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A PURELY METRIC PROOF OF THE CARISTI FIXED POINT THEOREM

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Abstract

In this paper we prove Caristi's fixed point theorem using only purely metric techniques.

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1. Introduction

We consider Caristi's fixed point theorem in its standard version.

THEOREM 1.1 (Caristi, 1976 [2]). *Let (M, d) be a complete metric space and assume that $\varphi : M \to \mathbb{R}^+$ is lower semicontinuous. Suppose that the mapping $T : M \to M$ satisfies*

$$d(x, Tx) \le \varphi(x) - \varphi(Tx) \tag{1.1}$$

for every $x \in M$. Then there exists $x_0 \in M$ such that

$$Tx_0 = x_0.$$

It is easy to see that Caristi's fixed point theorem is a generalisation of the Banach contraction principle by defining $\varphi(x) = (1 - k)^{-1} d(x, Tx)$, where $k < 1$ is the Lipschitz constant associated with the contraction T from Banach's principle. As proved by Kirk in [12], the validity of Caristi's fixed point theorem characterises completeness of M, while this is not the case with Banach's theorem (see [3] for a comprehensive discussion of this topic).

It has been proved that Caristi's fixed point theorem for nonlinear maps acting in complete metric spaces is equivalent to many results seemingly not related to fixed point theory, including the Ekeland variational principle [6] (see, for example, [14]).

THEOREM 1.2 (Ekeland, 1974 [6]). *Let (M, d) be a complete metric space and assume that $\varphi : M \to \mathbb{R}^+$ is lower semicontinuous. Define a partial order $\le$ on M as follows:*

$$x \le y \Leftrightarrow d(x, y) \le \varphi(x) - \varphi(y), \quad x, y \in M.$$

Then $(M, \le)$ has a maximal element.

Let us note in passing that the equivalence between Ekeland's and Caristi's results requires the assumption of some form of the axiom of choice. For an extensive discussion of this topic, we refer the reader to Kirk's recent retrospective paper on metric fixed point theory [13] and to an earlier paper by Jachymski [8]. See also some comments in the final section of the current paper. Caristi's fixed point theorem has been extended and generalised in many directions (see, for example, [3–5, 9] to mention just a few).

The original proof by Caristi invoked an iterated use of transfinite induction. Most later proofs of Caristi's fixed point theorem were based on the use of Zorn's lemma and the axiom of choice (or Zermelo's fixed point theorem) applied to the Brøndsted partial order [1]. As recently as 2014, Khamsi [10] (see also [11]) stated that "there are some trials of finding a pure metric proof of Caristi's fixed point theorem (without success so far)". In the current paper we propose such a proof. Its pure metric character is understood in the following sense.

(1) The proof uses only standard methods of metric spaces.
(2) The proof uses standard properties of the set of all real numbers.
(3) The proof uses mathematical induction for the construction of a sequence in a metric space.
(4) No partial order considerations are applied (the natural order in $\mathbb{R}$ is the only order used).
(5) No direct use of the axiom of choice, Zorn's lemma or equivalents are used.

A brief discussion of these restrictions against a broader axiomatic background is provided in the concluding section of the paper.

The next section is devoted to providing the metrical proof of Theorem 1.1 as announced above. The final section of the paper contains an analysis of this proof.

2. Proof of Caristi's theorem

First let us notice that from (1.1) it follows immediately that for every $x \in M$,

$$\varphi(Tx) \le \varphi(x). \tag{2.1}$$

For $x \in M$, set

$$\Pi(x) = \{y \in M : d(x, y) \le \varphi(x) - \varphi(y)\}.$$

Observe that $\Pi(x) \neq \emptyset$ because $x \in \Pi(x)$ and $Tx \in \Pi(x)$. Note also that $Ty \in \Pi(x)$ if $y \in \Pi(x)$, which follows from the following calculation:

$$d(x, Ty) \le d(x, y) + d(y, Ty) \le \varphi(x) - \varphi(y) + \varphi(y) - \varphi(Ty). \tag{2.2}$$

By the nonemptiness of $\Pi(x)$ for each $x \in M$ and by the nonnegativity of φ, the following function $p : M \to [0, \infty)$ is well defined:

$$p(x) = \inf\{\varphi(y) : y \in \Pi(x)\}.$$

It is immediate that for any $x \in M$,

$$0 \leq p(x) \leq \varphi(Tx) \leq \varphi(x).$$

Let us define by induction the following sequence $\{x_n\}$ of elements of M. Fix arbitrarily $x_1 \in M$ and assume that x_n has been constructed. From the definitions of $p(x_n)$ and $\Pi(x_n)$, it follows immediately that there exists $x_{n+1} \in \Pi(x_n)$ with

$$\varphi(x_{n+1}) \leq p(x_n) + \frac{1}{n}.$$

Since $x_{n+1} \in \Pi(x_n)$,

$$0 \leq d(x_n, x_{n+1}) \leq \varphi(x_n) - \varphi(x_{n+1}).$$

Hence $\{\varphi(x_n)\}$ is a nonincreasing sequence of nonnegative numbers and therefore there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \varphi(x_n) = r. \tag{2.3}$$

