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EINSTEIN-KAEHLER MANIFOLDS IMMERSED IN A COMPLEX PROJECTIVE SPACE

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A Kaehler manifold of constant holomorphic curvature is called a *complex space form*. By a Kaehler submanifold we mean a complex submanifold with the induced Kaehler metric. B. Smyth [**5**] has studied a complete Einstein-Kaehler hypersurface in a complete and simply connected complex space form and classified completely the hypersurface. The local version of this result has been shown to be true by S. S. Chern [**1**], and partially by T. Takahashi [**6**] independently. On the other hand, K. Ogiue has also proved an n-dimensional compact Einstein-Kaehler submanifold immersed in an N-dimensional complex projective space P_NC is totally geodesic or the Ricci tensor S satisfies $S \leqq (n/2)g$, where g is the induced Kaehler metric (cf. see [**4**]).

The purpose of this paper is to prove the following theorem. Throughout this paper, let $P_n(c)$ be an n-dimensional complex projective space of constant holomorphic curvature c.

THEOREM. *Let M be an $n(\geqq 2)$-dimensional Einstein-Kaehler submanifold immersed in $P_{n+p}(c)$. If the immersion is full and the second fundamental form is parallel, then the following are true:*

(1) *If $p < n/2$, then $p = 1$ and M is locally a complex quadric Q_n.*

(2) *If $p \geqq n(n+1)/2$, then $p = n(n+1)/2$ and M is locally $P_n(c/2)$.*

1. Preliminaries. In this section, we shall begin the self-contained discussion about Kaehler submanifolds in $P_{n+p}(c)$ for convenience, and prepare for necessary formulas for later use. Let M be an n-dimensional Kaehler submanifold immersed in $P_{n+p}(c)$. We choose a local field of unitary frames $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+p}$ in such a way that, restricted to M, $e_1, \ldots, e_n$ are tangent to M. Let $\omega^1, \ldots, \omega^n, \omega^{n+1}, \ldots, \omega^{n+p}$ be the field of its dual frames. Then the Kaehler metric $\tilde{g}$ of $P_{n+p}(c)$ is given by $\tilde{g} = 2\sum_A \omega^A \bar{\omega}^A$† and the structure equations of $P_{n+p}(c)$ are given by

$$(1.1) \qquad d\omega^A + \textstyle\sum_B \omega_B{}^A \wedge \omega^B = 0, \quad \omega_B{}^A + \bar{\omega}_A{}^B = 0,$$

$$(1.2) \qquad d\omega_B{}^A + \textstyle\sum_C \omega_C{}^A \wedge \omega_B{}^C = \Omega_B{}^A, \quad \Omega_B{}^A = \sum_{C,D} R^A{}_{BC\bar{D}} \omega^C \wedge \bar{\omega}^D.$$

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†Throughout this paper, we use the following convention on the range of indices, unless otherwise stated:

$A, B, C, \ldots = 1, \ldots, n, n+1, \ldots, n+p$

$i, j, k, \ldots = 1, \ldots, n$

$\alpha, \beta, \gamma, \ldots = n+1, \ldots, n+p$

Since the ambient space is a complex space form of constant holomorphic curvature c, we have

$$(1.3) \quad R_{BC\bar{D}}{}^{A} = \frac{c}{2}(\delta_B{}^A\delta_{CD} + \delta_C{}^A\delta_{BD}).$$

Restricting these forms to M, we have

$$(1.4) \quad \omega^\alpha = 0,$$

and the induced Kaehler metric g of M is given by $g = 2\sum_i \omega^i\bar{\omega}^i$. It follows from (1.4) and Cartan's lemma that (1.1) implies

$$(1.5) \quad \omega_i{}^\alpha = \sum_j h_{ij}{}^\alpha\omega^j, \quad h_{ij}{}^\alpha = h_{ji}{}^\alpha.$$

Moreover, we obtain

$$(1.6) \quad d\omega^i + \sum_j \omega_j{}^i \wedge \omega^j = 0, \quad \omega_j{}^i + \bar{\omega}_i{}^j = 0,$$

$$(1.7) \quad d\omega_j{}^i + \sum_k \omega_k{}^i \wedge \omega_j{}^k = \Omega_j{}^i, \quad \Omega_j{}^i = \sum_{k,l} R_{jk\bar{l}}{}^i\omega^k \wedge \bar{\omega}^l,$$

$$(1.8) \quad d\omega_\beta{}^\alpha + \sum_\gamma \omega_\gamma{}^\alpha \wedge \omega_\beta{}^\gamma = \Omega_\beta{}^\alpha, \quad \Omega_\beta{}^\alpha = \sum_{k,l} R_{\beta k\bar{l}}{}^\alpha\omega^k \wedge \bar{\omega}^l.$$

