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VARIATIONS ON MÜNTZ'S THEME

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ABSTRACT. We consider some variations of Müntz's classical theorem on when $\mathrm{Span}\{x^{\lambda_i}\}$ is dense in $C[0,1]$. We prove, for example, that if $\pi_n(\lambda_n) := \left\{\sum_{i=0}^n a_i x^{i\lambda_n}\right\}$ then the collection of spaces $\{\pi_n(\lambda_n)\}_{n=1}^\infty$ is dense in $C[0,1]$ if and only if $\limsup(\log n)/\lambda_n = \infty$. Another variation concerns the denseness of the union of spaces of the form $H_n := \left\{\sum_{i=0}^n a_i x^{\lambda_{i,n}}\right\}$. The derivations of these results require an examination of the location of the zeros of the associated Chebyshev polynomials.

1. **Introduction.** The purpose of this paper is to offer two Müntz type theorems. We denote the polynomials of degree n in the variable x^{λ_n} by $\pi_n(\lambda_n)$, that is,

$$(1.1)\qquad \pi_n(\lambda_n) := \{p_n(x^{\lambda_n}) \mid p_n \text{ an algebraic polynomial of degree at most } n\}.$$

and we denote

$$(1.2)\qquad H_n(\lambda_{1,n},\dots,\lambda_{n,n}) := \left\{\sum_{i=1}^n a_i x^{\lambda_{i,n}}\right\}.$$

The principal results of this paper are:

THEOREM 1. *Suppose $\delta \geq 0$ and $\lambda_n \geq 1$, for all n. Then $\bigcup_{n=1}^\infty H_n$ is dense in $C[\delta,1]$ and only if*

$$\limsup_n \frac{\log n}{\lambda_n} = \frac{1}{2}\log\frac{1}{\delta}.$$

In the context of this paper denseness of $\bigcup_{n=1}^\infty H_n$ is with respect to the uniform norm on $[\delta,1]$. (Note that $\overline{\bigcup_{n=1}^\infty H_n}$ is not usually a subspace of $C[\delta,1]$).

THEOREM 2. *Suppose $\{\alpha_i\}$ and $\{\beta_i\}$ are two monotone sequences. Suppose, for all n,*

$$0 < \alpha_i \leq \lambda_{i,n} \leq \beta_i, \quad 2 \leq i \leq n,$$
$$0 = \lambda_{1,n} < \lambda_{2,n} < \cdots < \lambda_{n,n}$$

and suppose

$$H_n := H_n(\lambda_{1,n},\dots,\lambda_{n,n}).$$

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Then

a) $\bigcup_{n=1}^{\infty} H_n$ *is dense in* $C[0,1]$ *if* $\sum \frac{1}{\beta_i} = \infty$.

b) $\bigcup_{n=1}^{\infty} H_n$ *is not dense in* $C[\delta,1]$ *for any* $\delta \leq 1$ *if* $\sum \frac{1}{\alpha_1} < \infty$.

The proofs of these two theorems both rely on a knowledge of the locations of the zeros of the associated Chebyshev polynomials. This is discussed in the next section.

The basic approximation theoretic facts, and two different proofs of Müntz's Theorem are available in [2] and [3].

2. **Chebyshev polynomials and denseness.** A subspace

$$H_n := \operatorname{Span}\{g_1, g_2, \dots, g_n\} \quad g_i \in C[a,b] \tag{2.1}$$

satisfies the Haar condition, if every element of H_n that vanishes at n distinct points of $[a,b]$ vanishes identically. The Chebyshev polynomial T_n, with respect to H_n on $[a,b]$, is the linear form

$$T_n := c\left[g_n + \sum_{i=1}^{n-1} c_i g_i\right] \tag{2.2}$$

where the c_i are chosen to minimize

$$\left\| g_n(x) + \sum_{i=1}^{n-1} c_i g_i(x) \right\|_{[a,b]} \tag{2.3}$$

and c is chosen so that

$$\|T_n\|_{[a,b]} = 1 \text{ and } T_n(b) > 0 \tag{2.4}$$

Here $\| \quad \|$ denotes the supremum norm. We make the following assumption throughout, about the Haar spaces we are considering:

ASSUMPTION 1. Assume that $1 \in H_n$, that each g_i is differentiable on (a,b), and if f' has $n-1$ zeros on (a,b), for $f \in H_n$, then f is identically constant.

This assumption is easily verified for systems of type (1.1) or (1.2), both of which are Descartes systems.

A property T_n shares with the usual Chebyshev polynomial (of degree $n-1$) is that T_n has exactly $(n-1)$ zeros on $[a,b]$ and that T_n oscillates between ± 1 exactly n times on $[a,b]$. Furthermore, under the normalization we have chosen, $T_n(b) = 1$ and T_n is unique.

