NONSINGULAR RINGS WITH ESSENTIAL SOCLES

Dedicated to the memory of Hanna Neumann

G. IVANOV

(Received 29 August 1972; revised 1 June 1973)

Communicated by M. F. Newman

This paper is a study of nonsingular rings with essential socles. These rings were first investigated by Goldie [5] who studied the Artinian case and showed that an indecomposable nonsingular generalized uniserial ring is isomorphic to a full blocked triangular matrix ring over a sfield. The structure of nonsingular rings in which every ideal generated by a primitive idempotent is uniform was determined for the Artinian case by Gordon [6] and Colby and Rutter [2], and for the semiprimary case by Zaks [12]. Nonsingular rings with essential socles and finite identities were characterized by Gordon [7] and the author [10]. All these results were obtained by representing the rings in question as matrix rings. In this paper a matrix representation of arbitrary nonsingular rings with essential socles is found (section 2). The above results are special cases of this representation. A general method for representing rings as matrices is developed in section 1.

The results of section 2 are used in section 3 to investigate the structure of nonsingular QF-3 rings with finite identities. Semi-primary QF-3 rings which are also partially PP rings were first studied by Harada [8], [9] who showed that each of these rings has a semi-simple Artinian ring as a left, and a right, injective hull and that a semi-primary hereditary QF-3 ring is generalized uniserial. Colby and Rutter [3] proved these results for the semi-perfect case and later [4] showed that an arbitrary nonsingular left and right QF-3 ring has a semi-simple Artinian ring as a left, and a right, essential extension. In section 3 a matrix representation of nonsingular QF-3 rings with finite identities is used to determine the structure of left and right QF-3 rings and of semi-perfect QF-3 rings with various abundances of projective ideals.

The following conventions, terminology and notation are used throughout the paper. All exceptions to these are specifically mentioned so there are, hopefully, no ambiguities. All rings have identities and all modules are unital. The adjective "left" is usually omitted from expressions such as "left module", "left ideal" unless doubt could arise or special emphasis is wanted. A *homomorphism* between R-modules is an *R-homomorphism*. Homomorphisms are written on the right, except those between right modules which are written on the left. Composition of homomorphisms is indicated by juxtaposition. The isomorphism symbol $\simeq$ is usually not subscripted but its meaning is always clear. The symbol $\oplus$ means *direct sum as modules.*

1. Matrix Representations of Rings

Let E be a right faithful two-sided ideal in a ring R and, for each $r \in R$, let ρ_r be that endomorphism of ${}_RE$ which maps $x \in E$ onto xr. Since E is right faithful $\rho_r = 0$ if, and only if, $r = 0$. Therefore the function $\Gamma : R \to \mathrm{End}({}_RE)$ which maps $r \in R$ onto ρ_r is a ring monomorphism. If $\rho_r \in E\Gamma$ and $\phi \in \mathrm{End}({}_RE)$, then $\rho_r\phi = \rho_{r\phi}$ and $\rho_{r\phi} \neq 0$ when $r \notin \ker \phi$. That is, $E\Gamma$ is a faithful right ideal of $\mathrm{End}({}_RE)$ and is, therefore, an essential left $E\Gamma$-submodule of $\mathrm{End}({}_RE)$. This proves the following theorem.

THEOREM 1.1. *Let E be a right faithful, two-sided ideal in a ring R. Then there is an embedding $\Gamma : R \to \mathrm{End}({}_RE)$ defined by: $r\Gamma$, $r \in R$, is the endomorphism which maps $x \in E$ onto xr. $E\Gamma$ is a faithful right ideal of $\mathrm{End}({}_RE)$ and an essential left $E\Gamma$-submodule of $\mathrm{End}({}_RE)$.*

Assume that E is a direct sum of finitely generated left ideals E_{ij}, $i \in I$, $j \in J(i)$, whose indexing satisfies the relation $E_{ij} \simeq E_{st}$ if, and only if, $i = s$ and let N be the disjoint union of the $J(i)$. Then $\mathrm{End}({}_RE)$ is isomorphic to the ring of all row-finite $N \times N$ matrices whose entries at the place (s, t), $s \in J(i)$, $j \in J(t)$, are elements of $\mathrm{Hom}(E_{is}, E_{jt})$. It is clear that, for fixed $i \in I$, all the rings $\mathrm{End}(E_{ij})$ are mutually isomorphic — isomorphic to H_{ii}, say. If $i, s \in I$, then it is also clear that the (H_{ii}, H_{ss})-bimodules $\mathrm{Hom}(E_{ij}, E_{st})$ are isomorphic — isomorphic to H_{is}, say. Moreover, these isomorphisms can be picked in such a way that they commute with the multiplication (composition of homomorphisms) on $\cup \mathrm{Hom}(E_{ij}, E_{st})$. By substituting the H_{ij} into the above matrix representation of $\mathrm{End}({}_RE)$, it can be seen that $\mathrm{End}({}_RE)$ is isomorphic to the ring H of all blocked row-finite $N \times N$ matrices whose $(i,j)^{th}$ block is a row-finite $J(i) \times J(j)$-matrix over H_{ij}. For simplicity of exposition, the rings $\mathrm{End}({}_RE)$ and H are usually identified and so are the (H_{ii}, H_{ss})-bimodules H_{is} and $\mathrm{Hom}(E_{ij}, E_{st})$. If E_{kj} is not finitely generated, then $\mathrm{End}({}_RE)$ is isomorphic to H with its $(k, t)^{th}$ blocks replaced by blocks which may have infinite rows.

DEFINITION. Let S be a set, $\{T(s) \mid s \in S\}$ a class of sets and $\{A_{st} \mid (s, t) \in S \times S\}$ a class of additive abelian groups which have the following properties.

(i) Each A_{ss} is a ring (with identity)

https://doi.org/10.1017/S1446788700017110 Published online by Cambridge University Press

(ii) Each A_{st} is a unital (A_{ss}, A_{tt})-bimodule.

(iii) There is a partial associative multiplication on $\bigcup A_{st}$ with the property that if $\phi \in A_{st}$ and $\psi \in A_{tu}$ then $\phi\psi$ is defined and is in A_{su}. This multilpication extends the ring multiplication on each A_{ss} and the bimodule multiplication of elements of A_{st} by elements of A_{ss} and A_{tt}. The *generalized matrix ring* GMR $(S, T(s), A_{st})$ is the ring of all blocked row-finite matrices whose $(s,t)^{th}$ block is an arbitrary row-finite $T(s) \times T(t)$-matrix over A_{st}.

A matrix is *finitary* if it has only a finite number of non-zero entries. The matrix in $\mathrm{GMR}(S, T(s), A_{st})$ whose only non-zero entry is the element ϕ of A_{st} at the place (p, q) of the $(s,t)^{th}$ block is denoted by $|\phi|_{sp}^{tq}$.

It is clear that the ring H, in the preceding discussion, is simply GMR $(I, J(i), H_{ij})$.

Assume that each summand E_{ij} of the ideal E is generated by an idempotent e_{ij} and that the e_{ij} are orthogonal. If χ is the endomorphism of E which is induced by right multiplication by $x \in e_{ij}Re_{pq}$, then $e_{ij}\chi = x$ and $e_{st}\chi = 0$, for $e_{st} \neq e_{ij}$. Therefore χ corresponds to the matrix $|\chi|_{ij}^{pq}$ of H. If $K \subseteq H$ is the image of R under the ring monomorphism induced by Γ, then clearly K contains all matrices of the form $|\alpha|_{kl}^{mn}$ and hence all finitary matrices of H. Moreover the identity of H is in K, since the identity of R induces the identity function on E.

