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TOWARDS A CLASSIFICATION OF CONVOLUTION-TYPE OPERATORS FROM l_1 TO l_∞

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1. Introduction. Let Z be the additive group of integer numbers with discrete topology, $l_1 \equiv L_1(Z)$ the space of complex-valued integrable functions on Z with respect to normalized Haar measure, $l_\infty \equiv L_\infty(Z)$ the space of bounded functions on Z. By $\mathcal{M}(l_1, l_\infty)$ we denote the set of convolution-type operators (or multipliers) from l_1 to l_∞; they are of the form $H_g(g \in l_\infty)$ with $H_g(f) = f * g(f \in l_1)$ where $*$ denotes convolution, so that $(f * g)(x) = \sum_{y \in Z} f(y)g(x-y)$.

We recall the following definitions about a bounded linear operator S from a Banach space X to a Banach space Y (to be found, e.g., in [3]): S is said to be *strictly singular* if whenever S has a bounded inverse on M, M a closed subspace of X, then M is finite dimensional. S is called *almost weakly compact* if whenever S has a bounded inverse on a closed subspace M of X, then M is reflexive.

We consider the following subsets of $\mathcal{M}(l_1, l_\infty)$: A_1, the set of compact operators; A_2, the set of weakly compact operators; A_3, the set of strictly singular operators; A_4, the set of almost weakly compact operators; A_5, the set of operators which do not have a bounded inverse on l_1; A_6 ($=\mathcal{M}(l_1, l_\infty) \setminus A_5$), the set of operators which do have a bounded inverse on l_1.

From the definitions we conclude that the inclusions $A_1 \subset A_2$ and $A_3 \subset A_4 \subset A_5$ are certainly true. That $A_2 \subset A_3$ follows easily from the fact that every infinite-dimensional subspace of l_1 is non-reflexive, and the obvious fact that a weakly compact operator can not be invertible on a non-reflexive subspace; the first observation also leads to $A_3 = A_4$.

A function g in l_∞ is called [weakly] almost periodic if the set $\{_a g : a \in Z\}$ of left translates is [weakly] relatively compact. The set of almost periodic functions on Z is a proper subset of the set of weakly almost periodic functions, since e.g., the function δ_0 which is one at 0 and zero at the other points of Z, is weakly almost periodic but not almost periodic. Since the [weakly] compact convolution operators H_g from l_1 to l_∞ are just those induced by the [weakly] almost periodic functions g, as shown in [2] and [7], we deduce $A_1 \subsetneqq A_2$.

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Hence, the foregoing observations lead to the following relations between the sets $A_1, \ldots, A_5$:

$$A_1 \subsetneqq A_2 \subset A_3 = A_4 \subset A_5.$$

In what follows we prove the additional results:

$$A_2 \subsetneqq A_3; \qquad A_4 \subsetneqq A_5; \qquad A_6 \neq \phi.$$

2. Main results

PROPOSITION 1. *There exists an operator H_g in $\mathcal{M}(l_1, l_\infty)$ which is an isometric embedding. In particular, $A_6 \neq \emptyset$.*

Proof. Denote by T the set of complex numbers z for which $|z| = 1$. For each positive integer k, the set T^k of all k-tuples of elements of T is separable; let $\{(z_{k,1}^{(i)}, z_{k,2}^{(i)}, \ldots, z_{k,k}^{(i)}\}_{i=1}^{\infty}$ be a countable dense subset of T^k. We choose a family $(B_i^j)_{i,j=1}^{\infty}$ of subsets of Z^+ with the following properties:

(i) for fixed j, each B_i^j consists of exactly $2j+1$ successive positive integers, say $B_i^j = [x_i^j, x_i^j + 1, \ldots, x_i^j + 2j]$.

(ii) if $i \neq i'$ or $j \neq j'$, then $B_i^j \cap B_{i'}^{j'} = \emptyset$.

This can be done by writing the sets B_i^j in a double array like an infinite matrix, and then choosing successively $B_1^1, B_1^2, B_2^1, B_3^1, B_2^2, B_1^3, B_1^4, \ldots$.

For each fixed j, we define g on $\bigcup_{i=1}^{\infty} B_i^j$ by means of

$$g(x_i^j) = z_{2j+1,2j+1}^{(i)}, \qquad g(x_i^j + 1) = z_{2j+1,2j}^{(i)}, \ldots, g(x_i^j + 2j) = z_{2j+1,1}^{(i)}.$$

We put $g(x) = 0$ for $x \in Z \setminus \bigcup_{i,j=1}^{\infty} B_i^j$.