For a sequence of Haar spaces $\{H_n\}$ on $[a,b]$ and their associated Chebyshev polynomials $\{T_n\}$ let the mesh of T_n be denoted by

$$M_n[c,d] := \max_{0 \leq i \leq m} |x_i - x_{i-1}| \tag{2.5}$$

where $x_1 < x_2 < \cdots < x_{m-1}$ are the zeros of T_n in $[c,d] \subset [a,b]$ and where $x_0 := c$ and $x_m := d$. This is the maximum gap between the zeros of T_n on the subinterval $[c,d]$. Let

$$Z_{\{H_n\}} := \{x \in [a,b] \mid T_n(x) = 0 \text{ for some } n\}$$

denote the set of zero set of the associated Chebyshev polynomials.

We now have the following theorem that relates denseness and the location of zeros of Chebyshev polynomials.

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THEOREM 3. *If* $\{H_n\}$ *is a collection of Haar spaces on* $[0,1]$ *(satisfying Assumption 1) and* $[\delta,\rho] \subset [0,1]$ *then*

a) $\{H_n\}$ *is dense in* $C[\delta,\rho]$ *if* $\liminf M_n[\delta,\rho] = 0$

b) $\{H_n\}$ *is not dense in* $C[\delta,\rho]$ *if* $\bar{Z}_{\{H_n\}} \subset [\delta,p]$ *and* $[\delta,p] - \bar{Z}_{\{H_n\}}$ *contains a non-trivial interval.*

This theorem is proved for infinite Markov systems, M, in [1]. The modifications to this case are entirely straightforward. (The key to the proof in [1] is Lemma 1 parts a] and c] and this lemma holds with H_n as above.) It should be noted that the proof of this theorem is elementary.

3. **Proofs.** The proof of Theorem 1 follows in a fairly straightforward way from Theorem 3. This is because we can explicitly write down the Chebyshev polynomials.

PROOF OF THEOREM 1. The Chebyshev polynomial S_n with respect to $\pi_n(\lambda_n)$ on $[0,1]$ is just

$$S_n(x) := T_n(x^{\lambda_n}) = \cos\left(n\cos^{-1}(2x^{\lambda_n}-1)\right)$$

where T_n is the usual Chebyshev polynomial on $[0,1]$.

The smallest positive zero, $x_{1,n}$, satisfies

$$\left(\frac{\gamma}{n^2}\right)^{\frac{1}{\lambda_n}} \leq x_{1,n} \leq \left(\frac{\beta}{n^2}\right)^{\frac{1}{\lambda_n}}$$

where $0 < \gamma$ and β do not depend on n.

In particular

$$\liminf_n x_{1,n} = \alpha \text{ iff } \limsup_n \frac{\log n}{\lambda_n} = \frac{1}{2}\log\frac{1}{\alpha}.$$

It is now just a check that if

$$\liminf_n x_{1,n} = \alpha$$

then

$$\liminf M_n[\alpha,1] = 0$$

and the result follows from Theorem 3. ■

The proof of Theorem 2, rests on the following interesting proposition concerning the relative location of the zeros of Chebyshev polynomials in Müntz systems.

THEOREM 4. *Let*

$$M : \mathrm{Span}\{1, x^{\lambda_2}, \ldots, x^{\lambda_n}\} \quad 0 = \lambda_1 < \lambda_2, \ldots$$

and

$$N : \mathrm{Span}\{1, x^{\gamma_2}, \ldots, x^{\gamma_n}\} \quad 0 = \gamma_1 < \gamma_2, \ldots$$

Suppose that $\lambda_i < \gamma_i$ *for* $i = 2, \ldots, n$. *(We denote this situation by* $M < N$*). Suppose that* T_M *is the Chebyshev polynomial with respect to* M*, with zeros* $\alpha_1 < \alpha_2 < \cdots < \alpha_n$

and T_N is the Chebyshev polynomial with respect to N with zeros $\beta_1 < \beta_2 < \cdots < \beta_n$, all in $[0, 1]$. *Then $\alpha_i \leq \beta_i$ for $i = 1 \ldots n$. (We say in this case that the zeros of T_M lie to the left of the zeros of T_N).*

PROOF. The proof rests on the following *improvement theorem* due to Smith [4] which in our context says the following: if $q > \gamma_n$ and

$$x^q - p_n(x) = x^q - q_n(x) = 0$$

at n distinct points in $(0, 1)$, where

$$q_n \in M \text{ and } p_n \in N$$

then

$$|q_n(x)| \geq |p_n(x)| \quad \forall x \in (0, 1)$$

(with strict inequality away from the roots).