Let G be a subring of H containing the finitary matrices and the identity and consider the left ideal $F = \sum_{i,j} Gf_{ij}$, where $f_{ij} = |1|_{ij}^{ij}$. If $x \in H$ then $f_{ij}x$ is the matrix which has the same entries as x at the places (j, l) in the $(i, k)^{th}$ blocks and whose other entries are zeros. As x is row-finite, this means that $f_{ij}x$ is a finitary matrix and is, therefore, in F. Hence, F is a right ideal of G. If x has a non-zero entry at the place (s, t) of the $(i,j)^{th}$ block, then $f_{is}x \neq 0$ and so $Fx \neq 0$. That is, F is right faithful. It follows from the above discussion that $\mathrm{End}({}_GE)$ is isomorphic to $\mathrm{GMR}(I, J(i), G_{ij})$ where $G_{ij} \simeq \mathrm{Hom}(Gf_{im}, Gf_{jn})$. But $G_{ij} \simeq f_{ij}Gf_{ij} \simeq H_{ij}$, therefore the rings $\mathrm{End}({}_GF)$ and H are isomorphic. This proves the following theorem.

THEOREM 1.2. *Let R be a ring and $\{e_{ij} \mid i \in I, j \in J(i)\}$ a set of orthogonal idempotents of R with the property that $\sum_{i,j} Re_{ij}$ is a faithful right ideal. Then R is isomorphic to a subring K of the ring $H = \mathrm{GMR}(I, J(i), H_{ij})$, where the groups H_{ij} and the subring K have the following properties.*

(i) *The identity of H is in K.*

(ii) *The finitary matrices of H are in K.*

(iii) *For each i, the rings $\mathrm{End}(Re_{ik})$ and H_{ii} are isomorphic and, for all i,j, these isomorphisms make each $\mathrm{Hom}(Re_{is}, Re_{jt})$ into an (H_{ii}, H_{jj})-bimodule isomorphic (as a bimodule) to H_{ij}.*

Conversely, if K and H satisfy the above conditions down to, and including, condition (ii), *then K has a set $\{f_{ij} \mid i \in I, J(i)\}$ of orthogonal idempotents with* *the property that* $\Sigma_{i,j} Kf_{ij}$ *is a faithful right ideal of* K. *Moreover, the rings* H *and* $\mathrm{End}_K(\Sigma_{i,j} Kf_{ij})$ *are isomorphic.*

DEFINITION. An idempotent is *primitive* if it is not the sum of two orthogonal idempotents. An idempotent is *finite* if it is a sum of (a finite number of) orthogonal idempotents. A ring is *indecomposable* if it is not a direct sum of two two-sided ideals.

When the ring R has a finite identity then Theorem 1.2 can be strengthened to the following characterization, which is proved in [10].

COROLLARY 1.3. *Every indecomposable ring with finite identity is isomorphic to a* $\mathrm{GMR}(I, J(i), H_{ij})$ *which satisfies the following conditions.*

(i) *The sets* I *and* $J(i)$, $i \in I$, *are finite.*

(ii) *Each* H_{ii} *has a unique idempotent (the identity).*

(iii) *For any* $(s,t) \in I \times I$ *there is a sequence* $s = r(1), r(2), \cdots, t(n) = t$ *of elements of* I *with the property that for each* $r(i)$ *either* $H_{r(i)r(i+1)} \neq 0$ *or* $H_{r(i+1)r(i)} \neq 0$.

Conversely, every $\mathrm{GMR}(I, J(i), H_{ij})$ *which satisfies the above conditions is indecomposable and has a finite identity.*

2. Nonsingular Rings with Essential Socles

DEFINITION. The *singular submodule* $Z(M)$ of an R-module M is the submodule of all elements of M which are annihilated by essential ideals of R. M is *nonsingular* if $Z(M) = 0$ and it is *singular* if $Z(M) = M$. The *left singular ideal* $Z_l(R)$ of a ring R is the ideal $Z({}_RR)$ and the *right singular ideal* $Z_r(R)$ is $Z(R_R)$. R is *left* (respectively, *right*) *nonsingular* if $Z_l(R) = 0$ (respectively, $Z_r(R) = 0$). The left and right socles of a ring R are denoted by $S_l(R)$ and $S_r(R)$, respectively. A module is *uniform* if all of its submodules are essential. The additive group of row-finite $c \times d$-matrices over a ring A is denoted by $M(A, c \times d)$.

The following well-known results are essential for the discussion in the remainder of the paper. Their proofs can be found in [10].

LEMMA 2.1. *If* N *is an essential submodule of an* R*-module* M, *then for any* $m \in M$ *there is an essential ideal* E *of* R *such that* $Em \subseteq N$.

LEMMA 2.2. *Let* M, N *be* R*-modules such that* $Z(N) = 0$. *If the kernel of a homomorphism* $\phi : M \to N$ *is essential, then* $\phi = 0$. *In particular, if* M *is uniform then every non-zero* $\phi : M \to N$ *is a monomorphism.*

LEMMA 2.3. *If* M *is a minimal ideal in a nonsingular ring* R *then there is a primitive idempotent* $e \in R$ *such that* $Re \simeq M$.

From now on let R be a nonsingular ring with essential socle. Let E_0 be a complement in $S_l(R)$ of the sum of all nilpotent minimal ideals. It is clear that E_0 is a direct sum of minimal ideals each generated by a primitive idempotent. Let E_1 be an ideal maximal with respect to not intersecting E_0 and the property that $E_0 \oplus E_1$ is a two-sided ideal. Since $S_l(R)$ is a two-sided ideal it is contained in $E = E_0 \oplus E_1$, and, therefore, E is right faithful. Hence it follows from Theorem 1.1 that there is an embedding of R into $\mathrm{End}({}_R E)$ and this embedding is characterized by the following theorem. To ensure that R is embedded in the smallest ring which is useful (for this approach) the ideal E, rather than $S_l(R)$, is used. This is particularly significant when R has a finite identity, for then $E = R$ and the above embedding is surjective.

THEOREM 2.4. *If R is a nonsingular ring with essential socle then it is isomorphic to a subring K of the ring $H = GMR(I, J(i), H_{ij})$. The rings K and H have the following properties.*

(i) *The integer 1 is an element of I and $J(1) = \{1\}$.*

(ii) *If $i \neq 1$ then H_{ii} is a sfield.*

(iii) *If for $i \neq 1$ the left H_{ii}-dimension of H_{i1} is b_i, then $H_{11} \subseteq \Pi_{i \neq 1} M\,(H_{ii}, b_i \times b_i)$.*

(iv) *$\sum_{i \neq 1} H_{i1}$ is a faithful right H_{11}-module.*

(v) *If $i \neq j$ and $j \neq 1$ then $H_{ij} = 0$.*

(vi) *The identity of H is in K.*

(vii) *For each $i \in I \backslash \{1\}$ and each $j \in J(i)$ there is a matrix $f_{ij} \in K$ whose non-zero entries are all in the j^{th} column of the $(i,i)^{th}$ block and whose entry at the place (j,j) in this block is the identity of H_{ii}.*

(a) *Each $f_{ij} \left| d \right|_{ij}^{ik}$ is in Kf_{ik}.*

(b) *Each $f_{ij} \left| d \right|_{ij}^{11}$ is in K.*

(c) *For each $i \in I \backslash \{1\}$, each $j \in J(i)$ and every non-zero $x \in K$ whose non-zero entries are all in the j^{th} column of the $(i,i)^{th}$ block, there is a matrix $y \in K$ with the property that $yx = f_{ij}$.*

Conversely, if K is a subring of $H = GMR(I, J(i), H_{ij})$ and K and H satisfy the above conditions then K is a nonsingular ring with essential socle.