For the function g so constructed we have $\|g\|_\infty = 1$; hence $\|f * g\|_\infty \leq \|f\|_1$ $(f \in l_1)$. To prove the converse inequality we may suppose that f has a compact support, since the set of those functions is dense in l_1. So let $f \neq 0$ be an element of l_1, with $f(x) = 0$ for $n \in Z \setminus [-n, +n]$, $n \in Z^+$. If y is an integer belonging to $[-n, +n]$ we put $a_y = \overline{\operatorname{sgn} f(y)}$ if $f(y) \neq 0^\perp$ and $a_y = 1$ if $f(y) = 0$. Then the $(2n+1)$-tuple $(a_{-n}, \ldots, a_0, \ldots, a_n)$ belongs to T^{2n+1}. Hence, given $\varepsilon > 0$ there exists an index i such that $|z_{2n+1,1}^{(i)} - a_{-n}| < \varepsilon, \ldots, |z_{2n+1,2n+1}^{(i)} - a_n| < \varepsilon$, and there exist points $x_i^n, \ldots, x_i^n + 2n$ such that

$$g(x_i^n) = z_{2n+1,2n+1}^{(i)}, \ldots, g(x_i^n + 2n) = z_{2n+1,1}^{(i)}$$

We so obtain

$$(f * g)(x_i^n + n) = \sum_{y=-n}^{n} f(y) g(x_i^n + n - y) = \sum_{y=-n}^{n} f(y) z_{2n+1,n+y+1}^{(i)}$$

from which we derive

$$(1) \qquad \left| (f * g)(x_i^n + n) - \sum_{y=-n}^{n} f(y) a_y \right| = \left| \sum_{y=-n}^{n} f(y) [z_{2n+1,n+y+1}^{(i)} - a_y] \right| \leq \varepsilon \, \|f\|_1.$$

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Since $\sum_{y=-n}^{n} f(y)a_y = \|f\|_1$, (1) leads to $|(f*g)(x_i^n+n)| \geq (1-\varepsilon)\|f\|_1$. This means that $\|f*g\|_\infty \geq (1-\varepsilon)\|f\|_1$, from which the result follows using the fact that ε was arbitrary. ■

PROPOSITION 2. *There exists an operator H_g in $\mathcal{M}(l_1, l_\infty)$ which is not strictly singular (= almost weakly compact) and which does not have a bounded inverse on l_1; i.e., $A_4 \subsetneqq A_5$.*

Proof. In Z^+ we choose a family $S = \{x_{ijk}\}$ of points were $1 \leq i$, $1 \leq j \leq i+1$, $1 \leq k \leq j$, where $x_{ijk} \neq x_{i'j'k'}$ if $(i, j, k) \neq (i', j', k')$, and where

$$x_{ijk} \leq x_{i'j'k'} \Leftrightarrow \text{either } i < i' \quad \text{or } i = i' \text{ and } j < j' \quad \text{or } i = i' \text{ and } j = j' \text{ and } k \leq k'.$$

We take care to construct S such that a finite sequence of 10^n $(n = 1, 2, \ldots)$ successive integers in Z does not contain more than $n+1$ elements from S and that, for $1 \leq k \leq j-1$, $x_{ij(j-k)} = x_{ijj} - 10^k + 1$.

We define the function $g \in l_\infty$ as follows:

(i) $g(x) = 0$ for $x \in Z \backslash S$

(ii) for $n = 1, 2, \ldots$, the set $\{(g(x_{in1}), g(x_{in2}), \ldots, g(x_{inn})) \colon i = n-1, n, \ldots\}$ is dense in T^n.

Put $A = \{10^n \colon n = 0, 1, 2, \ldots\}$, and $M = \{f \in l_1 \colon f(x) = 0 \text{ for } x \notin A\}$. Then M is an infinite dimensional closed subspace in l_1. Analogously as in proposition 1 it may be proved (using the special properties of S) that the convolution-operator H_g is an isomorphism on M. Hence H_g is not strictly singular. For each $n \in Z^+$ we define the function f_n on Z by

$$f_n(x) = \begin{cases} 10^{-n} & \text{for } 1 \leq x \leq 10^n \\ 0 & \text{elsewhere.} \end{cases}$$

Each f_n belongs to l_1, and $\|f_n\|_1 = 1$. If x is a point of Z we have

$$|(H_g(f_n))(x)| = \left| \sum_{1 \leq y \leq 10^n} 10^{-n} g(x-y) \right| \leq (n+1)10^{-n}.$$

Hence $\|H_g(f_n)\|_\infty \to 0$ for $n \to \infty$. This means that H_g does not have a bounded inverse on l_1. ■

PROPOSITION 3. *There exists an operator H_g in $\mathcal{M}(l_1, l_\infty)$ which is strictly singular but is not weakly compact; i.e., $A_2 \subsetneqq A_3$.*

In the proof use will be made of the following lemmas.