From the above equations, we have

$$(1.9) \quad \Omega_j{}^i = \sum_{k,l}\left\{\frac{c}{2}(\delta_j{}^i\delta_{kl} + \delta_k{}^i\delta_{jl}) - \sum_\alpha h_{jk}{}^\alpha\bar{h}_{il}{}^\alpha\right\}\omega^k \wedge \bar{\omega}^l.$$

Similarly, it follows from (1.2), (1.3), (1.4), (1.5) and (1.8) that we have

$$(1.10) \quad \Omega_\beta{}^\alpha = \sum_{k,l}\left(\frac{c}{2}\delta_\beta{}^\alpha\delta_{kl} + \sum_j h_{kj}{}^\alpha\bar{h}_{jl}{}^\beta\right)\omega^k \wedge \bar{\omega}^l.$$

Now, with respect to these frames, the Ricci tensor S of M can be expressed as follows;

$$(1.11) \quad S = \sum_{k,l}(S_{k\bar{l}}\omega^k \otimes \bar{\omega}^l + S_{\bar{k}l}\bar{\omega}^k \otimes \omega^l),$$

where $S_{k\bar{l}} = S_{\bar{l}k} = \bar{S}_{\bar{k}l}$ are given by

$$(1.12) \quad S_{k\bar{l}} = \frac{n+1}{2}c\delta_{kl} - \sum_{\alpha,j} h_{kj}{}^\alpha\bar{h}_{jl}{}^\alpha.$$

The scalar curvature R is also given by

$$(1.13) \quad R = n(n+1)c - 2\sum_{\alpha,k,l} h_{kl}{}^\alpha\bar{h}_{kl}{}^\alpha.$$

This implies that we have

$$(1.13)' \quad n(n+1)c - R \geqq 0,$$

where the equality is valid if and only if M is totally geodesic.

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If we define $h_{ijk}{}^{\alpha}$ and $h_{ij\bar{k}}{}^{\alpha}$ by

$$\sum_k h_{ijk}{}^{\alpha}\omega^k + \sum_k h_{ij\bar{k}}{}^{\alpha}\bar{\omega}^k = dh_{ij}{}^{\alpha} - \sum_k h_{kj}{}^{\alpha}\omega_i{}^k - \sum_k h_{ik}{}^{\alpha}\omega_j{}^k + \sum_\beta h_{ij}{}^{\beta}\omega_\beta{}^{\alpha},$$

then we can easily obtain

(1.14) $h_{ijk}{}^{\alpha} = h_{ijk}{}^{\alpha}$, $h_{ij\bar{k}}{}^{\alpha} = 0$.

Next we define $h_{ijkl}{}^{\alpha}$ and $h_{ijk\bar{l}}{}^{\alpha}$ as follows:

$$\sum_l h_{ijkl}{}^{\alpha}\omega^l + \sum_l h_{ijk\bar{l}}{}^{\alpha}\bar{\omega}^l = dh_{ijk}{}^{\alpha} - \sum_l h_{ljk}{}^{\alpha}\omega_i{}^l - \sum_l h_{ilk}{}^{\alpha}\omega_j{}^l - \sum_l h_{ijl}{}^{\alpha}\omega_k{}^l + \sum_\beta h_{ijk}{}^{\beta}\omega_\beta{}^{\alpha}.$$

Then, by the similar and easy calculation, we have

$$(1.15)\quad \begin{cases} h_{ijkl}{}^{\alpha} = h_{ijlk}{}^{\alpha}, \\ h_{ijk\bar{l}}{}^{\alpha} = \dfrac{c}{2}\,(h_{ij}{}^{\alpha}\delta_{kl} + h_{jk}{}^{\alpha}\delta_{il} + h_{ki}{}^{\alpha}\delta_{jl}) \\ \qquad - \displaystyle\sum_{\beta,h} (h_{hj}{}^{\alpha}h_{ik}{}^{\beta} + h_{ih}{}^{\alpha}h_{jk}{}^{\beta} + h_{ij}{}^{\beta}h_{hk}{}^{\alpha})\bar{h}_{hl}{}^{\beta}. \end{cases}$$