Therefore, $\{\varphi(x_n)\}$ is a Cauchy sequence of real numbers. Hence, for every $k \in \mathbb{N}$, there exists $N_k \in \mathbb{N}$ such that for every pair of natural numbers m, n with $m \geq n \geq N_k$,

$$0 \leq \varphi(x_n) - \varphi(x_m) < \frac{1}{k}.$$

Since

$$\varphi(x_{n+1}) \leq p(x_n) + \frac{1}{n} \leq \varphi(x_n) + \frac{1}{n},$$

it follows that

$$\lim_{n\to\infty} p(x_n) = r. \tag{2.4}$$

We claim that for $m \geq n \geq N_k$,

$$d(x_n, x_m) \leq \varphi(x_n) - \varphi(x_m) < \frac{1}{k}. \tag{2.5}$$

Since (2.5) is trivial for $m = n$, it is enough to prove it for $m > n$. In that case,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq \varphi(x_n) - \varphi(x_{n+1}) + \varphi(x_{n+1}) - \cdots - \varphi(x_m) = \varphi(x_n) - \varphi(x_m). \end{aligned} \tag{2.6}$$

From (2.5), it follows that $\{x_n\}$ is a Cauchy sequence and, by completeness of M, there exists $x_0 \in M$ such that

$$\lim_{n\to\infty} d(x_n, x_0) = 0. \tag{2.7}$$

Hence, for every $n \in \mathbb{N}$,

$$\lim_{m\to\infty} d(x_n, x_m) = d(x_n, x_0).$$

Using this, (2.6) and the lower semicontinuity of φ,

$$\begin{aligned} d(x_n, x_0) = \lim_{m\to\infty} d(x_n, x_m) &\leq \limsup_{m\to\infty} [\varphi(x_n) - \varphi(x_m)] \\ &\leq \varphi(x_n) - \liminf_{m\to\infty} \varphi(x_m) \leq \varphi(x_n) - \varphi(x_0), \end{aligned} \tag{2.8}$$

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which implies that $x_0 \in \Pi(x_n)$ for every $n \in \mathbb{N}$, and therefore that

$$p(x_n) \le \varphi(x_0) \le \varphi(x_n) - d(x_n, x_0) \tag{2.9}$$

for every $n \in \mathbb{N}$. By letting $n \to \infty$ in (2.9) and using (2.3), (2.4) and (2.7), we conclude that

$$\varphi(x_0) = r. \tag{2.10}$$

Since, as proved above, $x_0 \in \Pi(x_n)$ for every $n \in \mathbb{N}$, (2.2) implies that $Tx_0 \in \Pi(x_n)$ for every $n \in \mathbb{N}$. Therefore, by (2.1), we conclude from (2.10) that

$$p(x_n) \le \varphi(Tx_0) \le \varphi(x_0) = r. \tag{2.11}$$

Letting n to infinity in (2.11) and using (2.4), we obtain $\varphi(Tx_0) = \varphi(x_0)$. By (1.1) again,

$$0 \le d(x_0, Tx_0) \le \varphi(x_0) - \varphi(Tx_0) = 0.$$

Hence $Tx_0 = x_0$, as claimed.

3. Analysis of the proof

A brief analysis of the proof in the preceding section shows that, unsurprisingly, it depends mainly on the Caristi property (1.1), which, combined with the critical triangle property of the metric, gives the telescoping effect. Take as an example the key inequality (2.6), which, as we showed, gives immediately the Cauchy property of the constructed sequence and hence by completeness its limit x_0, which clearly becomes a candidate for a fixed point. This procedure parallels a quest in other proofs for a maximal element in the sense of the Brøndsted partial order. The lower semicontinuity of φ and the continuity of d with respect to itself taken together finish the job by proving (using simple analytical arguments in (2.8)) that $x_0 \in \Pi(x_n)$ for every $n \in \mathbb{N}$. The rest follows through almost automatically. The only point which remains to be added is the inherent importance of the lack of symmetry in the Caristi property, which immediately gives unusually strong properties (critical for the proof's success) of the function φ, such as the monotonicity of the sequence $\{\varphi(x_n)\}$, the fact that $\varphi(Tx) \le \varphi(x)$ and finally its role in showing that $Tx_0 = x_0$.

Much has been said in the literature about the role of the axiom of choice in the proof of Caristi's theorem. While this discussion is not the main subject of our considerations in this paper, we have to make a few comments related to this topic. As per the above dissection of the proof, all points characterising the 'pure metric proof' as described in the introductory section have been fulfilled. We have not used any partial order or transfinite induction based arguments and we have completely restricted our proof to the standard methods of the theory of metric spaces and real numbers including the supremum axiom of $\mathbb{R}$ (or equivalently the Dedekind completeness of $\mathbb{R}$). However, our application of mathematical induction to define the sequence $\{x_n\}$ requires some attention here. This method is commonly used in standard analysis and indeed its validity does not require any defence. However, it has to be noted that it actually requires some form of the axiom of choice. Typically in such situations a strictly weaker axiom than the axiom of choice (AC), the axiom of dependent choice (DC), is invoked. This fact is not surprising since DC is sometimes called the 'axiom of inductive definition of sequences' (see, for example, discussions in [7, 13, 15]), and that is exactly what we did in our proof.

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