LEMMA 1. *Let there be given two finite sets $\{c_j\}_{j=1}^n$ and $\{d_j\}_{j=1}^n$ of complex numbers such that $|c_j| \leq 1$ and $|d_j| \leq 1$ for each j. Then there exist complex*

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numbers $\{\alpha_j\}_{j=1}^n$ *with* $|\alpha_j|=1$ *for each* j *such that* $|\sum_{j=1}^n \alpha_j c_j| \le 2$ *and* $|\sum_{j=1}^n \alpha_j d_j| \le 2$.

Proof. Choose an element c_{j_1} of $\{c_j\}_{j=1}^n$ such that $|c_{j_1}| \ge |c_j|$ for each $j \in \{1, \ldots, n\}$, and choose $\alpha_{j_1}=1$. Let $d_{j_2}(j_2 \ne j_1)$ be an element of $\{d_j\}_{j=1}^n$ such that $|d_{j_2}| \ge |d_j|$ for each $j \in \{1, \ldots, n\} \setminus \{j_1\}$, and choose α_{j_2} such that $|\alpha_{j_2}|=1$ and $\operatorname{sgn} \alpha_{j_2} d_{j_2} = -\operatorname{sgn} d_{j_1}$ (if d_{j_1} is zero, choose $\alpha_{j_2}=1$). Choose then $c_{j_3} \in \{c_j\}_{j=1}^n$ ($j_3 \ne j_1$ and $j_3 \ne j_2$) such that $|c_{j_3}| \ge |c_j|$ for each $j \in \{1, \ldots, n\} \setminus \{j_1, j_2\}$, and choose α_{j_3} with $|\alpha_{j_3}|=1$ and $\operatorname{sgn} \alpha_{j_3} c_{j_3} = -\operatorname{sgn}(\alpha_{j_1} c_{j_1} + \alpha_{j_2} c_{j_2})$ (if $\alpha_{j_1} c_{j_1} + \alpha_{j_2} c_{j_2} = 0$, choose $\alpha_{j_3}=1$). And so on.

We now prove $|\sum_{k=1}^l \alpha_{j_k} c_{j_k}| \le 2$ for all $l \le n$.

This is obvious when $l=1, 2$. Now let l be any even number smaller than n such that the above inequality holds.

Since $\operatorname{sgn}(\alpha_{j_{l+1}} c_{j_{l+1}}) = -\operatorname{sgn}(\sum_{k=1}^l \alpha_{j_k} c_{j_k})$ we clearly have $|\sum_{k=1}^{l+1} \alpha_{j_k} c_{j_k}| \le 2$. If $l+1=n$, then the proof is complete. Otherwise, we consider two cases. First, if

$$|\alpha_{j_{l+1}} c_{j_{l+1}}| \le \left| \sum_{k=1}^{l} \alpha_{j_k} c_{j_k} \right|,$$

then

$$\left| \sum_{k=1}^{l+2} \alpha_{j_k} c_{j_k} \right| \le \left| \sum_{k=1}^{l} \alpha_{j_k} c_{j_k} + \alpha_{j_{l+1}} c_{j_{l+1}} \right| + |\alpha_{j_{l+2}} c_{j_{l+2}}|$$
$$= \left| \sum_{k=1}^{l} \alpha_{j_k} c_{j_k} \right| - |\alpha_{j_{l+1}} c_{j_{l+1}}| + |\alpha_{j_{l+2}} c_{j_{l+2}}|.$$

Since

$$|\alpha_{j_{l+1}} c_{j_{l+1}}| = |c_{j_{l+1}}| \ge |c_{j_{l+2}}| = |\alpha_{j_{l+2}} c_{j_{l+2}}|$$

we infer

$$\left| \sum_{k=1}^{l+2} \alpha_{j_k} c_{j_k} \right| \le \left| \sum_{k=1}^{l} \alpha_{j_k} c_{j_k} \right| \le 2.$$

If

$$|\alpha_{j_{l+1}} c_{j_{l+1}}| > \left| \sum_{k=1}^{l} \alpha_{j_k} c_{j_k} \right|,$$

then

$$\left| \sum_{k=1}^{l+2} \alpha_{j_k} c_{j_k} \right| \le \left| \sum_{k=1}^{l} \alpha_{j_k} c_{j_k} + \alpha_{j_{l+1}} c_{j_{l+1}} \right| + |\alpha_{j_{l+2}} c_{j_{l+2}}|$$
$$\le |\alpha_{j_{l+1}} c_{j_{l+1}}| + |\alpha_{j_{l+2}} c_{j_{l+2}}| \le 1 + 1 = 2.$$