Making use of the second equations of (1.14) and (1.15), we easily have

$$(1.16)\quad \sum_k h_{ijk\bar{k}}{}^{\alpha} = \frac{n+2}{2}\,ch_{ij}{}^{\alpha} - \sum_{\beta,k,l} (h_{ik}{}^{\beta}\bar{h}_{kl}{}^{\beta}h_{lj}{}^{\alpha} + h_{ik}{}^{\alpha}\bar{h}_{kl}{}^{\beta}h_{lj}{}^{\beta}) - \sum_{\beta,k,l} h_{kl}{}^{\alpha}\bar{h}_{kl}{}^{\beta}h_{ij}{}^{\beta}$$

and

$$(1.17)\quad \begin{aligned} \sum_{\alpha,i,j} h_{ijk\bar{l}}{}^{\alpha}\bar{h}_{ij}{}^{\alpha} &= \Big(\sum_{\alpha,i,j} h_{ij}{}^{\alpha}\bar{h}_{ij}{}^{\alpha}\Big)_{k\bar{l}} - \sum_{\alpha,i,j} h_{ijk}{}^{\alpha}\bar{h}_{ijl}{}^{\alpha} \\ &= \frac{c}{2}\Big(\sum_{\alpha,i,j} h_{ij}{}^{\alpha}\bar{h}_{ij}{}^{\alpha}\delta_{kl} + 2\sum_{\alpha,j} h_{kj}{}^{\alpha}\bar{h}_{jl}{}^{\alpha}\Big) - 2\sum_{\alpha,\beta,i,j,k} h_{ki}{}^{\beta}\bar{h}_{ij}{}^{\alpha}h_{jh}{}^{\alpha}\bar{h}_{hl}{}^{\beta} \\ &\quad - \sum_{\alpha,\beta}\Big(\sum_{i,j} h_{ij}{}^{\beta}\bar{h}_{ij}{}^{\alpha}\sum_h h_{kh}{}^{\alpha}\bar{h}_{hl}{}^{\beta}\Big). \end{aligned}$$

We define three kinds of matrices A, H, H_α by

$$A = (A_\beta{}^{\alpha}),\quad A_\beta{}^{\alpha} = \sum_{i,j} h_{ij}{}^{\alpha}\bar{h}_{ij}{}^{\beta},$$

$$H = (h_{(ij)}{}^{\alpha})\quad \text{for } i \leqq j,$$

$$H_\alpha = (h_{ij}{}^{\alpha})\quad \text{for a fixed } \alpha.$$

Then A is a $p \times p$-hermitian matrix, the second matrix H is $p \times n(n+1)/2$, H_α is an $n \times n$-matrix, and we have the following mutual relation

(1.18) $HH^* = A$.

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Using these matrices, we can express the last term of (1.17) with the following form

$$\frac{c}{2}\left(\mathrm{Tr}A\cdot I+2\sum_{\alpha}H_{\alpha}\bar{H}_{\alpha}\right)-2\sum_{\alpha,\beta}H_{\beta}\bar{H}_{\alpha}H_{\alpha}\bar{H}_{\beta}-\sum_{\alpha,\beta}A_{\alpha}{}^{\beta}H_{\alpha}\bar{H}_{\beta},$$

where I is an $n\times n$-unit matrix.

2. Einstein-Kaehler submanifolds. Let M be an n-dimensional Einstein-Kaehler submanifold immersed in $P_{n+p}(c)$. Since the Ricci tensor S of M satisfies

$$(2.1)\qquad S=\frac{R}{2n}g,\quad S_{k\bar{l}}=\frac{R}{2n}\delta_{kl},$$

where R is the scalar curvature, it follows from (1.12) and (2.1) that we have

$$(2.2)\qquad \sum_{\alpha}H_{\alpha}\bar{H}_{\alpha}=\frac{n(n+1)c-R}{2n}I,\quad \sum_{\alpha,j}h_{kj}{}^{\alpha}\bar{h}_{jl}{}^{\alpha}=\frac{n(n+1)c-R}{2n}\delta_{kl}.$$

It implies that we get

$$(2.3)\qquad \mathrm{Tr}\,A=\mathrm{Tr}\sum_{\alpha}H_{\alpha}\bar{H}_{\alpha}=\frac{n(n+1)c-R}{2}.$$

