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SPACES OF MAPS INTO EILENBERG-MACLANE SPACES

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1. Introduction. In this note we provide alternative and unified proofs for two theorems on the homotopy groups of spaces of (continuous) maps into Eilenberg–MacLane spaces. The first theorem is due to Thom, and independently Federer, and deals with spaces of maps into Eilenberg–MacLane spaces of type (π, n) for $n \geqq 1$ with π abelian. The second theorem is due to Gottlieb and deals with spaces of maps into Eilenberg–MacLane spaces of type $(\pi, 1)$ with π nonabelian. As a main tool we shall use the homotopy sequences for certain fibrations of spaces of maps.

2. Basic notation and some preliminary remarks. For any pair of connected CW-complexes X and Y with base points, we denote by $M(X, Y)$, respectively $F(X, Y)$, the space of free maps, respectively based maps, of X into Y. We assume that all domains in mapping spaces are locally compact, and that all mapping spaces are equipped with the compact-open topology. For any subcomplex A of X, the map $M(X, Y) \rightarrow M(A, Y)$, defined by restricting maps with domain X to A, is a Hurewicz fibration over its image. See e.g. Spanier ([**9**], Theorem 2, p. 97 and Corollary 2, p. 400). These are the fibrations we need.

Every (path-)component in $M(X, Y)$ contains a based map. In computing homotopy groups $\pi_i(M(X, Y), f)$ of $M(X, Y)$ we need therefore only to consider based maps $f:X \rightarrow Y$ as base points in $M(X, Y)$. When base points have been specified they will be omitted in homotopy groups. We shall compute the homotopy groups of $M(X, Y)$ in case Y is an Eilenberg–MacLane space of type (π, n) for $n \geqq 1$, i.e., π is a group, abelian for $n \geqq 2$, such that $\pi_n(Y) \cong \pi$ and $\pi_i(Y) = 0$ for $i \neq n$.

3. The abelian case. First we consider a theorem of Thom ([**10**], Theorem 2), and independently Federer ([**2**], p. 355). See also Dyer ([**1**], Corollary 8.4). In spirit our proof is closest to that of Thom.

THEOREM 1. *Suppose that Y is an Eilenberg–MacLane space of type (π, n) for $n \geqq 1$ with π abelian. Then*

$$\pi_i(M(X, Y), f) \cong H^{n-i}(X; \pi)$$

for all $i \geqq 1$.

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Proof. Since π is abelian, Y admits the structure of an H-space. Then $M(X, Y)$ inherits an H-space structure with pointwise multiplication of maps as multiplication. Therefore all the components in $M(X, Y)$ have the same homotopy type, and hence it suffices to consider the component $M_0(X, Y)$ of homotopically trivial maps in $M(X, Y)$ with the constant based map $f_0: X \to Y$ as base point. We note that $M_0(X, Y)$ is itself an H-space, such that $\pi_1(M_0(X, Y))$ is abelian.

Evaluation at the base point for X defines a Hurewicz fibration $p_0: M_0(X, Y) \to Y$ with the component $F_0(X, Y)$ of homotopically trivial maps in $F(X, Y)$ as fibre. Since p_0 has a section, namely the section of constant maps, the homotopy sequence for p_0 splits and we get

$$\pi_i(M_0(X, Y)) \cong \pi_i(Y) \oplus \pi_i(F_0(X, Y))$$

for all $i \geqq 1$. For $n = 1$ and $i = 1$ we use that $\pi_1(M_0(X, Y))$ is abelian.

Let $\wedge$ denote smash product, Ω^i the i-fold loop space functor, and $\pi(\cdot,\cdot)$ a homotopy set of based maps. With this notation there are canonical isomorphisms

$$\pi_i(F_0(X, Y)) \cong \pi(S^i \wedge X, Y) \cong \pi(X, \Omega^i Y)$$

for all $i \geqq 1$.

Observe now that $\Omega^i Y$ is a space of type $(\pi, n - i)$ for $1 \leqq i \leqq n - 1$, a discrete space for $i = n$, and a point for $i > n$. Consequently

$$\pi_i(F_0(X, Y)) \cong \begin{cases} H^{n-i}(X; \pi) & \text{for } 1 \leqq i \leqq n - 1 \\ 0 & \text{for } i \geqq n. \end{cases}$$

See e.g. [**9**], Theorem 10, p. 428. Since

$$\pi_i(Y) \cong \begin{cases} \pi = H^{n-i}(X; \pi) & \text{for } i = n \\ 0 & \text{for } i \neq n, \end{cases}$$

we then get immediately

$$\pi_i(M_0(X, Y)) \cong H^{n-i}(X; \pi)$$

for all $i \geqq 1$. This proves Theorem 1.

4. The nonabelian case. Next we consider a theorem, which seems first to have been observed by Gottlieb ([**3**], Lemma 2).