Hence $|\sum_{j=1}^n \alpha_j c_j| = |\sum_{k=1}^n \alpha_{j_k} c_{j_k}| \le 2$ and, similarly, $|\sum_{j=1}^n \alpha_j d_j| \le 2$. ■

LEMMA 2. *Let g be the function defined on Z by means of $g(n)=1$ for $n\geq 0$, $g(n)=0$ for $n<0$. Then g is not weakly almost periodic.*

Proof. If g were almost periodic, each sequence from the set of translates of g would possess a subsequence which converges weakly to an element of l_∞ (by the Eberlein–Šmulian theorem). So in particular the sequence $\{g_i: i=1,2,\ldots\}$ where $g_i(n)=g(n-i)$ would have a subsequence $\{g_{i_j}: j=1,2,\ldots\}$ converging weakly to $h\in l_\infty$.

Each $n\in Z$ defines a continuous linear functional F_n on l_∞ by means of $F_n(k)=k(n)$ $(k\in l_\infty)$. Hence, for each $n\in Z$ we would have $F_n(g_{i_j})\to F_n(h)$ for $j\to\infty$, or $g_{i_j}(n)\to h(n)$ for $j\to\infty$. Since for each n we have $n-i_j<0$ for all large enough values of j, $g_{i_j}(n)$ is zero for such j. Hence $h(n)=0$ for all n of Z, and so $h=0$.

Denote by βZ the Stone–Čech compactification of Z and let x_∞ in βZ be a cluster point of Z^+. For each fixed j, $g_{i_j}(x_\infty)=1$; so $F_{x_\infty}(g_{i_j})\to 1$ for $j\to\infty$. On the other hand $F_{x_\infty}(h)=h(x_\infty)=0$. This leads to a contradiction. ■

Proof of Proposition 3. From Lemma 2 and the connection between weakly almost periodic g and weakly compact H_g, it follows that the convolution operator H_g with g as defined in Lemma 2 is not weakly compact. We show that, however, H_g is strictly singular.

Let M be a closed infinite dimensional subspace of l_1, and $\varepsilon>0$ arbitrary. If f_1 is a function in l_1 with $\|f_1\|_1=1$, we may choose a compact subset K_1 of Z such that $\sum_{x\in Z\setminus K_1}|f_1(x)|\leq\varepsilon$. Since M has infinite dimension, there exist functions g_1 and g_2 in M which are different and such that $g_1=g_2$ on K_1. Putting $g_1-g_2=h$ we obtain a function $h\in M$ for which $\|h\|_1\neq 0$. Multiplying h with a constant leads to the following result: there exists a function $f_2\in M$ with $\|f_2\|_1=1$ and $f_2=0$ on K_1; for this function f_2 we may find a compact subset $K_2\supset K_1$ in Z such that $\sum_{x\in Z\setminus K_2}|f_2(x)|\leq\varepsilon$.

We use this procedure in the following manner. Let n and m be natural numbers, both not smaller than 2, and let $\varepsilon>0$ be arbitrary. For $1\leq i\leq m$ and $1\leq j\leq n$ we may then choose functions f_{ij} in M and strictly positive integers x_{ij} such that

(i) $x_{11}<x_{12}<\cdots<x_{1n}<x_{21}<\cdots<x_{2n}<x_{31}<\cdots<x_{mn}$.

(ii) $f_{ij}(x)=0$ for $x\in[-x_{i'j'}, x_{i'j'}]$, where $x_{i'j'}$ is the point in (i) just preceding x_{ij}, with the convention that $i'=j'=0$ if $i=j=1$, and $x_{00}=0$.

(iii) $\|f_{ij}\|_1=1$

(iv) $\sum_{x\in Z\setminus[-x_{ij},x_{ij}]}|f_{ij}(x)|\leq\varepsilon\cdot 2^{-n(i-1)-j}$.

For $1\leq i\leq m$, $1\leq j\leq n$ we put $C_{ij}=[-x_{ij}, -x_{i'j'}[$, $D_{ij}=]\,x_{i'j'}, x_{ij}]$ (with the same convention as in (ii)), and $c_{ij}=\sum_{x\in C_{ij}}f_{ij}(x)$, $d_{ij}=\sum_{x\in D_{ij}}f_{ij}(x)$.