Making use of (2.2), we can simplify equation (1.16) as follows:

$$(2.4)\qquad \Delta H=\frac{2R-n^{2}c}{2n}H-AH,\quad h_{ijk\bar{k}}{}^{\alpha}=\frac{2R-n^{2}c}{2n}h_{ij}{}^{\alpha}-\sum_{\beta}A_{\beta}{}^{\alpha}h_{ij}{}^{\beta}.$$

Moreover, since the scalar curvature R is constant and consequently the trace of the matrix A is also constant, we have from (1.17)

$$(2.5)\qquad -\sum_{\alpha,i,j}h_{ijk}{}^{\alpha}\bar{h}_{ijl}{}^{\alpha}=\frac{2R-n^{2}c}{2n}\cdot\frac{n(n+1)c-R}{2n}\delta_{kl}-\sum_{\alpha,\beta,j}A_{\beta}{}^{\alpha}h_{kj}{}^{\beta}\bar{h}_{jl}{}^{\alpha},$$

so that we get

$$(2.6)\qquad 0\geqq\frac{2R-n^{2}c}{2n}\mathrm{Tr}A-\mathrm{Tr}A^{2}.$$

Because of the definition of the hermitian matrix A, the hermitian transformation defined by A is positive semi-definite and it implies that eigenvalues of A are all non-negative. This means $\mathrm{Tr}\,A^{2}\leqq(\mathrm{Tr}\,A)^{2}$. Combining this inequality together with the inequality (2.3), we have

$$(2.7)\qquad 0\geqq(R-n^{2}c)\mathrm{Tr}\,A.$$

3. Proof of theorem. Since the second fundamental form is parallel, we have the following mutual relation between the matrices A and H:

$$(3.1)\qquad AH=\frac{2R-n^{2}c}{2n}H,$$

because of (2.4). From (1.18) and (3.1), we have

$$(3.2) \qquad A^2 = \frac{2R - n^2 c}{2n} A.$$

This means that if we take an eigenvalue λ of A, then $\lambda = 0$ or $(2R - n^2c)/2n$.

First of all, we consider the case where there exists a point x in M at which the matrix A has no non-zero eigenvalues. Then it is easily seen that A is a zero matrix, so that x is a geodesic point. It implies that $R = n(n + 1)c$ at x. Since the scalar curvature R is constant, the equation is true on M. Accordingly M is totally geodesic.

On the other hand, suppose that there does not exist a geodesic point. In other words, the matrix A has at least one non-zero eigenvalue $\lambda = (2R - n^2c)/2n$, so that we get

$$2R - n^2c > 0,$$

because the transformation defined by A is positive semi-definite. We investigate a property concerning the rank of matrices A and H. We denote by $r(x)$ the rank of the matrix A at any point x in M. Then the following result is verified.

LEMMA 3.1. *For any point x in M, we have*

$$(3.3) \qquad r(x) = \operatorname{rank} H = \frac{n\{n(n + 1)c - R\}}{2R - n^2c}.$$

Proof. From (1.18) and (3.1), we see easily that the rank of the matrix A is equal to that of the matrix H at any point in M. Since a non-zero eigenvalue $\lambda(x)$ of A at x satisfies $\lambda(x) = (2R - n^2c)/2n$, $\lambda(x)$ is constant on M, so that the multiplicity $r(x)$ of $\lambda(x)$ is constant, too. On the other hand, we get the trace of A from (2.3). Thus we have the relation

$$r(x)\lambda(x) = \frac{n(n + 1)c - R}{2},$$

and therefore it completes the proof.

Next we shall investigate the range of the scalar curvature.

LEMMA 3.2.

$$R = n^2c \quad \textit{or} \quad R \leqq \frac{n(3n + 2)}{4} c.$$

Proof. Since the second fundamental form is parallel, we get

$$\frac{c}{2} (h_{ij}{}^{\alpha}\delta_{kl} + h_{jk}{}^{\alpha}\delta_{il} + h_{ki}{}^{\alpha}\delta_{jl}) \\ - \sum_{\beta,h} (h_{hj}{}^{\alpha}h_{ik}{}^{\beta} + h_{ih}{}^{\alpha}h_{jk}{}^{\beta} + h_{ij}{}^{\beta}h_{hk}{}^{\alpha})\bar{h}_{hl}{}^{\beta} = 0,$$

because of (1.5). Transvecting $\bar{h}_{mi}{}^{\alpha}\bar{h}_{jk}{}^{\gamma}$ to this equation, from (2.2) and (3.1) we have

$$(3.4)\qquad \sum_{\alpha,\beta} \bar{H}_\alpha H_\beta \bar{H}_\gamma H_\alpha \bar{H}_\beta = \frac{1}{8n^2}\{2R^2 - n(3n+2)cR + n^2(n^2+2n+2)c^2\}\bar{H}_\gamma.$$

