RINGS OF QUOTIENTS OF RINGS OF DERIVATIONS

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The concept of a rational extension of a Lie module is defined as in the associative case [1, pp. 81 and 79]. It then follows from [3, Theorem 2.3] that any Lie module possesses a maximal rational extension (a rational completion), unique up to isomorphism. If now L and K are Lie rings with $L \subseteq K$, we call K a (Lie) ring of quotients of L if K, considered as a Lie module over L, is a rational extension of the Lie module L_L. Although we do not know if for every Lie ring L its rational completion can be given a Lie ring structure extending that of L (as is the case for associative rings), this is so, in any case, for abelian Lie rings (Propositions 2 and 4).

Let R be an associative ring, Q(R) its complete (maximal) ring of quotients. Tewari has shown [5, p.53] that every derivation of R has a unique extension to a derivation of Q(R). This implies that the Lie ring D(R) of all derivations of R can be faithfully embedded in the Lie ring D(Q(R)) of all derivations of Q(R). Since Q(R) is a ring of quotients of R, one may ask if D(Q(R)) is a (Lie) ring of quotients of D(R). Though this is the case for certain rings (e.g. $R = Z[x_1, \ldots, x_n]$), it is probably too much to expect in general. However, we show (Theorem 3) that under certain conditions (e.g. when R is an integral domain), the subring of D(Q(R)) of all "special" derivations is a (Lie) ring of quotients of D(R). (A derivation d of Q(R) is said to be <u>special</u> if $d = q_1 d_1 + \ldots + q_n d_n$, for some $q_i \in Q(R)$, $d_i \in D(Q(R))$, where the restrictions of the d_i to R are derivations of R.) Another result in this direction is Theorem 2, where we pick out, for each integer $n \geq 1$, subrings $D_n(R)$ and $D_n(Q(R))$ of D(R) and D(Q(R)) respectively (see p. 8), such that $D_n(Q(R))$ is a (Lie) ring of quotients of $D_n(R)$ (under conditions similar to those in Theorem 3).

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1. Let A_L, B_L be Lie L-modules[1] with $A_L \subseteq B_L$. B_L is said to be a rational extension of A_L if given any partial L-homomorphism $f: B_L \to B_L$ such that $A \subseteq \ker f$, then $\operatorname{im} f = 0$. We write $A_L \leq B_L$ (or $A \leq B$ if there is no ambiguity). (For the analogous definition in the associative case see [1, pp. 81 and 79).]

The category of Lie L-modules is isomorphic to the category of associative W(L)-modules[2], where W(L) is the universal enveloping ring of L [3, Theorem 2.3]. From this and the well known result in the associative case [1, pp. 83, 157, 158] we now get

PROPOSITION 1. Every Lie L-module M_L possesses a maximal rational extension N_L. Moreover, any rational extension of M_L is isomorphic to exactly one submodule of N_L. Thus a maximal rational extension of a Lie module is unique (up to isomorphism).

We shall call the maximal rational extension of M_L its rational completion.

Remark. The proof of the above proposition could have been obtained directly, without invoking the corresponding result in the associative case and the isomorphism of the categories.

If R is an associative ring, the rational completion of R_R can be given a ring structure faithfully extending that of R [1, p. 160]. We call this ring the complete or maximal ring of (right) quotients of R, and denote it by Q(R). For Lie rings we have only a partial result in this direction. First we prove

LEMMA 1. Let L be an abelian Lie ring ($ab = 0$ for all $a, b \in L$), M_L a trivial Lie L-module ($xa = 0$ for all $x \in M$, $a \in L$). Then N_L is a trivial Lie L-module for any rational extension N_L of M_L.

1. For the definitions of the basic concepts of "Lie module", Lie homomorphism", etc. see [2] or [3].

2. To distinguish between the Lie and associative cases, we call a module M_R over the associative ring R (in the usual sense) an associative module.

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Proof. Let a be a fixed element of L and define a mapping $f: N_L \to N_L$ by setting $f(y) = ya$ $(y \in N)$. Then $f(y_1+y_2) = f(y_1) + f(y_2)$ $(y_1, y_2 \in N)$, and $f(yb) = (yb)a = (ya)b - y(ab)$ (by the condition for a Lie module) $= (ya)b$ (since L is abelian) $= f(a)b$ $(y \in N,\ b \in L)$. Thus f is an L-homomorphism. Also $f(x) = xa = 0$ for all $x \in M$, hence $M \subseteq \ker f$. Since $M_L \leq N_L$, it follows that $\mathrm{im}\, f = 0$. That is, $ya = 0$ for all $y \in N$. Since $a \in L$ is arbitrary, the result follows.

PROPOSITION 2. If L is an abelian Lie ring, the rational completion N_L of L_L may be given a Lie ring structure faithfully extending that of L.

Proof. By the Lemma, the multiplication in N_L is trivial. We may thus extend the multiplication $N \times L \to \{0\}$ to $N \times N \to \{0\}$. Clearly N then becomes a Lie ring faithfully extending L.

We shall denote by $Q(L)$ the rational completion of L_L (even if L is not abelian).

Conjecture. Proposition 2 is not true for arbitrary Lie rings. That is, there exists a Lie ring L for which $Q(L)$ cannot be given a Lie ring structure faithfully extending that of L.

The concept of a ring of quotients, however, can always be defined. Thus if L and K are Lie rings with $L \subseteq K$, then K is said to be a (right) ring of quotients of L if K_L is a rational extension of L_L. Thus, if L is abelian, $Q(L)$ is a ring of quotients of L, called the maximal or complete ring of (right) quotients of L.

As a follow-up to the above conjecture, one may ask if the situation cannot be salvaged, in the following sense: does L always possess a "maximal" ring of quotients K, which may be smaller than the rational completion of L, but which is such that any ring of quotients of L is isomorphic to a unique subring of K?