Since $|c_{ij}|\leq 1$, $|d_{ij}|\leq 1$, we conclude from Lemma 1 that for each fixed $i\in\{1,\ldots,m\}$ there exist complex numbers α_{ij} $(1\leq j\leq n)$ where $|\alpha_{ij}|=1$ such

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that

$$\left|\sum_{j=1}^{n} \alpha_{ij} c_{ij}\right| \leq 2 \quad \text{and} \quad \left|\sum_{j=1}^{n} \alpha_{ij} d_{ij}\right| \leq 2.$$

If we put

$$f(x) = \sum_{i,j}^{m,n} \alpha_{ij} f_{ij}(x)$$

we obtain a function f belonging to M, and

$$\begin{aligned}\|f\|_1 &= \sum_{y \in Z} \left|\sum_{i,j}^{m,n} \alpha_{ij} f_{ij}(y)\right| \geq \sum_{i,j}^{m,n} \sum_{x \in C_{ij} \cup D_{ij}} \left|\sum_{r,s}^{m,n} \alpha_{rs} f_{rs}(x)\right| \\ &\geq \sum_{i,j}^{m,n} \sum_{x \in C_{ij} \cup D_{ij}} \left(|f_{ij}(x)| - \sum_{r,s}^{m,n} |f_{rs}(x)|\right) \\ &\geq \sum_{i,j}^{m,n} (1 - \varepsilon \cdot 2^{-n(i-1)-j}) - \sum_{i,j}^{m,n} \varepsilon \cdot 2^{-n(i-1)-j} \geq nm - 2\varepsilon.\end{aligned}$$

For the convolution operator H_g we have

$$(H_g(f))(x) = \sum\nolimits_{y \in Z} f(y) g(x-y) = \sum_{y \leq x} f(y).$$

Considering different cases (e.g., $x < -x_{mn}$, $x > x_{mn}$, $x \in C_{ij}$, $x \in D_{ij}$, $x = 0$) it can be shown that $\|H_g(f)\|_\infty \leq 4m + n + \varepsilon$.

Without putting in the laborious checking of all the cases, we show the way by noting that for all i

$$\begin{aligned}\left|\sum \left\{f(x) \colon x \in \bigcup_{j=1}^{n} C_{ij}\right\}\right| &\leq \left|\sum_{j=1}^{n} \sum_{x \in C_{ij}} \alpha_{ij} f_{ij}(x)\right| + \left|\sum_{j=1}^{n} \sum_{x \in C_{ij}} \sum_{(r,s) \neq (i,j)} \alpha_{rs} f_{rs}(x)\right| \\ &\leq \left|\sum_{j=1}^{n} \alpha_{ij} \sum_{x \in C_{ij}} f_{ij}(x)\right| + \sum_{r,s} \sum \left\{|\alpha_{rs} f_{rs}(x)| \colon x \in \bigcup_{j} C_{ij} \setminus C_{rs}\right\} \\ &\leq \left|\sum_{j=1}^{n} \alpha_{ij} c_{ij}\right| + \sum_{r,s} \sum \{|f_{rs}(x)| \colon x \in Z \setminus [-x_{rs}, +x_{rs}]\} \\ &\leq 2 + \sum_{r,s} \varepsilon \cdot 2^{-n(r-1)-s} \leq 2 + \varepsilon.\end{aligned}$$

Anyhow, we conclude

$$\frac{\|H_g(f)\|_\infty}{\|f\|_1} \leq \frac{4m + n + \varepsilon}{nm - 2\varepsilon},$$

which tends to zero for $n, m \to \infty$. From this we conclude that H_g does not have a bounded inverse on M, and so H_g is strictly singular. ■

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3. Remarks

3.1. It is easy to see that the following result is true: H_g belongs to $A_5 \Leftrightarrow$ for each $\varepsilon > 0$, there exists a finite set $\{c_i\}_{i=1}^n$ of complex numbers such that $\sum_{i=1}^n |c_i| = 1$, and a corresponding set $\{a_i\}_{i=1}^n$ of different points in Z such that $\|\sum_{i=1}^n c_i\, {}_{a_i}g\|_\infty < \varepsilon$.

3.2. As we mentioned in the introduction, the compact and weakly compact operators H_g in $\mathcal{M}(l_1, l_\infty)$ are completely determined by g being either almost periodic or weakly almost periodic. This is even true for more general locally compact groups (see the references). The problem of giving necessary and sufficient conditions on $g \in l_\infty$ for H_g to be in A_3 $(=A_4)$ remains unsolved.

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