

# VARIETIES OF TOPOLOGICAL GROUPS GENERATED BY LIE GROUPS

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## 1. Introduction

Varieties of topological groups have been investigated in several papers ((2) and (10)-(13)). In this note we investigate the varieties generated by classical Lie groups. In particular we show results of which the following is indicative: The variety generated by the unitary group  $U(n)$  contains  $U(m)$  if and only if  $m \leq n$ . *En route* we introduce the notion of a variety of topological Lie algebras which provides a convenient setting in which to answer our questions.

## 2. Definitions and preliminary results

A non-empty class of topological groups (not necessarily Hausdorff) is said to be a *variety of topological groups* if it is closed under the operations of taking subgroups, quotients, arbitrary cartesian products and isomorphic images. The smallest variety containing a class  $\mathcal{C}$  of topological groups is said to be the variety generated by  $\mathcal{C}$  and is denoted by  $V(\mathcal{C})$ , (or  $V(G)$  if  $\mathcal{C} = \{G\}$ ).

A *Banach Lie algebra*  $L$  is a Lie algebra (over the reals) and a Banach space such that there exists a constant  $c$  such that  $\|[x, y]\| \leq c \|x\| \|y\|$  for each  $x, y \in L$ . If  $B_i, i \in I$ , are Banach Lie algebras and  $E$  is a subalgebra of the product  $\prod_{i \in I} B_i$ , then  $E$  with the topology induced from the product is said to be a (locally convex) *topological Lie algebra*. (Cf. (8) pp. 252, 253 and the fact that any locally convex Hausdorff topological vector space is a subspace of a product of Banach spaces.)

A non-empty class of topological Lie algebras is to be a *variety of topological Lie algebras* if it is closed under the formation of cartesian products, subalgebras, separated quotients and isomorphic images.

Note that the "Varieties of linear topological spaces" considered in (3) and (4) are varieties of topological Lie algebras.

If  $\mathcal{C}$  is a class of topological groups then  $Q\mathcal{C}$  denotes the class of all topological groups isomorphic to quotients of members of  $\mathcal{C}$ . Similarly we define the operators  $S, \bar{S}, \bar{Q}, C$  and  $D$  where they respectively denote subgroup, closed subgroup, separated quotient, arbitrary cartesian product and finite product.

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The operators  $S$ ,  $\bar{S}$ ,  $\bar{Q}$ ,  $C$  and  $D$  are defined similarly on classes of topological Lie algebras.

We now present two basic theorems, the first is proved in (2) and the second can be proved in a like manner.

**Theorem 2.1.** *If  $\mathcal{C}$  is a class of topological groups and  $G$  is a Hausdorff group in  $V(\mathcal{C})$ , then  $G \in SC\bar{Q}\bar{S}D(\mathcal{C})$ .*

**Theorem 2.2.** *If  $\mathcal{C}$  is a class of topological Lie algebras and  $V(\mathcal{C})$  is the variety generated by  $\mathcal{C}$ , then  $V(\mathcal{C}) = SC\bar{Q}\bar{S}D(\mathcal{C})$ .*

The next theorem, proved in a similar manner to Theorem 4.1 of (4) has a very useful corollary.

**Theorem 2.3.** *Let  $\mathcal{C}$  be any class of topological Lie algebras and let  $B$  be a Banach Lie algebra in  $V(\mathcal{C})$ . Then  $B \in \bar{Q}\bar{S}D(\mathcal{C})$ . In particular, this is the case for finite-dimensional topological Lie algebras.*

**Corollary 2.4.** *Let  $\mathcal{C}$  be a class of topological Lie algebras of dimension  $\leq m$ , for some integer  $m$ . Then every finite-dimensional simple topological Lie algebra  $L$  in  $V(\mathcal{C})$  has dimension  $\leq m$ .*

**Proof.** From Theorem 2.3 we see that  $L \in \bar{Q}\bar{S}D(\mathcal{C})$ . Using Theorem 3.1 of (7) this implies that  $L \in \bar{S}D(\mathcal{C})$ . Finally, noting that  $L$  is simple we have that  $L \in S(\mathcal{C})$ , which ensures that the dimension of  $L \leq m$ .

Finally, we mention

**Theorem 2.5.** *Let  $G$  be a connected locally compact Hausdorff group. Then*

- (i)  $V(G) \cong V(T)$ , where  $T$  is the circle group,
- (ii)  $G$  is compact if and only if  $V(G) \cong V(R)$ , where  $R$  is the group of reals,
- (iii)  $G$  is abelian if and only if  $V(G) \subseteq V(R)$ ,
- (iv) If  $G$  is compact, then the variety of groups, (14), generated by  $G$  is either the class of all groups or the class of all abelian groups. In the latter case

$$V(G) = V(T).$$

**Proof.** Suppose  $G$  is non-compact. Then, in view of § 4.13 of (9),  $G$  has  $R$  as a subgroup. Therefore  $V(G) \cong V(R) \supset V(T)$ . If  $G$  is abelian, then Theorem 5.8 of (12) yields that  $V(G) = V(R)$ .