Remark. Any right ring of quotients of L is also a left ring of quotients of L and conversely. This follows from the observation that for Lie modules M_L and N_L we have $M_L \leq N_L$ if and only if ${}_LM \leq {}_LN$ (the right module M_L may be considered as a left module ${}_LM$ by defining $ax = -xa$, $x \in M$, $a \in L$, and any L-homomorphism $f: M_L \to N_L$ is also an L-homomorphism: ${}_LM \to {}_LN$, and conversely, since $f(ax) = f(-xa) = -f(xa) = -f(x)a = af(x)$).

We shall now show that to determine the maximal ring of quotients $Q(L)$ of an abelian Lie ring L, it suffices to determine the maximal rational extension of the additive group of L. To make this precise, let A be an abelian group (written additively). If we define multiplication in A by: $xy = 0$ for all $x, y \in A$, then A becomes an abelian Lie ring. Denote this ring by $L_0(A)$. If B is another abelian group and $f:A \to B$ a homomorphism, then f can also be regarded as a Lie ring homomorphism: $L_0(A) \to L_0(B)$, since $f(xy) = f(0) = 0 = f(x)f(y)$ $(x, y \in L_0(A))$. Thus, we have a mapping F from the category $\mathcal{A}$ of abelian groups to the category $\mathcal{L}$ of abelian Lie rings $(F(A) = L_0(A)$, $F(f) = f)$, which is easily seen to be a functor. Conversely, if L is an abelian Lie ring, let $(L, +)$ denote its additive group. If K is another abelian Lie ring and $g:L \to K$ a Lie ring homomorphism, then g is also (by restriction) a group homomorphism: $(L, +) \to (K, +)$. The mapping G: $\mathcal{L} \to \mathcal{A}$ $(G(L) = (L, +), G(g) = g)$ is also a functor. Moreover, $FG(L) = F(L, +) = L_0(L, +) = L$ and $GF(A) = G(L_0(A)) = (L_0(A), +) = A$. Also clearly $FG(g) = g$, $GF(f) = f$. We thus have the following

PROPOSITION 3. <u>The category of abelian Lie rings is isomorphic to the category of abelian groups.</u>

An abelian group may be considered as an (associative) module over the ring Z of integers. Thus, if A and B are abelian groups, it is meaningful to speak of B being a rational extension of A. Similarly, if L and K are Lie rings, we say that K is a rational extension of L if $L_L \leq K_L$. We now show that in this sense rational extensions are preserved under the above isomorphism of the categories.

LEMMA 2. *If* $A, B \in \mathcal{A}$ *with* B *a rational extension of* A, *then* $F(B)$ *is a rational extension of* $F(A)$. *Conversely, if* $L, K \in \mathcal{L}$ *with* K *a rational extension of* L, *then* $G(K)$ *is a rational extension of* $G(L)$.

Proof. We note first that $A \subseteq B$ implies $F(A) \subseteq F(B)$, and $L \subseteq K$ implies $G(L) \subseteq G(K)$. Let now $A_Z \leq B_Z$. We wish to show that $L_0(A)_{L_0(A)} \leq L_0(B)_{L_0(A)}$. Thus, let M be an $L_0(A)$-submodule of $L_0(B)$ with $L_0(A) \subseteq M$, and let $f:M \to L_0(B)$ be an $L_0(A)$-homomorphism such that $L_0(A) \subseteq \ker f$. Then $(L_0(A), +) \subseteq (M, +) \subseteq (L_0(B), +)$. That is, $A \subseteq (M, +) \subseteq B$ $((L_0(A), +) = (F(A), +) = GF(A) = A)$. Also $f:(M, +) \to B$ is clearly a Z-homomorphism with $f(A) = 0$. Since $A_Z \leq B_Z$, hence $f(M, +) = f(M) = 0$, and $L_0(A) \leq L_0(B)$. The proof that $L \leq K$ implies $G(L) \leq G(K)$ is similar.

PROPOSITION 4. $Q(F(A)) = F(Q(A))$ *and* $Q(G(L)) = G(Q(L))$ $(A \in \mathcal{A}$, $L \in \mathcal{L}$). *That is,* $Q(L_0(A)) = L_0(Q(A))$ *and* $Q(L, +) = (Q(L), +)$.

Proof. Since Q(A) is a rational extension of A, F(Q(A)) is a rational extension of F(A) (Lemma 2). Also if K is any rational extension of F(A), then (also by Lemma 2) G(K) is a rational extension of GF(A) = A. Since Q(A) is the maximal rational extension of A, it follows that $G(K) \subseteq Q(A)$, hence $K = FG(K) \subseteq F(Q(A))$. Thus, F(Q(A)) is the maximal rational extension of F(A); i.e. F(Q(A)) = Q(F(A)). Similarly one shows that Q(G(L)) = G(Q(L)).

Remarks. (i) To be precise, one should use isomorphism in place of equality in the above; however, there is no loss in generality. Also the equality (isomorphism) Q(F(A)) = F(Q(A)), is by the above proof, that between F(A)-modules. But since Q(F(A)) and F(Q(A)) are abelian Lie rings (Proposition 2 for Q(F(A))), and hence are trivial F(A)-modules, the isomorphism can be extended to a ring isomorphism.

(ii) Since L = FG(L), hence Q(L) = Q(FG(L)) = F(QG(L)) = $F(Q(L, +)) = L_0(Q(L, +))$. That is, to obtain the maximal ring of quotients Q(L) of an abelian Lie ring L, it suffices to find the maximal rational extension of its additive group.