PROOF. As mentioned above the isomorphism between R and K is obtained by applying Theorem 1.1. To do this, express E_0 as a direct sum

$$\bigoplus_{\substack{i \in I' \\ j \in J(i)}} Re_{ij}$$

of minimal ideals generated by primitive idempotents which are indexed in such a way that $Re_{ij} \simeq Re_{st}$ if, and only if, $i = s$. Assume, moreover, that I' does not contain the integer 1 and let $I = I' \cup \{1\}$, $J(1) = \{1\}$ and $E_{11} = E_1$. It follows from the discussion following Theorem 1.1 that $\mathrm{End}({}_RE)$ is isomorphic to a ring H' which is a slightly modified $\mathrm{GMR}(I, J(i), H_{ij})$, where for fixed i,j, H_{ij} is an identification of all the groups $\mathrm{Hom}_R(E_{is}, E_{jt}) - E_{pq}$ stands for Re_{pq}, if $p \neq 1$. The modification is that the $(1,p)^{th}$ blocks in elements of H' need not be row-finite. As before, $\mathrm{End}({}_RE)$ and H' are identified so that, by Theorem 1.1, R is isomorphic to a subring K of H'. So it only remains to show that K satisfies conditions (vi) and (vii) and that it is contained in a subring H of H' which satisfies conditions (ii)—(v).

(ii) Each Re_{ij} is a minimal ideal so if $i \neq 1$ then H_{ii} is a sfield.

(iii) Clearly b_i is the cardinal of a set $T(i)$ of images of Re_{ij} in E_{11} which is maximal with respect to the sum of the elements of $T(i)$ being direct. Since every minimal ideal of R is isomorphic to an Re_{ij} (Lemma 2.3) the socle of E_{11} is the direct sum $\oplus_{i\neq 1} \hat{T}(i)$, where $\hat{T}(i)$ is the sum of all minimal ideals in $T(i)$. By Lemma 2.2 the restriction to $S(E_{11})$ of a non-zero endomorphism of E_{11} is non-zero. Therefore H_{11} can be regarded as a subring of $\Pi_{i\neq 1} M(H_{ii}, b_i \times b_i)$ which is isomorphic to $\mathrm{End}(S(E_{11}))$.

(iv) Each Re_{ij} is a direct summand of R, so $S(E_{11}) = R(\Sigma_{i\neq 1} H_{i1})$. As $S(E_{11})$ is not annihilated by non-zero endomorphisms of E_{11}, the product $(\Sigma_{i\neq 1} H_{i1})h$ is non-zero, for each non-zero $h \in H_{11}$.

(v) If $i \neq j$ and $i,j \neq 1$ then $Re_{is} \not\simeq Re_{jt}$ and, therefore, $H_{ij} = 0$. For non-zero $x \in Re_{pq}$ if $E_{11}x \neq 0$ then, since $Rx = Re_{pq}$ is projective, $E_{11} = N \oplus M$ where $Mx \neq 0$ and $M \simeq Rx$. By definition of E_0, M is nilpotent and so $MRx = 0$: a contradiction. Hence $E_{11}x = 0$ for all $x \in Re_{pq}$ and, therefore, K is contained in the subring H of H' of all matrices whose $(1,p)^{th}$ blocks, $p \neq 1$, are zero.

(vi) Since the identity endomorphism on H can be extended to the identity on R, the identity of E is in K.

(vii) The idempotent e_{ij} induces, by multiplication on the right, a homomorphism $E \to Re_{ij}$ which is the identity on Re_{ij}. Therefore e_{ij} is mapped, by the embedding of R into H, onto a matrix $f_{ij} \in K$ whose non-zero entries are all in the j^{th} column of the $(i,i)^{th}$ block and whose entry at the place (j,j) is the identity of H_{ii}.

(vii) (a) Every homomorphism $d : Re_{ij} \to Re_{ik}$ can be extended to the endomorphism $|d|_{ij}^{ik}$ of E. Therefore Re_{ij} is mapped by $|d|_{ij}^{ik}$ into Re_{ik}, that is, $x|d|_{ij}^{ik} \in Kf_{ik}$, for every $x \in Kf_{ij}$.

(vii) (b) A similar argument shows that for every $h \in H_{i1}$ the matrix $f_{ij}|h|_{ij}^{11}$ is in K.

(vii) (c) The ideal E is mapped by x into Re_{ij}, so $x(1-f_{ij})$ annihilates E. Hence Lemma 2.2 implies that $x(1-f_{ij}) = 0$ and so $Kx \subseteq Kf_{ij}$. As Kf_{ij} is a minimal ideal, there is a matrix $y \in K$ such that $yx = f_{ij}$.

The ring K has an identity (condition (vi)), so to prove the converse it is necessary to show only that the socle of K is essential and that K is nonsingular.

https://doi.org/10.1017/S1446788700017110 Published online by Cambridge University Press

The former is true if every non-zero principal ideal Kx contains a minimal ideal. By (vii) (c), Kf_{ij} is a minimal ideal, so if $f_{ij}Kx \neq 0$ then, Kx contains a minimal ideal isomorphic to Kf_{ij}. If x has a non-zero entry at the place (j,k) of the $(i,t)^{th}$ block, $i \neq 1$, then $f_{ij}x \neq 0$; and if $x = |h|_{11}^{11}$ then, by (iv), there is a $g \in H_{i1}$, for some $i \neq 1$, such that $gh \neq 0$ and so $f_{ij}|g|_{ij}x \neq 0$. That is, Kx always has a minimal ideal: therefore $S_l(K)$ is essential in K. To show that K is nonsingular it is sufficient to prove that $S_l(K)x \neq 0$, for every non-zero $x \in K$. If $x \neq 0$, then one of the ideals $Kf_{ij}x$, $Kf_{ij}|h|_{1j}^{11}x$ is non-zero and so, since Kf_{ij} and $Kf_{ij}|h|_{ij}^{11}$ $(h \neq 0)$ are minimal ideals, $S_l(K)x \neq 0$: therefore K is nonsingular.

Dorroh's extension A^* of a ring A with characteristic t is the unital ring on $Z_t \times A$, Z_t being the integers modulo t, whose addition is component-wise and whose multiplication is given by $(m,x)(n,y) = (mn, my + nx + xy)$, for all m, $n \in Z_t$, x, $y \in A$. It is easy to see that the isomorphic image $\{0\} \times A$ of A is an essential ideal of A^*. Therefore, A is nonsingular ring with essential socle if, and only if, A^* is. Consequently, the next result follows immediately from conditions (ii) and (iii) of Theorem 2.4.

