^{i\theta})|^2 d\theta\right\}^{1/2} \\ &\leqq (1+\rho^{1/2})^{2n} I(2, \rho, n) \|R(u)/(1+u)^n\|_{L_\infty[0,\infty)} \\ &= (1+\rho^{1/2})^{2n} I(2, \rho, n) \|P\|_{L_\infty[0,1]}.\end{aligned}$$

The remaining inequalities of Theorem 1.1 follow (2.5) and (2.6). □

PROOF OF THEOREM 1.2. Let R be defined by (2.8). Then for $0 < \rho \leqq 1, 0 \leqq j \leqq \mu n$,

$$\begin{aligned}|c_j| &= \left|\frac{1}{2\pi i}\int_{|t|=\rho} R(t)/t^{j+1} dt\right| \\ &\leqq \rho^{-j}(1+\rho^{1/2})^{2n} I(1, \rho, n) \|R(u)/(1+u)^n\|_{L_\infty[0,\infty)} \\ &\quad \text{(by Theorem 2.2)} \\ &\leqq \rho^{-\mu n}(1+\rho^{1/2})^{2n} \min\left\{1,\; 2\sqrt{2}[\pi\lambda(\rho)]^{-1/2}\frac{\Gamma(n+1)}{\Gamma(n+3/2)}\right\} \|P\|_{L_\infty[0,1]},\end{aligned} \tag{2.11}$$

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by (2.4), (2.5) and as in the previous proof. Choosing $\rho^{1/2} := \mu/(1-\mu)$, (which yields the minimal value for $\rho^{-\mu n}(1+\rho^{1/2})^{2n}$, $\rho \in (0,\infty)$) and then using (1.2), we obtain (1.12) for $0 \leqq j \leqq \mu n$. For $n(1-\mu) \leqq j \leqq n$, we need only note that

$$P(1-x) = \sum_{k=0}^{n} c_{n-k} x^k (1-x)^{n-k},$$

and apply our previous inequality for $0 \leqq k \leqq \mu n$. □

PROOF OF THEOREM 1.3. Let $0 \leqq j < n$. We first show that $\{x^k(1-x)^{n-k} : k = 0, 1, \ldots, j-1, j+1, \ldots, n\}$ is a Chebyshev system on [0, 1]. To do this, we need only show that if

$$S(x) := \sum_{\substack{k=0 \\ k\neq j}}^{n} d_k x^k (1-x)^{n-k},$$

and S is not identically zero, then S can have at most $n-1$ zeros in [0, 1] (see [1]). Suppose S has at least n distinct zeros in [0, 1] — say l at $1 (l = 0$ or $1)$ and $n-l$ in [0, 1). Let

$$R(u) := \sum_{\substack{k=0 \\ k\neq j}}^{n} d_k u^k.$$

Then $S(x) = (1-x)^n R(x/(1-x))$, and it follows that $R(u)$ has at least $n-l$ distinct zeros in $[0,\infty)$. If $l = 1$, then $d_n = S(1) = 0$, and then R has degree $n-1$. So, as an ordinary polynomial, R has degree at most $n-l$. Hence R is a linear combination of the $n-l$ functions $\{u^k : k = 0, 1, \ldots, j-1, j+1, \ldots, n-l\}$ (recall here that $j < n$ so $j \leqq n-l$), which form a Chebyshev system on $[0,\infty)$ [5, pp. 9-10]. Then necessarily all $d_k = 0$, so S is identically zero.

Next, let us define

$$E_{n,j} = \min_{d_k} \|x^j(1-x)^{n-j} - \sum_{\substack{k=0 \\ k\neq j}}^{n} d_k x^k (1-x)^{n-k}\|_{L_\infty[0,1]}. \tag{2.12}$$

Let us denote the unique $\{d_k\}$ minimizing this expression by $\{d_k^*\}$ and let

$$r(x) := x^j(1-x)^{n-j} - \sum_{\substack{k=0 \\ k\neq j}}^{n} d_k^* x^k (1-x)^{n-k}.$$

By the alternation theorem [1], there exist $0 \leqq x_1 < x_2 < \ldots < x_{n+1} \leqq 1$ such that

$$r(x_{i+1}) = -r(x_i) = \pm E_{n,j}, \; i = 1, 2, \ldots, n+1.$$

It follows that $r(x)$ has n distinct zeros in [0, 1], so is a polynomial of degree n. Note here that r cannot vanish identically, for $\{x^k(1-x)^{n-k}\}_{k=0}^{n}$ is a basis for the polynomials of degree at most n. We can then write $r(x) = c\{x^n - p(x)\}$, where $c \neq 0$, and $p(x)$ is a polynomial of degree at most n. Since $x^n - p(x)$ alternates at least $n+1$ times in $[0, 1]$, the alternation theorem and uniqueness of best polynomial approximations [1] imply that

$$r(x)/c = 2^{-2n+1}T_n(2x-1). \tag{2.13}$$

Also then from (1.8) to (1.9), comparing coefficients of $x^j(1-x)^{n-j}$ on both sides of (2.13), $1/c = 2^{-2n+1}d_{n,j}$.