(iii) If R is an associative ring, we associate with it a Lie ring C(R) whose additive group is that of R, with multiplication defined by the additive commutator: $[a, b] = ab - ba$ $(a, b \in C(R))$. This turns C(R) into a Lie ring. If R is commutative, then clearly $C(R) \in \mathcal{L}$. If R = Z, then $Q(C(Z)) = L_0(Q(C(Z), +))$ (by (ii)) $= L_0(Q(Z, +))$ ((C(Z), +) $= (Z, +)) = L_0(Q(Z_Z))$ (Q(Z, +) means $Q(Z_Z)$, by definition) = C(Q(Z)) (where Q(Z) is the rational completion of Z as a ring, which is known to be the ring of rational numbers, while $Q(Z_Z)$, the rational completion of the module Z_Z, is the additive group of rationals; thus $L_0(Q(Z_Z)) = C(Q(Z))$). Hence we have Q(C(Z)) = C(Q(Z)). In general it is not true that Q(C(R)) = C(Q(R)) for an arbitrary commutative ring R.

(iv) It may be noted that, in fact, every abelian Lie ring L is a subring of a ring of the form C(R), where R is an associative and commutative ring. Just let R = W(L), the universal enveloping ring of L. Then L is (isomorphic to) a subring of C(W(L)), and L abelian implies W(L) commutative (see, for example, [3, p. 32]).

2. Let R be an associative ring. A mapping $d: R \to R$ is called a derivation of R if

(i) $$d(x+y) = d(x) + d(y)$$

(ii) $$d(xy) = d(x)y + xd(y), \text{ for all } x, y \in R .$$

It is easily verified that the set D(R) of all derivations of R forms a Lie ring, with the usual addition of mappings and the commutator multiplication : $[d_1, d_2] = d_1 d_2 - d_2 d_1$.

If R is an integral domain, then the rational completion $Q(R)$ of R is just the field of quotients of R [1, p. 164]. That is $Q(R) = \{\frac{x}{y} : x, y \in R, \ y \neq 0\}$, with the usual addition and multiplication. It is then easily shown [6, p. 120] that any derivation d of R can be extended uniquely to a derivation $\bar{d}$ of $Q(R)$, namely:

$$\bar{d}\left(\frac{x}{y}\right) = \frac{yd(x) - xd(y)}{y^2} .$$

The corresponding result for an arbitrary associative ring is due to Tewari.

THEOREM 1. Let R be an associative ring, $Q(R)$ its complete ring of quotients. Then any derivation of R can be extended uniquely to a derivation of $Q(R)$.

For the proof see [5, p. 53].

We shall now discuss some examples. First we note that if R is a ring with identity 1, and $d \in D(R)$ then $d(1) = 0$. For $d(1) = d(1.1) = d(1)\cdot 1 + 1\cdot d(1) = d(1) + d(1)$.

1. Let $R = Z$ and suppose $d \in D(Z)$. Since $d(1) = 0$ and d is additive, hence $d(z) = 0$ for all $z \in Z$. Thus $d = 0$ (the zero derivation), hence $D(Z) = 0$. Here $Q(Z)$ is the field of rational numbers. We also have $D(Q(Z)) = 0$. For, if $q \in Q(Z)$, $q = \frac{x}{y}$ $(x, y \in Z, \ y \neq 0)$, then $x = qy$, hence $d(x) = d(qy) = d(q)y + qd(y)$ for any $d \in D(Q(Z))$. Since $d(1) = 0$, it follows that $d(x) = 0 = d(y)$, and as $y \neq 0$, $d(q) = 0$.

2. The same situation as above obtains for $R = Z_n$ (the ring of integers modulo n). That is, $D(Z_n) = 0 = D(Q(Z_n))$.

3. Let $R = Z[x_1, \ldots, x_n]$, the commutative ring of polynomials in the indeterminates $x_1, \ldots, x_n$ over Z. For each $i = 1, 2, \ldots, n$ define a mapping $d_i : Z[x_1, \ldots, x_n] \to Z[x_1, \ldots, x_n]$ as follows: if

$$f(x_1, \ldots, x_n) = \Sigma z_{k_1, \ldots, k_n} x_1^{k_1} \ldots x_i^{k_i} \ldots x_n^{k_n} \in Z[x_1, \ldots, x_n] \text{ then}$$

$$d_i f(x_1, \ldots, x_n) = \Sigma z_{k_1, \ldots, k_n} k_i x_1^{k_1} \ldots x_i^{k_i - 1} \ldots x_n^{k_n} .$$

(That is, the d_i are the "partial derivatives" with respect to the x_i.) A straightforward computation shows that the d_i are derivations of $Z[x_1, \ldots, x_n]$. In fact, it is not difficult to show [6, p. 122] that every derivation of

$Z[x_1, \ldots, x_n]$ is of the form $f_1 d_1 + \ldots + f_n d_n$, where $f_i \in Z[x_1, \ldots, x_n]$.

The rational completion of $Z[x_1, \ldots, x_n]$ is the field of rational functions $Q(Z)(x_1, \ldots, x_n)$ in the x_i over $Q(Z)$ (the field of rational numbers). The derivations $d_1, \ldots, d_n$ of $Z[x_1, \ldots, x_n]$ extend uniquely to derivations $\bar{d}_1, \ldots, \bar{d}_n$ of $Q(Z)(x_1, \ldots, x_n)$. Also in this case every derivation of $Q(Z)(x_1, \ldots, x_n)$ is of the form $g_1 \bar{d}_1 + \ldots + g_n \bar{d}_n$, where $g_i \in Q(Z)(x_1, \ldots, x_n)$.