COROLLARY 2.5. *The characteristic of a nonsingular ring with essential socle (but not necessarily with identity) is not divisable by p^2, for any prime p.*

The statement of the next result requires some more notation. Consider the ring $H = \mathrm{GMR}(I, J(i), H_{ij})$ of Theorem 2.4 and let $e \in H_{11}$ be an idempotent and $I_0 \subseteq I \backslash \{1\}$. Let $f(I_0, e)$ denote that matrix in H whose only non-zero entries are the identities of (the appropriate) H_{ii} on the diagonals of the blocks (i,j), $i \in I_0$, and the idempotent e in the block $(1,1)$.

COROLLARY 2.6. *The ring R of Theorem 2.4 is indecomposable if, and only if, for every proper subset $I_0 \subseteq I \backslash \{1\}$ and every idempotent $e \in H_{11}$ with the properties that $H_{i1}e = H_{i1}$, for $i \in I_0$, and $H_{i1}e = 0$, for $i \notin I_0$, the matrix $f(I_0, e)$ is not in K.*

PROOF. If R is decomposable then it has an idempotent f such that R is the sum of the non-zero two-sided ideals Rf and $R(1-f)$, that is, $R = fRf \oplus (1-f) \cdot R(1-f)$. Therefore each e_{ij} is in fRf or in $(1-f)R(1-f)$. If $M \simeq Re_{ij}$ and $e_{ij} \in fRf$ then, since $e_{ij}M \neq 0$, $fM \neq 0$ and so $M \subseteq fRf$. Therefore there is a proper subset $I_0 \subseteq I \backslash \{1\}$ with the property that a minimal ideal is in fRf if, and only if, it is isomorphic to an Re_{ij}, $i \in I_0$.

Conversely, if for a proper subset $I_0 \subseteq I \backslash \{1\}$ there is an idempotent $f \in R$ with the property that a minimal ideal is in Rf if, and only if, it is isomorphic to an Re_{ij}, $i \in I_0$, then R is the sum of the non-zero two-sided ideals Rf, $R(1-f)$ and so is decomposable. The ideals Rf, $R(1-f)$ are two-sided since $fR(1-f) = (1-f)Rf = 0$. For if $fR(1-f) \neq 0$ then there is a non-zeroh omomorphism $\phi : Rf \to R(1-f)$ which, by Lemma 2.2, does not kill a minimal ideal in Rf.

https://doi.org/10.1017/S1446788700017110 Published online by Cambridge University Press

Therefore $R(1-f)$ contains a minimal ideal isomorphic to a minimal ideal in Rf: a contradiction to the hypothesis on f. Therefore $fR(1-f) = 0$ and, similarly, $(1-f)Rf = 0$. Since I_0 is a proper subset of $I\ \{1\}$, both Rf and $R(1-f)$ contain an Re_{ij} and so Rf and $R(1-f)$ are non-zero.

It follows that R is indecomposable if, and only if, it does not have an idempotent f with the above properties. If R does contain such an idempotent f, then because f is a left and a right identity for each Re_{ij} $(i \in I_0)$ and all its homomorphic images in R, f is mapped, by the embedding of R into H, onto a matrix $f(I_0, e)$, where $e \in H_{11}$ is an idempotent with the properties that $H_{i1}e = H_{i1}$, for $i \in I_0$, and $H_{i1}e = 0$, for $i \notin I_0$. Conversely, if K contains an idempotent $f = f(I_0, e)$ then $K = fKf \oplus (1-f)K(1-f)$ and so K is decomposable.

Corollary 2.7. *The ideal* $E_0 = \Sigma_{i,j} Re_{ij}$ *of the ring* R *of Theorem* 2.4 *is generated by a set of orthogonal primitive idempotents if, and only if, condition* (vii) *of Theorem* 2.4 *can be replaced by the condition that all of the finitary matrices with zero in the block* (1, 1) *are in* K.

Proof. Necessity. Clearly it can be assumed that the idempotents e_{ij} are orthogonal. Therefore their images, the idempotents f_{ij}, in K are orthogonal and so $f_{ij} = |1|_{ij}^{ij}$. Hence conditions (vii) (a) and (vii) (b) merely state that the finitary matrices with zero in the block $(1, 1)$ are in K. If $x \in K$ has non-zero entries only on the j^{th} column of the $(i, i)^{th}$ block and the non-zero entry d at the place (k, j) then $|d^{-1}|_{ij}^{ik}x = |1|_{ij}^{ij}$. So condition (vii) (c) is satisfied.

Sufficiency. It was just shown that the above condition implies condition (vii), so it remains to show only that the complement, in $S_l(R)$ of the sum of all nilpotent minimal ideals of K is generated by a set of orthogonal primitive idempotents. It is clear that $\{|1|_{ij}^{ij}\}$ is such a generating set.

Theorem 2.8. *If R is a nonsingular ring with essential socle and an essential (left) ideal which is generated by a set of orthogonal primitive idempotents and which is also a right ideal then R is isomorphic to a subring K of $H =$* GMR $(I, J(i), H_{ij})$, *where K and H have the following properties.*

(i) *I is the disjoint union of a non-empty subset I_0 and a subset I_1.*

(ii) *If $i \in I_0$ then H_{ii} is a sfield.*

(iii) *If the H_{ii}-dimension of H_{ij} is b_{ij} then $H_{pq} \subseteq \Pi_{i \in I_0} M(H_{ii}, b_{ip} \times b_{iq})$.*

(iv) *If $j \in I_0$ and $i \neq j$ then $H_{ij} = 0$.*

(v) *If $\phi \in H_{jk}$ is non-zero then there is an $i \in I_0$ such that $H_{ij}\phi \neq 0$.*

(vi) *The identity of H is in K.*

(vii) *The finitary matrices of H are in K.*

Conversely, if K is a subring of $H =$ GMR$(I, J(i), H_{ij})$ *and K and H satisfy the above condition, then K is a nonsingular ring with essential socle and has a right ideal which is also an essential left ideal and is generated, as a left ideal, by a set of orthogonal primitive idempotents.*

PROOF. Let E be the left essential two-sided ideal of R generated by orthogonal idempotents e_{ij}, $i \in I$, $j \in J(i)$, which are indexed in the usual way. Lemma 2.3 implies that there is a subset I_0 of I with the properties that Re_{ij}, $i \in I_0$, is a minimal ideal and every minimal ideal is isomorphic to an Re_{ij}, $i \in I_0$. It follows from Theorems 1.2 and 2.4 and Corollary 2.7 that R can be represented as a matrix ring K which has all the above properties except (iii) and (v), so it remains to prove that K satisfies (iii) and (v).

(iii) The socle of Re_{jl} is a direct sum of ideals L_{ijl}, $i \in I_0$, where L_{ijl} is the sum of all minimal ideals in Re_{jl} which are isomorphic to an Re_{is}, $i \in I_0$. As L_{ijl} is a sum of minimal ideals it can be expressed as a direct sum of b_{ij} (say) minimal ideals. By Lemma 2.2, different homomorphisms from Re_{jl} to Re_{km} have different restrictions to $S(Re_{jl})$. Therefore $\mathrm{Hom}(Re_{jl}, Re_{km}) \simeq H_{jk}$ can be embedded in $\mathrm{Hom}(S(Re_{jl}), S(Re_{km})) \simeq \Pi_{i \in I_0} M(H_{ii}, b_{ij} \times b_{ik})$.