Now we define a matrix $G_{\alpha\beta\gamma}$ by

$$G_{\alpha\beta\gamma} = H_\alpha \bar{H}_\beta H_\gamma + H_\gamma \bar{H}_\beta H_\alpha - \frac{2\{n(n+1)c - R\}}{n^3c - (n-2)R}(A_\beta{}^\alpha H_\gamma + A_\beta{}^\gamma H_\alpha).$$

By direct calculation, it follows from (2.2), (3.2) and (3.4) that we obtain

$$\sum_{\alpha,\beta,\gamma} G_{\alpha\beta\gamma}{}^*G_{\alpha\beta\gamma} = \frac{n+2}{8n^3} \times \frac{\{n(3n+2)c - 4R\}(n^2c - R)\{n(n+1)c - R\}\{n(n+2)c - R\}}{n^3c - (n-2)R} I.$$

Since the trace of the matrix on the left hand side is non-negative, the conclusion of this lemma follows immediately from (1.13)′ and (2.7).

Taking account of Lemmas 3.1 and 3.2, we have the following equations:

$$(3.5)\qquad R = \frac{n^2(n+r+1)}{n+2r}c,$$

$$(3.6)\qquad r = 1 \text{ or } r \geqq \frac{n}{2}.$$

LEMMA 3.3. *There exists an $(n+r)$-dimensional totally geodesic submanifold M' in $P_{n+p}(c)$, in which the given submanifold M is immersed, where* $r =$ rank $A > 0$.

Proof. For the unitary frame $\{e_i, e_\alpha\}$ at any point x, we define the normal space to M at x, which is denoted by N_x, by

$$N_x = \left\{\sum_\alpha \xi^\alpha e_\alpha : \xi^\alpha \in C\right\},$$

where C is the complex field. We define a mapping f of $N_x \times N_x$ into C by

$$f(X, Y) = \sum_{\alpha,\beta} A_\beta{}^\alpha \bar{\xi}^\alpha \eta^\beta, \quad \text{where } X = \sum_\alpha \xi^\alpha e_\alpha \text{ and } Y = \sum_\beta \eta^\beta e_\beta.$$

Let H_p be a set of all hermitian matrices of order p, which is considered as a complex vector space. Then the unitary group $U(p)$ operates on H_p as follows: For any hermitian matrix H in H_p and any unitary matrix U in $U(p)$,

$$U(H) = U^*HU.$$

Since the matrix A is invariant under $U(p)$, the mapping f is well-defined and it is a positive semi-definite hermitian form of order r, so that it can be normalized. This means that we can choose a new unitary frame $\{e_i, e_\alpha, e_\lambda\}$ such that

(3.7) $\quad \omega_i^\alpha \neq 0, \omega_i^\lambda = 0 \quad$ for $n + 1 \leqq \alpha \leqq n + r, n + r + 1 \leqq \lambda \leqq n + p.$

By the definition of h_{ijk}^λ, we have

$$\sum_{\alpha=n+1}^{n+r} h_{ij}^\alpha \omega_\alpha^\lambda = 0 \text{ for } \lambda \geqq n + r + 1.$$

It implies that

(3.8) $\quad \omega_\alpha^\lambda = 0 \quad$ for $\alpha \leqq n + r, \lambda \geqq n + r + 1.$

From (3.7) and (3.8) we can consider a distribution $\tilde{\mathscr{M}}$ defined by

$$\tilde{\omega}^\lambda = 0, \tilde{\omega}_i^\lambda = 0, \tilde{\omega}_\alpha^\lambda = 0 \quad \text{for } \alpha \leqq n + r, \lambda \geqq n + r + 1.$$