4. If $\{x_\alpha\}_{\alpha \in A}$ is a collection of indeterminates indexed by a set A, and we let $R = Z[\{x_\alpha\}]$, the ring of polynomials in the (infinite) set of indeterminates x_α, we can also here define, for each $\beta \in A$, the "partial derivatives" d_β with respect to x_β by $d_\beta(x_\alpha) = \begin{cases} 1, & \text{if } \alpha = \beta \\ 0, & \text{if } \alpha \neq \beta \end{cases}$ (extending to all of $Z[\{x_\alpha\}]$ in the obvious way - as in the above example). The d_β are derivations of $Z[\{x_\alpha\}]$ which, however, do not "span" $D(Z[\{x_\alpha\}])$ (contrary to the case in the previous example). Thus, the derivation $d: Z[\{x_\alpha\}] \to Z[\{x_\alpha\}]$ given by $d(x_\beta) = 1$ for all $\beta \in A$ (this mapping is a derivation when extended to all of $Z[\{x_\alpha\}]$ in the obvious way) clearly cannot be written in the form $h_1 d_{\alpha_1} + \ldots + h_m d_{\alpha_m}$, where $h_i \in Z[\{x_\alpha\}]$ and d_{α_i} are partial derivatives (as defined above). The same situation carries over to the field of quotients (rational completion) $Q(Z)(\{x_\alpha\})$ of $Z[\{x_\alpha\}]$.

Theorem 1 implies that the Lie ring $D(R)$ of derivations of R is (isomorphic to) a subring of the Lie ring $D(Q(R))$ of derivations of $Q(R)$. Since $Q(R)$ is the maximal ring of quotients of R, one may ask if $D(Q(R))$ is the maximal (Lie) ring of quotients (or, at least, a ring of quotients) of $D(R)$. That is, is $D(Q(R)) = Q(D(R))$, or, at least, $D(Q(R)) \subseteq Q(D(R))$? The former is clearly the case in examples 1 and 2 above, and it can be shown that at least the latter is true of example 3. We now proceed to discuss two results (Theorems 2 and 3), both of which generalize this special result in the case of example 3.

Thus, let R be an associative ring, and let $d \in D(R)$. Given any $r \in R$, one can define a mapping $rd: R \to R$ by setting $(rd)(x) = rd(x)$ $(x \in R)$.

LEMMA 3. <u>If</u> R <u>is commutative then</u> $rd \in D(R)$ <u>for any</u> $r \in R$, $d \in D(R)$.

Proof. For any $x, y \in R$ clearly $(rd)(x+y) = (rd)(x) + (rd)(y)$. Also $(rd)(xy) = r(d(x)y + xd(y)) = rd(x)y + xrd(y) = (rd)(x)y + x(rd)(y)$.

A non-zero element of R is said to be *regular* if it is not a zero divisor.

Remark. If $d \in D(R)$ is such that $d(R)$ contains a regular element, then R is commutative if and only if $rd \in D(R)$ for all $r \in R$. This is easily verified.

We assume from now on that R is a commutative ring with identity 1.

With example 3 above in mind, let $d_1, \ldots, d_n$ be n derivations of R such that $[d_i, d_j] = 0$ $(i, j = 1, 2, \ldots, n)$. Let $D_n(R) = \{r_1d_1 + \ldots + r_nd_n : r_i \in R\}$. It follows from lemma 3 (and since $D(R)$ is closed under addition) that $D_n(R)$ is a subset of $D(R)$. It is, in fact, a subring of $D(R)$, as can easily be verified (this is where one uses $[d_i, d_j] = 0$). Let now $\bar{d}_1, \ldots, \bar{d}_n$ be the unique extensions of $d_1, \ldots, d_n$ respectively, to $Q(R)$ (Theorem 1), and set $D_n(Q(R)) = \{q_1\bar{d}_1 + \ldots + q_n\bar{d}_n : q_i \in Q(R)\}$. Since R is commutative so is $Q(R)$ (1, p. 163), hence $q_i\bar{d}_i \in D(Q(R))$. Since $[\bar{d}_i, \bar{d}_j](x) = [d_i, d_j](x)$ for all $x \in R$, it follows from Theorem 1 that also $[\bar{d}_i, \bar{d}_j] = 0$, and hence one can show that $D_n(Q(R))$ is a subring of $D(Q(R))$. Clearly, then, $D_n(R)$ is a subring of $D_n(Q(R))$ (of course, with Theorem 1 in mind, and the consequent identification of $r_1d_1 + \ldots + r_nd_n \in D_n(R)$ with $r_1\bar{d}_1 + \ldots + r_n\bar{d}_n \in D_n(Q(R))$, which we shall always make without explicit mention). We now show that, with certain assumptions, $D_n(Q(R))$ is a (Lie) ring of quotients of $D_n(R)$.

First, we recall that a submodule N_R of an R-module M_R is said to be dense if M_R is a rational extension of N_R. (An ideal I of R is dense if R_R is a rational extension of I_R.) If now E_R is a dense submodule of $Q(R)_R$ then, for any $q \in Q(R)$, $qE = 0$ implies $q = 0$. For, the mapping $f: Q(R) \to Q(R)$ given by $f(q') = qq'$ is an R-homomorphism such that $f(E) = 0$, hence $f(Q(R)) = 0$. That is, $qq' = 0$ for all $q' \in Q(R)$, hence $q = 0$ (since R has an identity).

THEOREM 2. Let R be an associative and commutative ring with identity 1. Let $d_1, \ldots, d_n$ be derivations of R such that $[d_i, d_j] = 0$ $(i, j = 1, \ldots, n)$, and let $\bar{d}_1, \ldots, \bar{d}_n$ be their unique extensions to derivations of $Q(R)$. Let $D_n(R) = \{r_1 d_1 + \ldots + r_n d_n : r_i \in R\}$, $D_n(Q(R)) = \{q_1 \bar{d}_1 + \ldots + q_n \bar{d}_n : q_i \in Q(R)\}$. Suppose that the following conditions hold:

(i) $2(=1+1)$ is a regular element of R.