(viii) If $\phi : Re_{jl} \to Re_{km}$ is non-zero then, by Lemma 2.2, $L_{ijl}\phi \neq 0$ for some $i \in I_0$. Therefore there is a homomorphism $\psi : Re_{is} \to Re_{jl}$ with the property that $\psi\phi \neq 0$.

When R has a finite identity the embedding described by Theorem 2.8 is surjective. This leads to the following result which was proved independently by Gordon[7] and the author [10], although Gordon stated it only for the semi-perfect case. It is included here for the reader's convenience as it will be used throughout the rest of the paper.

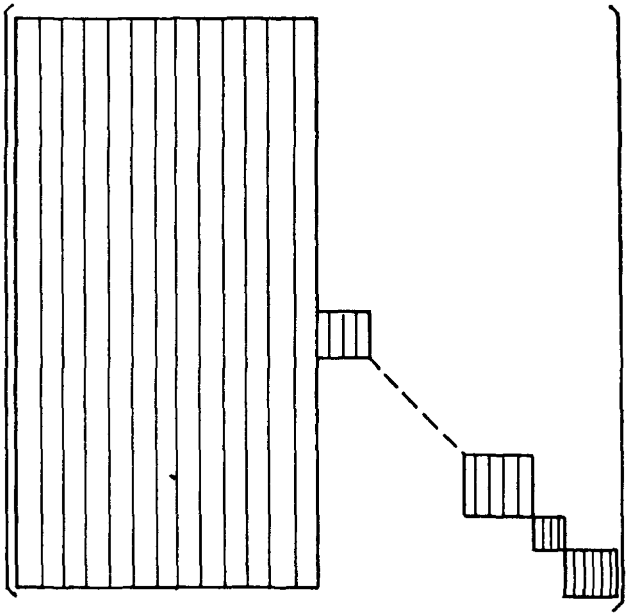
THEOREM 2.9. *If R is an indecomposable nonsingular ring with essential socle and a finite identity, then R is isomorphic to the ring $H = \mathrm{GMR}(I, J(i), H_{ij})$ of Theorem 2.8 which now satisfies the usual indecomposability condition and has the additional property that the sets I, $J(i)$, $(i \in I)$ are finite. The decomposability condition can be stated in terms of the minimal ideals of H as follows. If $s, t \in I$ then there is a sequence $s = r(1), \cdots, r(n) = t$ of elements of I with the property that for each $r(i)$ there is a $j \in I_0$ such that either $H_{jr(i)} \neq 0$ or $H_{jr(i+1)} \neq 0$.*

The converse is also true.

REMARK. The ring H of the preceding theorem can be represented diagrammatically as follows, on the next page, where blanks denote zeros and the squares on the diagonal denote sfields.

DEFINITION. A (*left*) *T-ring* is a ring whose non-zero (left) modules have non-zero socles.

Alin and Armenderiz [1] have investigated the structure of T-rings whose singular simple modules are injective. When such a ring has a finite identity Theorem 2.9 provides a quick determination of its structure.



LEMMA 2.10. [1, Theorem 1.1(a)]. *If the singular simple modules of a T-ring are injective then the ring is nonsingular.*

THEOREM 2.11. *If R is an indecomposable T-ring whose identity is finite and whose singular simple modules are injective then R is isomorphic to the ring H of Theorem 2.9 and H has the following additional properties.*

(i) *Each H_{ii} is a sfield.*

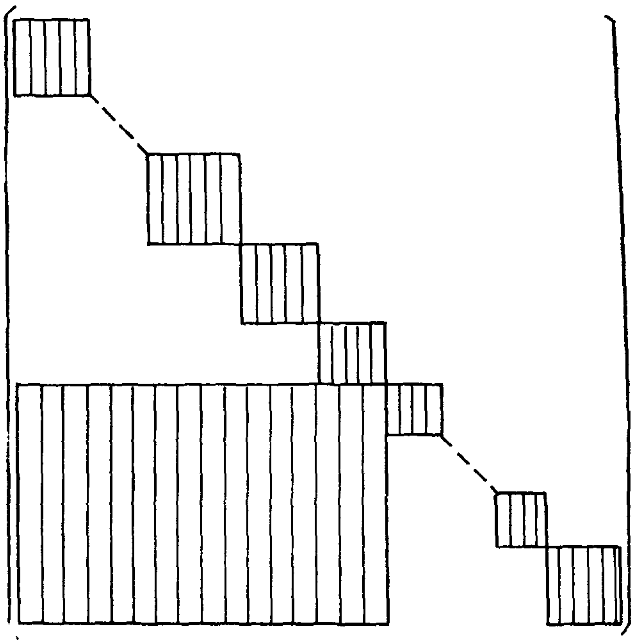
(ii) *If $i \in I_1$ and $i \neq j$ then $H_{ij} = 0$.*

The converse is also true.

REMARK. A diagrammatic representation of H is given on the next page, where the blanks stand for zeros and the squares on the diagonal for sfields.

PROOF. Let $\bar{R} = R/S_l(R)$ and note that, since $S_l(R)$ is a two-sided ideal of R, $\bar{R}$ is a ring whose ideals coincide with its R-submodules. Since $\bar{R}$ is a singular R-module, every minimal ideal of $\bar{R}$ is a direct summand and so is generated by an idempotent. Let $e \in R$ be a primitive idempotent which is not in $S_l(R)$. By hypothesis, $\bar{R}$ has essential socle, so it has a minimal ideal $M \subseteq \overline{Re} = Re/S(Re)$. If M is generated by an idempotent f then $\bar{e}f \neq 0$, since $f\bar{e} = f$ and $f^2 = f$. Therefore M is generated by the idempotent f then $\bar{e}f \neq 0$, since $f\bar{e} = f$ and $f^2 = f$. Therefore M is generated by the idempotent $f_1 = \bar{e}f \in \overline{eRe}$. By Lemma 2.3, $eS_l(R) = 0$ which implies that $eRe \cap S_l(R) = 0$ and, therefore, eRe and $\overline{eRe}$ are isomorphic rings. Since e is the only idempotent in eRe, $f_1 = \bar{e}$. Therefore $\overline{Re}$ is a minimal ideal. It follows that $S(Re)$ is the unique maximal submodule of Re and that eRe is a

https://doi.org/10.1017/S1446788700017110 Published online by Cambridge University Press



sfield. Consequently, if $e_1 \in R$ is also a primitive idempotent, then $eRe_1 \neq 0$ if, and only if, $Re \simeq Re_1$. This proves that H satisfies conditions (i) and (ii).

Now consider the converse. It is clear that every factor module of ${}_H H$ has non-zero socle, hence every cyclic H-module has non-zero socle and, therefore, H is a T-ring. Every singular simple H-module M is isomorphic to $H|1|_{ij}^{ij} / S(H|1|_{ij}^{ij})$, for some $i \in I_1$. Consequently, the only ideals which have M as homomorphic images are the $H|1|_{ij}^{ij}$, $j \in J(i)$, and their sums. Hence every homomorphism from an ideal of H to M can be extended to a homomorphism from H to M. Therefore M is injective.

The case when R has an infinite identity poses difficult problems. It is still true that if $e \in R$ is a primitive idempotent then eRe is a sfield and $S(Re)$ is the unique maximal submodule of Re, but these properties are no longer sufficient to guarantee the converse.

3. Nonsingular QF-3 Rings with Finite Identities

Definition. A ring R is a *(left) QF-3 ring* if it has a faithful left module which is (isomorphic to) a direct summand of every faithful left R-module.