Then it follows from the structure equations that we obtain

$$\begin{aligned}
d\tilde{\omega}^\lambda &= - \sum_{i=1}^{n} \tilde{\omega}_i^\lambda \wedge \tilde{\omega}^i - \sum_{\alpha=n+1}^{n+r} \tilde{\omega}_\alpha^\lambda \wedge \tilde{\omega}^\alpha - \sum_{\mu=n+r+1}^{n+p} \tilde{\omega}_\mu^\lambda \wedge \tilde{\omega}^\mu \quad \lambda \geqq n + r + 1 \\
&\equiv 0 \pmod{\tilde{\omega}^\lambda, \tilde{\omega}_i^\lambda, \tilde{\omega}_\alpha^\lambda}, \\
d\tilde{\omega}_i^\lambda &= - \sum_{j=1}^{n} \tilde{\omega}_j^\lambda \wedge \tilde{\omega}_i^j - \sum_{\alpha=n+1}^{n+r} \tilde{\omega}_\alpha^\lambda \wedge \tilde{\omega}_i^\alpha - \sum_{\mu=n+r+1}^{n+p} \tilde{\omega}_\mu^\lambda \wedge \tilde{\omega}_i^\mu + \Omega_i^\lambda \\
&\qquad \lambda \geqq n + r + 1 \\
&\equiv 0 \pmod{\tilde{\omega}^\lambda, \tilde{\omega}_i^\lambda, \tilde{\omega}_\alpha^\lambda}, \\
d\tilde{\omega}_\alpha^\lambda &= - \sum_{i=1}^{n} \tilde{\omega}_i^\lambda \wedge \tilde{\omega}_\alpha^i - \sum_{\alpha=n+1}^{n+r} \tilde{\omega}_\beta^\lambda \wedge \tilde{\omega}_\alpha^\beta - \sum_{\mu=n+r+1}^{n+p} \tilde{\omega}_\mu^\lambda \wedge \tilde{\omega}_\alpha^\mu + \Omega_\alpha^\lambda \\
&\qquad \lambda \geqq n + r + 1 \\
&\equiv 0 \pmod{\tilde{\omega}^\lambda, \tilde{\omega}_i^\lambda, \tilde{\omega}_\alpha^\lambda}.
\end{aligned}$$

Therefore a distribution $\mathscr{M}$ becomes an $(n + r)$-dimensional completely integrable distribution. For any point x, we consider the maximal integral submanifold $M'(x)$ of $\mathscr{M}$ through x. Then $M'(x)$ is of $(n + r)$-dimensional and by the construction it is totally geodesic in $P_{n+p}(c)$. Moreover M is immersed in $M'(x)$. This completes the proof.

The immersion of M into $P_{n+p}(c)$ is said to be *full*, if M cannot be immersed in an $(n + q)$-dimensional totally geodesic submanifold in $P_{n+p}(c)$, where $p > q \geqq 0$. The assertion (1) of the theorem follows immediately from (3.6), Lemma 3.3 and a theorem due to Nomizu and Smyth [**3**].

We shall prove the other one. In this case, we may suppose $p = r = n(n + 1)/2$, because of the full immersion. This means that by virtue of (3.5) we have

(3.9) $\quad R = \dfrac{n(n + 1)}{2} c.$

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We define a tensor $Z_{i\bar{j}k\bar{l}}$ by

$$Z_{i\bar{j}k\bar{l}} = \sum_{\alpha} h_{ik}{}^{\alpha} \bar{h}_{jl}{}^{\alpha} - \frac{c}{4}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}).$$

Then we get

$$\sum_{i,j,k,l} Z_{i\bar{j}k\bar{l}} \bar{Z}_{i\bar{j}k\bar{l}} = \mathrm{Tr}A^2 - c\mathrm{Tr}A + \frac{n(n+1)}{8}c^2.$$

Taking account of equations (2.3), (3.2) and (3.9), we see that the right hand side vanishes identically, so that $Z_{i\bar{j}k\bar{l}} = 0$. It implies M is of constant holomorphic curvature $c/2$. This concludes the proof.

Remark 1. As it can easily be imagined from the main theorem, a complex quadric Q_n and a complex projective space are trivial examples of Einstein-Kaehler manifolds immersed holomorphically in a complex projective space. We can take the following other examples:

(1) $P_n(c) \times P_n(c)$ in $P_{n^2+n}(c)$.

(2) Compact irreducible hermitian symmetric spaces.

Remark 2. The estimate of the codimension in assertion (1) of the theorem is best possible. In particular, we point out expressly the fact that the codimension is greater than or equal to half the dimension of imbedded manifolds in the above examples except for the complex quadric.. The equality holds only in the following two cases; $SU(5)/S(U(3) \times U(2))$ in P_9C and $SO(10)/U(5)$ in $P_{15}C$. In these cases, the second fundamental forms are both parallel. See the forthcoming paper [**2**] along this line.

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