(ii) $\bigcup_{i=1}^{n} \bar{d}_i(Q(R))$ contains a regular element.

(Alternatively: $\bigcup_{i=1}^{n} \bar{d}_i(Q(R))$ contains a dense R-submodule of $Q(R)$.)

Then $D_n(Q(R))$ is a (Lie) ring of quotients of $D_n(R)$.

Proof. We shall prove the theorem for $n = 2$, the proof in the general case being similar. Thus, suppose that B is a $D_2(R)$-submodule of $D_2(Q(R))$ containing $D_2(R)$, and let $f: B \to D_2(Q(R))$ be a $D_2(R)$-homomorphism such that $f(D_2(R)) = 0$. We wish to show that $f(B) = 0$. Thus, let $b \in B$, $f(b) = b'$. Then $b = q_1\bar{d}_1 + q_2\bar{d}_2$, $b' = q_3\bar{d}_1 + q_4\bar{d}_2$, for some $q_i \in Q(R)$, and we want $q_3\bar{d}_1 + q_4\bar{d}_2 = 0$. For any $q \in Q(R)$, let $q^{-1}R = \{r \in R : qr \in R\}$. Then $q^{-1}R$ is a dense ideal of R [4, p.40]. Set $I = q_1^{-1}R \cap q_2^{-1}R \cap \bar{d}_1(q_1)^{-1}R \cap \bar{d}_1(q_2)^{-1}R \cap \bar{d}_2(q_1)^{-1}R \cap \bar{d}_2(q_2)^{-1}R$. Since the intersection of a finite number of dense ideals of R is dense [4, p.37], hence I is a dense ideal of R.

By a straightforward calculation we arrive at the following result for multiplication in $D_2(R)$, which we set down for reference: if $r_1d_1+r_2d_2$, $s_1d_1+s_2d_2 \in D_2(R)$ then $[r_1d_1+r_2d_2, s_1d_1+s_2d_2] = (r_1d_1(s_1) - s_1d_1(r_1) + r_2d_2(s_1) - s_2d_2(r_1))d_1 + (r_1d_1(s_2) - s_1d_1(r_2) + r_2d_2(s_2) - s_2d_2(r_2))d_2$. A similar formula holds for multiplication in $D_2(Q(R))$.

Let now x be an arbitrary element of I. Then

$$[q_1\bar{d}_1+q_2\bar{d}_2, x^2\bar{d}_1+0\bar{d}_2] = (q_1\bar{d}_1(x^2) - x^2\bar{d}_1(q_1) + q_2\bar{d}_2(x^2))\bar{d}_1 + (-x^2\bar{d}_1(q_2))\bar{d}_2$$

$$= (q_1 2x\bar{d}_1(x) - x^2\bar{d}_1(q_1) + q_2 2x\bar{d}_2(x))\bar{d}_1 + (-x^2\bar{d}_1(q_2))\bar{d}_2$$

$$= t_1\bar{d}_1 + t_2\bar{d}_2 \text{ say.}$$

By definition of I and since $x \in I$, it follows that $t_1, t_2 \in R$, hence $t_1\bar{d}_1 + t_2\bar{d}_2 \in D_2(R)$. We thus have $[q_1\bar{d}_1 + q_2\bar{d}_2, x^2\bar{d}_1 + 0\bar{d}_2] \in D_2(R)$, hence $f[q_1\bar{d}_1 + q_2\bar{d}_2, x^2\bar{d}_1 + 0\bar{d}_2] = 0$. But $f[q_1\bar{d}_1 + q_2\bar{d}_2, x^2\bar{d}_1 + 0\bar{d}_2] = [f(q_1\bar{d}_1 + q_2\bar{d}_2), x^2\bar{d}_1 + 0\bar{d}_2]$. Hence

$$\begin{aligned} 0 &= [q_3\bar{d}_1 + q_4\bar{d}_2, x^2\bar{d}_1 + 0\bar{d}_2] \\ &= (q_3\bar{d}_1(x^2) - x^2\bar{d}_1(q_3) + q_4\bar{d}_2(x^2))\bar{d}_1 + (-x^2\bar{d}_1(q_4))\bar{d}_2 \\ &= (2xq_3\bar{d}_1(x) - x^2\bar{d}_1(q_3) + 2xq_4\bar{d}_2(x))\bar{d}_1 + (-x^2\bar{d}_1(q_4))\bar{d}_2 . \end{aligned}$$

This gives the relation

(1) $2x(q_3\bar{d}_1(x) + q_4\bar{d}_2(x))\bar{d}_1 = x^2(\bar{d}_1(q_3)\bar{d}_1 + \bar{d}_1(q_4)\bar{d}_2)$, for all $x \in I$.

If we now consider $0\bar{d}_1 + x^2\bar{d}_2 \in D_2(R)$ in place of $x^2\bar{d}_1 + 0\bar{d}_2$, then also $[q_1\bar{d}_1 + q_2\bar{d}_2, 0\bar{d}_1 + x^2\bar{d}_2] \in D_2(R)$, hence $[q_3\bar{d}_1 + q_4\bar{d}_2, 0\bar{d}_1 + x^2\bar{d}_2] = 0$. This, in turn, yields a relation analogous to (1), namely

(1') $2x(q_3\bar{d}_1(x) + q_4\bar{d}_2(x))\bar{d}_2 = x^2(\bar{d}_2(q_3)\bar{d}_1 + \bar{d}_2(q_4)\bar{d}_2)$, for all $x \in I$.