In this section all rings have finite identities, although some results are valid without this assumption. The following characterization of arbitrary QF-3 rings, due to Colby and Rutter, is fundamental to the work in this section.

https://doi.org/10.1017/S1446788700017110 Published online by Cambridge University Press

THEOREM 3.1. [4, Theorem 1]. *If R is a QF-3 ring (not necessarily with finite identity), then it has a finite set of orthogonal primitive idempotents $\{e_1, \cdots, e_k\}$ with the properties that $\Sigma_1^k Re_i$ is injective, has essential socle, and is the (up to isomorphism) unique minimal faithful R-module. Conversely, if a ring R has a set $\{e_1, \cdots, e_k\}$ of orthogonal primitive idempotents such that $\Sigma_1^n Re_i$ is a faithful, injective ideal with essential socle, then R is a QF-3 ring.*

LEMMA. *If R is a nonsingular QF-3 ring (not necessarily with finite identity), then R has essential socle and every minimal ideal of R is isomorphic to the socle of one of the ideals Re_i of Theorem 3.1.*

PROOF. Let $Re = \Sigma_1^k Re_i$ be the minimal faithful ideal of R given by Theorem 3.1. If $x \in R$ is non-zero, then $xRe \neq 0$ and so there is a non-zero homomorphism $\phi : Rx \to Re$ (given by $rx\phi = rxa$, for some $a \in Re$ satisfying $xa \neq 0$). Since Re has essential socle, there is a minimal ideal $M \subseteq Rx\phi$. Let $K = M\phi^{-1}$. Then $\ker\phi|_K$ is a maximal submodule of K but is, by Lemma 2.2, not essential in K. Therefore $K = L \oplus \ker\phi|_K$ for some minimal ideal $L \simeq M$. Therefore, every ideal of R contains a minimal ideal and so the socle of R is essential. Since M is isomorphic to an $S(Re_i)$, every minimal ideal of R is isomorphic to the socle of an Re_i.

It is clear from Theorem 3.2 that a nonsingular QF-3 ring with finite identity is a direct product of a finite number of indecomposable nonsingular QF-3 rings, so it is sufficient to study only the indecomposable ones.

THEOREM 3.3. *If R is an indecomposable nonsingular QF-3 ring with a finite identity then it is isomorphic to the ring H described by Theorem 2.9 and the following additional properties.*

(i) *There are positive integers $k, n, n \geqq 2k-1$, such that $I = \{1, \cdots, n\}$, $I_0 = \{m, \cdots, n\}$, where $m = n-k+1$. Denote $\{1, \cdots, k\}$ by I_2.*

(ii) *If $i \in I_2$ then H_{ii} is a sfield.*

(iii) *If $i \in I_2$ and $j = n-i+1$ then $H_{ii} = H_{jj} = H_{ji}$.*

(iv) *If $i \in I_2$ and $j = n-i+1$ then $H_{si} = M(H_{jj}, b_{js} \times 1)$, for every $s \in I$.*

(v) *If $i \in I_2$ and $i \neq j$ then $H_{ij} = 0$.*

The converse is also true.

PROOF. Necessity. Let R be as in the theorem and let

$$R = Re_{11} \oplus \cdots \oplus Re_{i1} \oplus \cdots \oplus Re_{iv(i)} \oplus \cdots \oplus Re_{nv(n)} \tag{3.4}$$

be a decomposition of R such that the e_{ij} are orthogonal primitive idempotents with the property that $Re_{ij} \simeq Re_{st}$ if, and only if, $i = s$. For simplicity of notation, denote e_{i1} by e_i. In view of Theorem 3.1, it can be assumed that for a positive integer $k \leqq n$ each Re_i, $1 \leqq i \leqq k$, is an injective ideal and $\Sigma_1^k Re_i$ is the minimal faithful R-module. It follows from Lemmas 2.3 and 3.2 that for a positive integer

$m \leqq n$ each Re_i, $m \leqq i \leqq n$, is a minimal ideal and every minimal ideal is isomorphic to one of these. Since the Re_i, $1 \leqq i \leqq k$, are non-isomorphic injective ideals their socles are non-isomorphic minimal ideals. Therefore $n - m = k - 1$ and it can be assumed that the indexing of the Re_i is done in such a way that if $m \leqq i \leqq n$ and $1 \leqq j \leqq k$ then $Re_i \simeq S(Re_j)$ if, and only if, $i = n - j + 1$. It follows from Theorem 2.9 that R is isomorphic to a generalized matrix ring H which satisfies all the conditions stated in the theorem except (ii), (iii), (iv), (v) and the inequality $n \geqq 2k - 1$. So it is sufficient to show that H also satisfies these conditions.

(ii) If $i \in I_2$ then Re_i is an indecomposable injective ideal whose socle is, therefore, a minimal ideal. Since Re_i is non-singular it follows, from Lemma 2.2, that $\operatorname{End}(Re_i) \simeq \operatorname{End}(S(Re_i))$. Therefore H_{ii} is a sfield.

(iii) If $i \in I_2$ and $j = n - 1 + 1$, then $Re_j \simeq S(Re_i)$ and so H_{ji} is a one dimensional vector space over H_{jj}. Moreover it follows, from the fact that End (Re_i) $\simeq \operatorname{End}(S(Re_i))$, that $H_{ii} \simeq H_{jj}$. If ϕ, $\psi \in H_{ji}$ are non-zero then $Re_j\phi \simeq Re_j\psi$ and so there is a homomorphism $\alpha : Re_j\phi \to Re_j\psi$ satisfying $\phi\alpha = \psi$. Since Re_i is injective α can be extended to an element of H_{ii}. Therefore H_{ji} is a one dimensional right vector space over H_{ii}. Since H_{ji} is a one dimensional left vector space over H_{jj} and since $H_{ii} \simeq H_{jj}$, it follows that H_{ii}, H_{jj} and H_{ji} can be identified.

(iv) It follows from the indexing of the Re_i that if $i \in I_2$ then $b_{ji} \neq 0$ if, and only if, $j = n - i + 1$ and then $b_{ji} = 1$. As Re_i is injective, every homomorphism $S(Re_s) \to Re_i$ can be extended to a homomorphism $Re_s \to Re_i$. Therefore, by (iv), $H_{si} = M(H_{jj}, b_{js} \times 1)$, where $j = n - i + 1$.

(v) If $i \in I_2$ then, by Lemma 2.2, every non-zero homomorphism $\phi : Re_i \to Re_j$ is a monomorphism. As Re_i is injective, $Re_i\phi$ is a direct summand of Re_j and so $Re_i\phi = Re_j$. Therefore, $i = j$ and $H_{ij} = 0$ if $i \neq j$.

$n \geqq 2k - 1$. If $n < 2k - 1$ then $k > m$ and conditions (v) and (iv) of 2.8 are incompatible with the indecomposability condition of 2.9: a contradiction. Therefore $n \geqq 2k - 1$.

Sufficiency. Let H be a matrix ring with all the properties stated in the theorem. Then it follows from Theorem 2.9 that H is an indecomposable, nonsingular ring with essential socle and a finite identity. So it remains to show that H is a QF-3 ring. In view of Theorem 3.1, it is sufficient to prove that $\sum_1^k H|1|_{i1}^{i1}$ is a faithful, injective ideal.