We now proceed to increase the coefficients of T_M, one at a time, starting with the largest, then the second largest, etc. At each stage we show that the new associated Chebyshev polynomial has zeros to the right of the old one.

Suppose that only one exponent of N is different from those of M (i.e. for exactly one $i, \lambda_i < \gamma_i$). If $i = n$ then T_N has a larger lead term than T_M and hence,

$$\lim_{x \to \infty} T_N(x) - T_M(x) = \infty. \tag{3.2}$$

If $i \neq n$ then we deduce from the *improvement theorem,* that the best approximation to

$$x^{\lambda_n} \text{ from span } \{1, x^{\lambda_2}, \ldots, x^{\lambda_{n-1}}\}$$

is not as good as the best approximation to

$$x^{\lambda_n} \text{ from span } \{1, x^{\gamma_2}, \ldots, x^{\gamma_{n-1}}\}.$$

So, in particular, the coefficient of x^{λ_n} in T_M must be less than the coefficient of x^{λ_n} in T_N and once again (3.2) holds. (Note that in each case, up to normalization, x^{λ_n} minus the best approximation is the Chebyshev polynomial. Note also that T_M and T_N are both monotone on $[1, \infty)$).

Suppose now that

$$T_N(x) \text{ has zeros } \alpha_1, \ldots, \alpha_n$$

and that

$$T_M(x) \text{ has zeros } \beta_1, \ldots, \beta_n.$$

Suppose also that

$$\alpha_i < \beta_i \quad i = k + 1, \ldots, n$$

but

$$\alpha_k \geq \beta_k.$$

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Consider $S := T_N - T_M$. Then S has at most n zeros (counting multiplicity) on $(0, \infty)$ since it is in a Haar system of order $n+1$. Furthermore, between the n extrema of each of T_M and T_N there lies a zero of S. Also $S(1) = S(0) = 0$. In particular no interval between sucessive extrema can contain two zeros of S. However, under the above assumption, the interval (α_k, α_{k+1}) contains two zeros (note that T_M and T_N have the same sign on (α_k, α_{k+1})) and these do not lie between sucessive extrema of T_M. This contradiction finishes the proof. ■

PROOF OF THEOREM 2. Part a). It is possible, and straightforward, to construct a sequence of spaces

$$H'_n := \mathrm{Span}\{1, x^{\gamma_{2,n}}, \ldots, x^{\gamma_{n,n}}\}$$

where

1) $H'_n \subset H_m$ for some $m > n$
2) $\delta_{i-1} < \gamma_{i,n} \leq \delta_i \quad \forall n, i, 2 \leq i \leq n$.
3) $\sum \frac{1}{\delta_i}$ diverges.

One does this, inductively, by first picking $\delta_1 = \limsup_n x^{\lambda_{1,n}}$. Then δ_2 is chosen by taking the lim sup over those indices appearing in the first lim sup, etc. It now suffices to show that $\{H'_n\}$ is dense in $C[0, 1]$.

Consider

$$\bigcup_n := \mathrm{Span}\{1, x^{\delta_2}, \ldots, x^{\delta_n}\}$$

and let S_n be the associated Chebyshev polynomial. Then, by Proposition 1, the zeros of T_n, the Chebyshev polynomial with respect to H'_n, are to the left of the zeros of S_n. However H'_n contains a polynomial $V_{n-1} := c_0 + \sum_{i=3}^{n} c_i x^{\delta_i}$ which is a Chebyshev polynomial of degree $n-1$ with respect to $\{1, x^{\delta_3}, x^{\delta_4}, \ldots, x^{\delta_n}\}$ and which has its zeros to the right of the zeros of S_{n-1}. Furthermore the zeros of T_n and V_{n-1} interlace (otherwise on subtraction $T_n - V_{n-1}$ would have too many zeros).

Now, since $\sum \frac{1}{\alpha_n} = \infty$, by Müntz's Theorem $\{1, x^{\alpha_2}, \ldots\}$ is dense in $C[0, 1]$ and hence, by Theorem 3, the mesh of the $\{S_n\}$ tends to zero on $[0, 1]$. Since the zeros of S_n and S_{n-1} interlace it follows that the mesh of the $\{T_n\}$ also tends to zero. This, with Theorem 3, finishes the proof of a).

Part b). The zeros of the Chebyshev polynomials associated with the H_n are bounded below by the zeros of the Chebyshev polynomial associated with a non-dense Markov system. In this case, as in [1], the zeros all avoid an interval $[0, \delta]$ and the result follows from Theorem 3. ■

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