Let now q' be an arbitrary but fixed element of $Q(R)$, and set $J = I \cap q'^{-1}R \cap \bar{d}_1(q')^{-1}R \cap \bar{d}_2(q')^{-1}R \cap (q_1q')^{-1}R \cap (q_2q')^{-1}R$. Then J is also a dense ideal of R. Now, for any $y \in J$, $yq' \in R$, hence $y^2q'\bar{d}_1 + 0\bar{d}_2 \in D_2(R)$. A similar calculation to the above shows that $[q_1\bar{d}_1 + q_2\bar{d}_2, y^2q'\bar{d}_1 + 0\bar{d}_2] \in D_2(R)$, hence

$$\begin{aligned} 0 = f[q_1\bar{d}_1 + q_2\bar{d}_2, y^2q'\bar{d}_1 + 0\bar{d}_2] &= [q_3\bar{d}_1 + q_4\bar{d}_2, y^2q'\bar{d}_1 + 0\bar{d}_2] \\ &= q'2y(q_3\bar{d}_1(y) + q_4\bar{d}_2(y))\bar{d}_1 - q'y^2(\bar{d}_1(q_3)\bar{d}_1 + \bar{d}_1(q_4)\bar{d}_2) \\ &\quad + y^2(q_3\bar{d}_1(q') + q_4\bar{d}_2(q'))\bar{d}_1 . \end{aligned}$$

By relation (1), the first two terms disappear (since $J \subseteq I$, (1) holds with $x \in I$ replaced by $y \in J$). We therefore get

$y^2(q_3\bar{d}_1(q') + q_4\bar{d}_2(q'))\bar{d}_1(q) = 0$, for all $q \in Q(R)$. If we let

$t = (q_3\bar{d}_1(q') + q_4\bar{d}_2(q'))\bar{d}_1(q)$ (keeping q fixed for the moment), then

$y^2 t = 0$ for all $y \in J$. If z is any other element of J, then also $z^2 t = 0$ and $(y+z)^2 t = 0$. That is, $y^2 t + 2yzt + z^2 t = 0$, hence $2yzt = 0$. By condition (i) of the theorem we now get $yzt = 0$. (It should be noted that since z is a regular element of R, it is also a regular element of $Q(R)$, as can easily be shown.) Since this holds for every $y, z \in J$, hence $JJt = J(Jt) = 0$, so that $Jt = 0$, and finally $t = 0$. We thus have

(2) $$(q_3\bar{d}_1(q') + q_4\bar{d}_2(q'))\bar{d}_1(q) = 0, \text{ for all } q \in Q(R).$$

By the same arguments as in the preceding paragraph, but now using $0\bar{d}_1 + y^2 q'\bar{d}_2$ in place of $y^2 q'\bar{d}_1 + 0\bar{d}_2$ (with relation (1') replacing (1)), we obtain

(2') $$(q_3\bar{d}_1(q') + q_4\bar{d}_2(q'))\bar{d}_2(q) = 0, \text{ for all } q \in Q(R).$$

From condition (ii) of the theorem (or its alternative) and relations (2) and (2') it now follows that $q_3\bar{d}_1(q') + q_4\bar{d}_2(q') = 0$. Since $q' \in Q(R)$ is arbitrary, hence $q_3\bar{d}_1 + q_4\bar{d}_2 = 0$. This completes the proof of the theorem.

Conjecture. $D_n(Q(R))$ is, in fact, the maximal (Lie) ring of quotients of $D_n(R)$.

Remarks. (i) The above theorem also holds for the case of an infinite number of derivations. Thus, let $\{d_\alpha\}_{\alpha \in A}$ (A some index set) be a collection of derivations of R such that $[d_\alpha, d_\beta] = 0$ for all $\alpha, \beta \in A$, with unique extensions $\bar{d}_\alpha \in D(Q(R))$. Let

$D_A(R) = \{\sum_{\beta \in F} r_\beta d_\beta : r_\beta \in R$, F ranging over all finite subsets of $A\}$,

$D_A(Q(R)) = \{\sum_{\beta \in F} q_\beta \bar{d}_\beta : q_\beta \in Q(R)$, F as in $D_A(R)\}$. Then $D_A(Q(R))$ is a (Lie) ring of quotients of $D_A(R)$ provided that

(i) 2 is a regular element of R

(ii) $\bigcup_{\beta \in F} \bar{d}_\beta(Q(R))$ contains a regular element, for some finite subset F of A.

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The proof is similar to the finite case. We also note that $D_A(R) = \bigcup_F D_F(R)$, $D_A(Q(R)) = \bigcup_F D_F(Q(R))$, where the union, in each case, ranges over all finite subsets F of A. Thus, this is, in a sense, a limiting case of the above theorem. We shall discuss another type of "limiting case" of Theorem 2 subsequently.

(ii) Conditions (i) and (ii) of Theorem 2 hold, of course, if R has no zero divisors. The condition $[d_i, d_j] = 0$ is not used in the proof; it only ensures that $D_n(R)$ is a Lie ring (a subring of $D(R)$).

(iii) Applying Theorem 2 to example 3 above, where $R = Z[x_1, \ldots, x_n]$, we pick the $d_1, \ldots, d_n$ to be the partial derivatives as defined therein. Then, as shown, $D_n(R) = D(R)$, $D_n(Q(R)) = D(Q(R))$. Thus we get that $D(Q(Z)(x_1, \ldots, x_n))$ is a (Lie) ring of quotients of $D(Z[x_1, \ldots, x_n])$. (Conditions (i) and (ii) of the theorem hold by the above remark, while $[d_i, d_j] = 0$ is easily verified.) Remark (i) above can be applied to example 4, where we pick $\{d_\beta\}_{\beta \in A}$ to be the partial derivatives of $Z[\{x_\alpha\}]$ with respect to x_β, and obtain that $D_A(Q(Z)(\{x_\alpha\}))$ is a (Lie) ring of quotients of $D_A(Z[\{x_\alpha\}])$. We recall, however, that here $D_A(Z[\{x_\alpha\}]) \neq D(Z[\{x_\alpha\}])$ (also $D_A(Q(Z)(\{x_\alpha\})) \neq D(Q(Z)(\{x_\alpha\}))$).