For each $i \in I_0$, let e_i denote the element $|1|_{i1}^{i1}$ of H. If $\phi \in H_{st}$ is non-zero then, by (iii) of 2.8, $\phi = \sum_m^n \phi_i$, where $\phi_i \in M(H_{ii}, b_{is} \times b_{it})$, and so one ϕ_i, say ϕ_j, is non-zero. Therefore $\phi_j M(H_{jj}, b_{jt} \times 1) \neq 0$ and so if $i = m - j + 1$ then it follows from (iv) that $|\phi|_{sp}^{tq} He_i \neq 0$. It can, similarly, be shown that if x is a non-zero element of H, then $x(\sum_1^k He_i) \neq 0$. That is, $\sum_1^k He_i$ is faithful.

To show that He_i, $i \in I_2$, is injective, it is sufficient to prove that for any ideal L of H every homomorphism $L \to He_i$ can be extended to a homomorphism $H \to He_i$. Since $S(L)$ is essential in L and is a direct summand of $S_l(H)$, it follows from Lemma 2.2 that it is necessary to show only that every homomorphism $\phi : S_l(H) \to He_i$ can be extended to a homomorphism $\phi : H \to He_i$. Let $\phi_{st} : S(H) \to He_i$ be the map which agrees with ϕ on $S(He_{st})$ and kills the other $S(He_{pq})$. The image of ϕ_{st} is in $S(He_i) \simeq He_j$, where $j = n - i + 1$, so ϕ_{st} is determined by its action on L_j, the sum of the images of He_j in He_{st}. As L_j is a direct sum of b_{js} minimal ideals (Theorem 2.9), ϕ_{st} can be regarded as an $f \in M(H_{jj}, b_{js} \times 1)$. The matrix $|f|_{st}^{i1}$ is in H (condition (iv)) and induces, by multiplication on the right, a homomorphism $\phi'_{st} : H \to He_i$ which agrees with ϕ_{st}. Consequently, $\phi = \sum_{s,t}\phi_{st}$ can be extended to a homomorphism $\phi' = \sum_{s,t}\phi'_{st}$ from H to He_i and, therefore, He_i is injective. This proves the theorem.

Corollary 3.5. *A nonsingular QF-3 ring with finite identity is right nonsingular and has essential right socle.*

Proof. This is an immediate consequence of Theorem 3.3 and the right dual of Theorem 2.9.

Theorem 3.6. *Let R be an indecomposable, nonsingular ring which has a finite identity and is a left, and a right, QF-3 ring. Then $S_l(R)$, and $S_r(R)$, are direct sums of a finite number of minimal left, and right, ideals, respectively, and R is isomorphic to a matrix ring H which is described by Theorem 3.3 and the following additional property.*

If $i \in I_0$ then each H_{ij} has finite left dimension over H_{ii}.
(The converse is also true.)

Proof. Let H be the representation of R afforded by Theorem 3.3 and let e_i denote the matrix $|1|_{i1}^{i1}$ of H. It is clear that e_iH, $i \in I_2$, are minimal right ideals and that every minimal right ideal is isomorphic to such an e_iH. Since these e_iH are non-isomorphic, it follows from the right dual of Lemma 3.2 that every e_jH, $j \in I_0$, is an injective right ideal. For $i \in I_2$ let c_{si} be the right dimension of H_{si} over H_{ii}. Since right homomorphisms are written on the left, it follows from condition (iv) of 3.3 and its right dual, that if $i \in I_2$ and $j = n - i + 1$ then $H_{si} = M(H_{jj}, b_{js} \times 1)$ and $H_{js} = M(H_{ii}, c_{si} \times 1)$. If b_{js} is infinite then, from the first equation, $c_{si} > b_{js}$ and, from the second equation, $b_{js} < c_{si}$: a contradiction. Consequently, b_{js} is finite, $b_{js} = c_{si}$, and both $S_l(R)$ and $S_r(R)$ are finite dimensional. The converse to the theorem follows immediately from Theorem 3.3 and its right dual.

A completely analogous argument shows that even without the assumption that the identity of R is finite, the above theorem is still true. It is, of course, necessary to use a different representation of R. If H' is the representation of R afforded by the decomposition $R = Re_i \oplus \cdots \oplus Re_k \oplus Re_m \oplus \cdots \oplus Re_n \oplus R(1-e)$, where $e = \sum_1^k e_i + \sum_m^n e_j$, then it can easily be shown that $e_i H'$, $i \in I_2$, is a minimal right ideal and that H' satisfies (an analogue of) condition (iv) of 3.3. The rest of the proof is verbatim.

COROLLARY 3.7. *A nonsingular QF-3 ring whose socle is a direct sum of a finite number of minimal ideals is a right QF-3 ring.*

DEFINITION. A ring R with finite identity is a (*left*) *partially PP ring* if its identity has a decomposition $1 = \sum e_i$ into orthogonal primitive idempotents with the property that for every non-zero $x \in e_i R e_j$ the ideal Rx is projective. For simplicity, only such decompositions of the identity of a partially PP ring will be discussed. The ring R is a (*left*) *PP ring* if its principal ideals are projective. R is (*left*) *semi-hereditary* if its finitely generated ideals are projective. A module is *locally cyclic* if its finitely generated submodules are cyclic. A ring R is *semi-perfect* if its identity is a sum of a (finite) number of orthogonal primitive idempotents e_i such that each $e_i R e_i$ is a local ring. This definition is equivalent to the usual one [11].

LEMMA 3.8. *A partially PP ring is nonsingular.*

PROOF. Let $1 = \sum e_i$ be a decomposition, into orthogonal primitive idempotents, of the identity of a partially PP ring R and let $x_{ij} \in e_i R e_j$ be non-zero. If $l(x_{ij})$ is the left annihilator of x_{ij}, then $Rx_{ij} \simeq R/l(x_{ij})$ and, since Rx_{ij} is projective, $l(x_{ij})$ is a direct summand of R. That is, $l(x_{ij})$ is not essential. Every $x \in R$ is a sum of x_{ij}: therefore, R is nonsingular.

Let R be an indecomposable QF-3 and partially PP ring, and consider the decomposition (3.4) and the representation H, afforded by Theorem 3.3, of R. The image of a non-zero homomorphism $\phi : Re_i \to Re_j$ is projective, so its kernel is a direct summand of Re_i and, therefore, ϕ is a monomorphism. Therefore it follows from conditions (iv) of 3.3 and (v) of 2.8 that every Re_i is isomorphic to a submodule of an Re_j, $i \in I_2$. Hence, the socle of each Re_i is indecomposable and since, by the indecomposability condition of 2.9, $k = 1$ and $m = n$, every $b_{ns} = 1$. That is, all $S(Re_i)$ are isomorphic. It follows from condition (iv) of 3.3 that $H_{i1} = H_{ni} = H_{nn}$, for all i, and from condition (iii) of 2.8 that $H_{ij} \subseteq H_{nn}$, for all i, j.