Then

$$\begin{aligned} E_{n,j} &= \|r\|_{L_\infty[0,1]} \\ &= c2^{-2n+1}\|T_n(2x-1)\|_{L_\infty[0,1]} = 1/d_{n,j}. \end{aligned}$$

Next, if P has the representation (1.1), and $c_j \neq 0$, the

$$\begin{aligned} \|P\|_{L_\infty[0,1]} &\geqq |c_j| \|x^j(1-x)^{n-j} \\ &\quad + \sum_{\substack{k=0 \\ k\neq j}}^{n} \{c_k/c_j\} x^k(1-x)^{n-k}\|_{L_\infty[0,1]} \\ &\geqq |c_j| E_{n,j} = |c_j|/d_{n,j}. \end{aligned}$$

This yields (1.13) if $c_j \neq 0$. Of course, if $c_j = 0$, then (1.13) is trivial. Finally, if $j = n$, we may apply (1.13) to $P(1-x)$, and use the symmetry, as well as the fact that $d_{n,n} = d_{n,0} = 1$. The case of equality may be handled much as above. □

Finally, we turn to the

PROOF OF (1.10). Let

$$R_n(u) := \sum_{j=0}^{n} d_{n,j}(-u)^j, \quad n \geqq 1,$$

and consider the transformation (2.9). As at (2.10), we see that

$$\begin{aligned} T_n(2x-1) &= (1-x)^n(-1)^n R_n\left(\frac{x}{1-x}\right) \\ &= (1+u)^{-n}(-1)^n R_n(u). \end{aligned}$$

Let

$$\varphi(z) := z + (z^2-1)^{1/2}, \quad z \in \mathbf{C},$$

denote the usual conformal map of $\mathbf{C}\setminus[-1, 1]$ onto $\{w : |w| > 1\}$. The branch of the square root is chosen so that $(z^2-1)^{1/2} > 0$, $z \in [1, \infty)$. It is well known [3, p. 116] that

$$T_n(z)/\varphi(z)^n = \frac{1}{2}\{1 + \varphi(z)^{-2n}\} \longrightarrow \frac{1}{2} \text{ as } n \longrightarrow \infty,$$

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uniformly in closed subsets of $\mathbf{C}\setminus[-1,1]$. Then

$$\lim_{n\to\infty}\left|R_n(u)\Big/\left\{(1+u)^n\varphi\left(\frac{u-1}{u+1}\right)^n\right\}\right|$$
$$=\lim_{n\to\infty}\left|T_n\left(\frac{u-1}{u+1}\right)\Big/\varphi\left(\frac{u-1}{u+1}\right)^n\right|=1/2,$$

uniformly in closed subsets of $\mathbf{C}\setminus[0,\infty)$. But for $u\in\mathbf{C}\setminus[0,\infty)$,

$$(1+u)\varphi\left(\frac{u-1}{u+1}\right)=u-1+(1+u)\sqrt{\frac{-4u}{(u+1)^2}}$$
$$=u-1-2\sqrt{-u}=-(1+\sqrt{-u})^2,$$

so

$$\lim_{n\to\infty}|R_n(u)/(1+\sqrt{-u})^{2n}|=\frac{1}{2}, \tag{2.14}$$

uniformly in closed subsets of $\mathbf{C}\setminus[0,\infty)$. Then given $\rho>0$, as $n\to\infty$,

$$\sum_{j=0}^{n}d_{n,j}^2\rho^{2j}=\frac{1}{2\pi}\int_0^{2\pi}|R_n(\rho e^{i\theta})|^2d\theta$$
$$=\frac{1}{2\pi}\int_{\pi/2}^{3\pi/2}|1+\sqrt{-\rho e^{i\theta}}|^{4n}d\theta\left(\frac{1}{4}+o(1)\right)$$
$$+0\left(\int_0^{\pi/2}+\int_{3\pi/2}^{2\pi}|1+\sqrt{-\rho e^{i\theta}}|^{4n}d\theta\right)$$
(by (2.14) and Lemma 2.1)
$$=(1+\rho^{1/2})^{4n}\left(\frac{4}{\pi}\int_0^{\pi/8}[1-\lambda(\rho)\sin^2 y]^{2n}dy\left(\frac{1}{4}+o(1)\right)\right.$$
$$\left.+0\left(\int_{\pi/8}^{\pi/4}[1-\lambda(\rho)\sin^2 y]^{2n}dy\right)\right),$$

exactly as in the proof of Theorem 2.2. Now given $\eta\in(0,\pi/4]$, for $y\in[\eta,\pi/4]$,

$$[1-\lambda(\rho)\sin^2 y]^{2n}\leqq[1-\lambda(\rho)\sin^2(\eta)]^{2n},$$

which decreases geometrically to zero as $n\to\infty$. Hence from (2.6) and (1.3),

$$\left\{\sum_{j=0}^{n}d_{n,j}^2\rho^{2j}\right\}^{1/2}=(1+\rho^{1/2})^{2n}I(2,\rho,n)\left(\frac{1}{2}+o(1)\right),\ n\to\infty.$$

Finally, since $\|T_n(2x-1)\|_{L_\infty[0,1]} = 1$, (1.10) follows. □

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