(iv) Let $d \in D(R)$ be any non-zero derivation, and pick $r_1, \ldots, r_n \in R$ such that $d(r_j) = 0$ $(j = 1, \ldots, n)$. (Such r_j always exist; e.g. let $r_1 = 1$, $r_2 = 2(=1+1), \ldots, r_n = n$.) Put $d_i = r_i d$ $(i = 1, \ldots, n)$. Then, for any $x \in R$, $[d_i, d_j](x) = [r_i d, r_j d](x) = r_i d(r_j d(x)) - r_j d(r_i d(x)) = r_i d(r_j) d(x) + r_i r_j d^2(x) - r_j d(r_i) d(x) - r_j r_i d^2(x) = 0$. That is, $[d_i, d_j] = 0$. Thus for any commutative ring R with identity we can always find subrings $D_n(R)$ and $D_n(Q(R))$ of $D(R)$ and $D(Q(R))$ respectively.

(v) If we pick different sets of derivations $d_1, \ldots, d_n$ and $d'_1, \ldots, d'_n$ of R such that $[d_i, d_j] = 0$, $[d'_i, d'_j] = 0$, it can be shown that the Lie rings $D_n(R)$ and $D'_n(R)$ are, in general, not isomorphic (but the corresponding rings of quotients $D_n(Q(R))$ and $D'_n(Q(R))$ may be isomorphic).

The subring of $D(R)$ generated by (that is, the smallest subring of $D(R)$ containing) all the $D_n(R)$ (for every choice of $d_1, \ldots, d_n$ and every n - in fact, it suffices to take $n = 1$) is clearly $D(R)$ itself. The subring of $D(Q(R))$ generated by all the $D_n(Q(R))$ (this is, in general, a proper subring of $D(Q(R))$, as we shall see later) will now be shown to be a ring of quotients of $D(R)$ (under certain conditions).

We continue to assume that R is a commutative ring with identify. A derivation $d \in D(Q(R))$ will be called <u>special</u> if $d = q_1\bar{d}_1 + \ldots + q_k\bar{d}_k$, for some $q_i \in Q(R)$, $d_i \in D(R)$ (where $\bar{d}_i$ denotes, as usual, the unique derivation of $Q(R)$ extending d_i). Let $D_0(Q(R))$ denote the set of all special derivations of $Q(R)$. It is easy to verify that $D_0(Q(R))$ is a subring of $D(Q(R))$ (this is, in fact, the subring of $D(Q(R))$ generated by all the $D_n(Q(R))$). Since R has an identity, $D(R)$ is a subring of $D_0(Q(R))$.

THEOREM 3. <u>Let</u> R <u>be an associative and commutative ring with identity satisfying the following conditions</u>:

(i) 2 <u>is a regular element of</u> R

(ii) $\bigcup_{d^* \in D(R)} \bar{d}^*(Q(R))$ <u>contains a regular element</u>. (<u>Alternatively</u>: $\bigcup_{d^* \in D(R)} \bar{d}^*(Q(R))$ <u>contains a dense</u> R-<u>submodule of</u> $Q(R)$.)

<u>Then</u> $D_0(Q(R))$ <u>is a (Lie) ring of quotients of</u> $D(R)$.

<u>Proof</u>. Let B be a $D(R)$-submodule of $D_0(Q(R))$ containing $D(R)$, and suppose $f: B \to D_0(Q(R))$ is a $D(R)$-homomorphism with $f(D(R)) = 0$. We wish to show that $f(B) = 0$. Let $d \in B$, $f(d) = d'$, and suppose that d^* is an arbitrary but fixed element of $D(R)$. Then $d = \sum_{i=1}^{k} q_i\bar{d}_i$, $[d, d^*] = \sum_{j=1}^{m} q_j'\bar{d}_j'$, for some $q_i, q_j' \in Q(R)$, $d_i, d_j' \in D(R)$.

Set $I = (\bigcap_{i=1}^{k} q_i^{-1}R) \cap (\bigcap_{j=1}^{m} q_j'^{-1}R)$. Thus I is a dense ideal of R, and $xq_i, xq_j' \in R$ for any $x \in I$, hence $xd, x[d, d^*] \in D(R)$. One then easily verifies that $[d, x^2d^*] \in D(R)$. Hence $0 = f[d, x^2d^*] = [f(d), x^2d^*] = [d', x^2d^*] = 2xd'(x)d^* + x^2[d', d^*]$, or

(1) $$2xd'(x)d^* = -x^2[d', d^*], \quad \text{for all } x \in I.$$

Let now q' be an arbitrary but fixed element of $Q(R)$. Set $J = I \cap q'^{-1}R \cap d(q')^{-1}R \cap (\bigcap_{i=1}^{k} (q'q_i)^{-1}R)$. Then J is a dense ideal of R, and $yq', yd(q') \in R$, $yq'd \in D(R)$, for any $y \in J$. An easy verification then shows that $[d, y^2q'd^*] \in D(R)$, hence $0 = [d', y^2q'd^*] = y^2d'(q')d^* + 2yq'd'(y)d^* + y^2q'[d', d^*]$. By relation (1), the last two terms disappear, hence we have $y^2d'(q')d^* = 0$. That is, $y^2d'(q')\bar{d}^*(q) = 0$, for all $q \in Q(R)$ (if d is the zero derivation on R, $\bar{d}$ is, by Theorem 1, the zero derivation on $Q(R)$). By the same argument as in Theorem 2 it now follows (using condition (i) of the theorem) that $d'(q')\bar{d}^*(q) = 0$. Then, by condition (ii) (or its alternative), we have $d'(q') = 0$. Since $q' \in Q(R)$ is arbitrary, $d' = 0$ and the proof is complete.