THEOREM 2. *Suppose that Y is an Eilenberg-MacLane space of type $(\pi, 1)$ with π nonabelian. For any based map $f: X \to Y$, consider the induced homomorphism $f_*: \pi_1(X) \to \pi_1(Y)$ and denote by $C(\pi; f)$ the centralizer for $f_*(\pi_1(X))$ in $\pi_1(Y) \cong \pi$. For any finite dimensional CW-complex X we then get*

$$\pi_i(M(X, Y), f) \cong \begin{cases} C(\pi; f) & \textit{for } i = 1 \\ 0 & \textit{for } i > 1. \end{cases}$$

Proof. Up to homotopy type X admits a filtration $X^0 \subseteq X^1 \subseteq \ldots \subseteq X^n = X$, where X^0 is a point, X^1 is a wedge of circles, and X^k for $k \geqq 2$ is obtained by attaching a number of k-cells to X^{k-1}.

For each k, $1 \leqq k \leqq n$, restriction of maps defines a Hurewicz fibration (over its image) $M(X^k, Y) \to M(X^{k-1}, Y)$, the fibre of which is easily seen to have the homotopy type of the product of a number of copies of the space of k-loops $\Omega^k Y$ corresponding to the number of k-cells we attach to X^{k-1} to obtain X^k.

For $k \geqq 2$, this fibre is contractible, and hence, if f^1 denotes the restriction of f to X^1, we get

$$\pi_i(M(X, Y), f) \cong \pi_i(M(X^1, Y), f^1)$$

for all $i \geqq 1$.

For $k = 1$, we have the evaluation fibration $p: M(X^1, Y) \to Y$ with fibre $F(X^1, Y)$. Now X^1 is a wedge of circles, say $X^1 = \bigvee_{\gamma \in \Gamma} S_\gamma^1$. Therefore $F(X^1, Y)$ can be identified with the product $\times_{\gamma \in \Gamma} (\Omega^1 Y)_\gamma$, and hence

$$\pi_i(F(X^1, Y), f^1) = 0 \quad \text{for all } i \geqq 1.$$

By the homotopy sequence for p we get then immediately

$$\pi_i(M(X, Y), f) \cong \pi_i(M(X^1, Y), f^1) \cong \pi_i(Y) = 0$$

for all $i \geqq 2$.

In dimension 1, p induces a monomorphism

$$p_*: \pi_1(M(X^1, Y), f^1) \to \pi_1(Y).$$

Hence

$$\pi_1(M(X, Y), f) \cong \pi_1(M(X^1, Y), f^1) \cong \text{Image } (p_*).$$

For each $\gamma \in \Gamma$, let α_γ denote the generator for $\pi_1(X)$ given by the inclusion of the circle S_γ^1 into X. Proceeding exactly as in ([**7**], Proof of Proposition 1), it is easy to see that $\alpha \in \pi_1(Y)$ is in the image for p_* if and only if the Whitehead product $[\alpha, f_*(\alpha_\gamma)] = 1$ for all $\gamma \in \Gamma$, and therefore if and only if α commutes with every element in the image for $f_*: \pi_1(X) \to \pi_1(Y)$. But this says precisely that Image $(p_*) = C(\pi; f)$, and therefore

$$\pi_1(M(X, Y), f) \cong C(\pi; f)$$

as asserted. This proves Theorem 2.

5. Some applications of theorem 2. For X an aspherical space, i.e., a compact polyhedron which is also an Eilenberg–MacLane space of type $(\pi, 1)$, Theorem 2 applies to the space of self-mappings on X. For 1_X, the identity map on X, observe that $C(\pi, 1_X)$ is equal to the center $C(\pi)$ of π. Hence Theorem 2 has the following

COROLLARY 1. *For an aspherical space X with fundamental group $\pi = \pi_1(X)$, the identity component $M_1(X, X)$ in the space of self-mappings on X is an Eilenberg–MacLane space of type $(C(\pi), 1)$.*

This result is originally due to Gottlieb ([**4**], Theorem III.2). In particular we get

COROLLARY 2. *Let S be a closed surface, except the sphere, the torus, the projective plane and the Klein bottle. Then the identity component $M_1(S, S)$ in the space of self-mappings on S is contractible.*

Corollary 2 follows from Corollary 1, since $\pi_1(S)$ has trivial center, see e.g. [**5**], Theorem 4.4. More generally, the identity component in the space of self-mappings on a closed Riemannian manifold with strictly negative sectional curvature is contractible, since such a manifold is an aspherical space, where the fundamental group has trivial center. The Riemannian geometry necessary to prove these assertions can be found in e.g. [**6**], Section 7.2.

Finally we mention that the result in Theorem 2 was used by the author in [**8**] to give a proof of a theorem of Al'ber on spaces of maps into a manifold, which admits a Riemannian metric with strictly negative sectional curvature. The theorem says that for such spaces of maps a component will alway have the homotopy type of a point, a circle, or the target.

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