An equivalence relation can be defined on the set $\{Re_i\}$ by relating two ideals if, and only if, they contain isomorphic copies of each other. The resulting set of equivalences classes $\{[Re_i]\}$ is partially ordered by the relation: $[Re_i] \leqq [Re_j]$ if, and only if, Re_i is isomorphic to a submodule of Re_j. This order has a unique minimal element, $[Re_n]$, and a unique maximal element, $[Re_1]$. If $[Re_i]$ and $[Re_j]$ have an equal number of classes smaller than themselves, then they are not com-

https://doi.org/10.1017/S1446788700017110 Published online by Cambridge University Press

parable: that is, $e_iRe_j = e_jRe_i = 0$. It follows that there is a sequence of integers $1 < h(1) < \cdots < h(t) < n$, with the properties that if $i \leqq h(s) < j$, then $[Re_i] \nleqq [Re_j]$; and if $h(s) < i, j \leqq h(s+1)$ then different $[Re_i]$ and $[Re_j]$ are not comparable. This proves the necessity of the following characterization of R. The sufficiency is clear.

THEOREM 3.9. *If R is an indecomposable QF-3 and partially PP-ring then it is isomorphic to the ring H of Theorem 3.3 and H has the following additional properties.*

(i) *I is partitioned into subsets $I(1) = \{1, 2, \cdots h(1)\}, \cdots, I(s) = \{h(s-1)+1, \cdots, h(s)\}, \cdots, I(t) = \{h(t-1)+1, \cdots, n\}$ which have the following properties.*

(a) *If $i \in I(s_1)$ and $j \in I(s_2)$, $s_1 < s_2$, then $H_{ij} = 0$.*

(b) *If $i, j \in I(s)$ and $H_{ij} \neq 0$ then $H_{ji} \neq 0$.*

(ii) *Each H_{ij} is contained in a sfield D and, for each $i \in I$, $H_{i1} = H_{ni} = D$. The converse is also true.*

COROLLARY 3.10. *A (left) QF-3 and (left) partially PP ring is a right QF-3 and right partially PP-ring.*

Now let the ring H, of Theorem 3.9, be a semi-perfect PP-ring. Since H is semi-perfect its projective modules are direct sums of isomorphic copies of the He_i, $i \in I$ [11]. It follows that if $a \in e_iH$ and $b \in e_jH$, $i \neq j$, are non-zero elements with the property that $Ha \cap Hb \neq 0$ then $H(a+b) = Ha + Hb$ is isomorphic to one of He_i, He_j. That is, $H(a+b) \in \{Ha, Hb\}$. Therefore either there is an element $c \in e_iHe_j$ such that $a = cb$ or there is an element $d \in e_jHe_i$ such that $b = da$.

This shows that if $\alpha \in H_{ik}$ and $\beta \in H_{jk}$ then either there is a $\gamma \in H_{ij}$ such that $\alpha = \gamma\beta$ or there is a $\delta \in H_{ji}$ such that $\beta = \delta\alpha$, but since $He_i \not\approx He_j$, not both. If $H_{ij} = 0$ then $Hb \subseteq Ha$ and, by restricting a, b to He_1, it can be seen that $H_{ji} = D$. As $He_i \not\approx He_j$ no element of H_{ij} has an inverse in H_{ji}, so if $H_{ji} = D$ then $H_{ij} = 0$. This proves the first part of Theorem 3.11.

The converse of 3.11 can be checked by noting that if $a = \sum_{s \in I} |\alpha_s|_{i1}$, $b = \sum_{t \in I} |\beta_t|_{i1}^{t1}$ and $Ha \cap H_b \neq 0$ then it follows, from condition (v) and the fact that all H_{st} are in D, that either there is a $\gamma \in H_{ij}$ such that each $\alpha_i = \gamma\beta_i$ or there is a $\delta \in H_{ji}$ such that each $\beta_i = \delta\alpha_i$. Hence one of the ideals Ha, Hb is contained in the other. It follows that every principal ideal of H is projective.

THEOREM 3.11. *If R is an indecomposable semi-perfect QF-3 and PP-ring then it is isomorphic to the ring H of Theorem 3.9 and H satisfies the following additional conditions.*

(i) *Each H_{ii} is a local ring.*

(ii) *If $i > j$ then $H_{ij} \neq 0$.*

(iii) *If $i, j \in I(s)$ then $H_{ij} \neq 0$. If moreover $h(s+1) > h(s)+1$ then $H_{ij} \neq D$.*

https://doi.org/10.1017/S1446788700017110 Published online by Cambridge University Press

(iv) *If* $i > h(s) \geqq j$ *then* $H_{ij} = D$.

(v) *If* $\alpha \in H_{ik}$, $\beta \in H_{jk}$ *then either* $\alpha \in H_{ij}\beta$ *or* $\beta \in H_{ij}\alpha$.

The converse is also true.

If the ring H, of 3.11, is semi-hereditary then every indecomposable finitely generated ideal is principal. This leads to the following result.

THEOREM 3.12. *A ring is an indecomposable semi-perfect semi-hereditary QF*-3 *ring if, and only if, it is isomorphic to the ring H of* 3.11 *and H has the following additional property.*

Every H_{ij} *is a locally cyclic* H_{ii}*-module.*

QUESTION. Are Theorems 3.11 and 3.12 (without condition (i) of 3.11) still valid if the semi-perfect condition is weakened to the condition that the identity of H is finite? This will be true if every indecomposable principal (respectively, finitely generated) ideal of H is isomorphic to one of the He_i, $i \in I$.

Acknowledgements

This paper is part of the author's Ph.D. thesis which was written under the supervision of the late Professor Hanna Neumann and Dr Carl Christensen and was submitted to the Australian National University in July, 1972. The author is indebted to Professor Neumann for her help and friendship and offers thanks Dr Christensen for his help.

The author offers thanks to the referee for pointing out a minor error and for his suggestions on improving the presentation of the paper.

References

[1] J. S. Alin and E. P. Armendariz, 'A class of rings having all singular simple module injective', *Math. Scand.* 23 (1968), 233–240.

[2] R. R. Colby and E. A. Rutter Jr., 'The structure of certain Artinian rings with zero singular ideal', *J. Algebra* 8 (1968), 156–164.

[3] R. R. Colby and E. A. Rutter Jr., 'Semi-perfect *QF*–3 and *PP*-rings', *Osaka J. Math.* 5 (1968), 99–102.

[4] R. R. Colby and E. A. Rutter Jr., '*QF*–3 rings with zero singular ideal', *Pac. J. Math.* 28 (1969), 303–308.

[5] A. W. Goldie, 'Torsion-free modules and rings', *J. Algebra* 1 (1964), 268–297.

[6] R. Gordon, 'Rings faithfully represented on their left socles', *J. Algebra* 7 (1967), 303–342.

[7] R. Gordon, 'Rings defined by *R*-sets and a characterisation of a class of semi-perfect rings', *Trans. Amer. Math. Soc.* 155 (1971), 1–17.

[8] M. Harada, '*QF*–3 and semi-primary *PP*-rings I', *Osaka J. Math.* 2 (1965), 357–368.

[9] M. Harada, '*QF*–3 and semi-primary *PP*-rings II', *Osaka J. Math.* 3 (1966), 21–27.

[10] G. Ivanov, 'Rings with zero singular ideal', *J. Algebra* 16 (1970), 340–346.
[11] B. J. Müeller, 'On semi-perfect rings', *Ill. J. Math.* 14 (1970), 464–467.
[12] A. Zaks, 'Semiprimary rings of generalized triangular type', *J. Algebra* 9 (1968), 54–78.

Department of Pure Mathematics
School of General Studies
Australian National University
Box 4, Post Office
Canberra, A.C.T. 2600.
Australia

Present address:

School of Mathematics
Institute for Advanced Study
Princeton, N.J. 08540
U.S.A.