Conjecture. $D_0(Q(R))$ is, in fact, the maximal (Lie) ring of quotients of $D(R)$.

Remark. This theorem, too, may be applied to the example $R = Z[x_1, \ldots, x_n]$ to show that $D(Q(Z)(x_1, \ldots, x_n))$ is a (Lie) ring of quotients of $D(Z[x_1, \ldots, x_n])$. In this case $D(R) = D_n(R)$, $D(Q(R)) = D_0(Q(R)) = D_n(Q(R))$ (with the choice of $d_1, \ldots, d_n$ as the partial derivatives). We now show that in general $D_n(R) \subsetneqq D(R)$, $D_n(Q(R)) \subsetneqq D_0(Q(R)) \subsetneqq D(Q(R))$ (for every choice of $d_1, \ldots, d_n$ such that $[d_i, d_j] = 0$).

First, we show that $D_0(Q(R)) \neq D(Q(R))$. Thus let $R = Z[\{x_i\}]$, the ring of polynomials in the countably infinite number of indeterminates $x_1, x_2, x_3, \ldots$. Then $Q(R) = Q(Z)(\{x_i\})$, the field of quotients of $Z[\{x_i\}]$. Define a mapping $d: Q(Z)(\{x_i\}) \to Q(Z)(\{x_i\})$ by setting $d(x_j) = \frac{1}{x_j}$ $(j = 1, 2, \ldots)$, and extending to $Z[\{x_i\}]$, then to $Q(Z)(\{x_i\})$ in the obvious way (so as to make d a derivation). Thus $d \in D(Q(Z)(\{x_i\}))$, and we show that d is not special. For, suppose $d = q_1\bar{d}_1 + \ldots + q_m\bar{d}_m$ for some $q_k \in Q(Z)(\{x_i\})$, $d_k \in D(Z[\{x_i\}])$ $(k = 1, \ldots, m)$. Then $d(x_j) = \frac{1}{x_j} = q_1d_1(x_j) + \ldots + q_nd_n(x_j)$ $(j = 1, 2, \ldots)$. Suppose that $q_1, \ldots, q_m$ are functions of at most the first t interdeterminates $x_1, \ldots, x_t$. Then $\frac{1}{x_{t+1}} = q_1d_1(x_{t+1}) + \ldots + q_md_m(x_{t+1})$. But this is clearly impossible since $d_k(x_{t+1}) \in Z[\{x_i\}]$ $(k = 1, \ldots, m)$, and $q_1, \ldots, q_m$ are not functions of x_{t+1}. Thus $d \notin D_0(Q(Z)(\{x_i\}))$.

Next we consider the case $D_n(R) \neq D(R)$. As above, also here we let $R = Z[\{x_i\}]$. Let $d_1, \ldots, d_n$ be any derivations of $Z[\{x_i\}]$ such that $[d_i, d_j] = 0$. For each $j = 1, 2, \ldots$, $d_1(x_j), \ldots, d_n(x_j)$ are functions of at most the first k_j indeterminates $x_1, x_2, \ldots, x_{k_j}$. Define a mapping $d: Z[\{x_i\}] \to Z[\{x_i\}]$ by setting

$$d(x_j) = \begin{cases} x_j, & \text{if } x_j \neq x_1, \ldots, x_{k_j} \\ x_{k_j+1}, & \text{otherwise} \end{cases} \qquad (j = 1, 2, \ldots).$$

(What should be noted is that $d(x_j) = x_{s_j}$, where $s_j \geq j$, and $d_1(x_j), \ldots, d_n(x_j)$ are not functions of x_{s_j}, for each $j = 1, 2, \ldots$.) We now extend d to a derivation of $Z[\{x_i\}]$, and claim that $d \notin D_n(Z[\{x_i\}])$. For, if $d = r_1 d_1 + \ldots + r_n d_n$, for some $r_1, \ldots, r_n \in Z[\{x_i\}]$, then $r_1, \ldots, r_n$ are functions of at most the first t indeterminates $x_1, \ldots, x_t$. Now, $d(x_{t+1}) = r_1 d_1(x_{t+1}) + \ldots + r_n d_n(x_{t+1})$. Since $d(x_{t+1}) = x_{s_{t+1}}$, where $s_{t+1} \geq t+1$, hence $r_1, \ldots, r_n$ are not functions of $x_{s_{t+1}}$; but nor are $d_1(x_{t+1}), \ldots, d_n(x_{t+1})$. This gives rise to a contradiction which proves that $D_n(Z[\{x_i\}])$ is a proper subring of $D(Z[\{x_i\}])$.

Exactly the same considerations apply to $D_n(Q(R)) \neq D_0(Q(R))$. The R is the same as above, and we extend the derivation d of $Z[\{x_i\}]$, as defined above, to the derivation $\bar{d}$ of $Q(Z)(\{x_i\})$ (Theorem 1). Then clearly $\bar{d}$ is special. To show that $\bar{d} \notin D_n(Q(Z)(\{x_i\}))$, suppose that $\bar{d} = q_1 \bar{d}_1 + \ldots + q_n \bar{d}_n$ ($q_1, \ldots, q_n \in Q(Z)(\{x_i\})$), and assume that $q_1, \ldots, q_n$ are functions of at most the first t indeterminates $x_1, \ldots, x_t$. The proof now is as above (with q_j replacing r_j).

As a corollary to the previous conjecture (p. 14) and the above remark, we have the following

Conjecture. $D(Q(R))$ is, in general, not a ring of quotients of $D(R)$.

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