Now suppose that  $G$  is compact. Then by § 4.6 of (9),  $G$  has a proper normal subgroup  $N$  such that  $G/N$  is a Lie group. This implies (§ 4.13 of (9))  $G/N$  has  $T$  as a subgroup. Consequently,  $V(G) \cong V(T)$ . Corollary 3 of (2) states that a locally compact group in a variety generated by compact groups is compact. Thus  $R \notin V(G)$ .

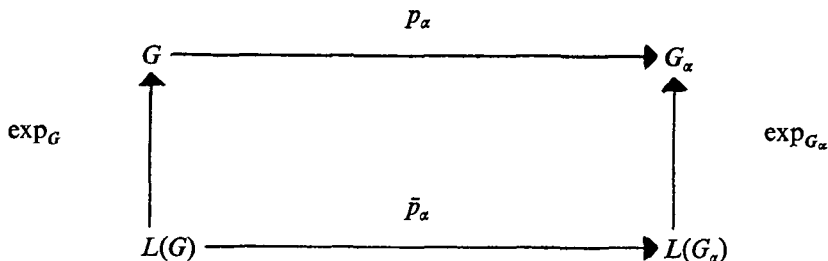
If  $G$  is compact and non-solvable, then the main result of (1) implies that the variety of groups generated by  $G$  is the class of all groups.

If  $G$  is compact and solvable, then by Theorem 29.44 of (6),  $G$  is abelian. Therefore  $V(G) = V(T)$ , and the variety of groups generated by  $G$  is the class of all abelian groups. This completes the proof.

3. The main results

**Lemma 3.1.** *Let  $G$  and  $\{G_\alpha: \alpha \in I\}$  be Lie groups. If  $G$  is a subgroup of the product  $\prod_{\alpha \in I} G_\alpha$ , then  $L(G)$  is isomorphic to a subalgebra of  $\prod_{\alpha \in I} L(G_\alpha)$ , where  $L(G)$  and  $L(G_\alpha)$  are the Lie algebras of  $G$  and  $G_\alpha, \alpha \in I$ , respectively.*

**Proof.** Let  $p_\alpha$  be the projection mapping of  $G$  into  $G_\alpha$  and let  $\bar{p}_\alpha$  be the induced homomorphism of  $L(G)$  into  $L(G_\alpha)$ . Consider the diagram:



Clearly it commutes. Since there are enough maps  $p_\alpha$  to separate points of  $G$ , there are enough maps  $\bar{p}_\alpha$  to separate points of  $L(G)$ . Therefore the mapping of  $L(G)$  into  $\prod_{\alpha \in I} L(G_\alpha)$ , defined to be the product of the mappings  $\bar{p}_\alpha, \alpha \in I$ , is an isomorphism of  $L(G)$  on to its image.

**Theorem 3.2.** *Let  $G$  be a Lie group and  $L(G)$  its topological Lie algebra (that is, the Lie algebra of  $G$  given the unique vector space topology it admits).*

(i) *If  $H$  is a Lie group in  $V(G)$ , then its topological Lie algebra  $L(H)$  is in  $V(L(G))$ .*

(ii) *If  $L$  is any finite-dimensional topological Lie algebra in  $V(L(G))$ , then there is a Lie group  $H$  in  $V(G)$  such that  $L(H)$  is isomorphic to  $L$ .*

**Proof.** (i) By Theorem 2.1,  $H \in SC\bar{Q}\bar{S}D(G)$ . That is,  $H \leq \prod_{\alpha \in I} G_\alpha$ , where each  $G_\alpha \in \bar{Q}\bar{S}D(G)$ . Then clearly  $L(G_\alpha) \in \bar{Q}\bar{S}D(L(G))$ . By Lemma 3.1, this implies that  $L(H)$  is isomorphic to a subalgebra of  $\prod_{\alpha \in I} L(G_\alpha)$ , and hence is in  $V(L(G))$ .

(ii) If  $L$  is a finite-dimensional topological Lie algebra in  $V(L(G))$ , then, by Theorem 2.3,  $L \in \bar{Q}\bar{S}D(L(G))$ . This implies that there is an  $H$  in  $\bar{Q}\bar{S}D(G)$  such that  $L(H)$  is isomorphic to  $L$ . The proof is complete.

**Corollary 3.3.** *Let  $G$  be a Lie group of dimension  $n$  with a simple Lie algebra, and  $\{G_\alpha: \alpha \in I\}$  be Lie groups of dimension  $< n$ . Then  $G \notin V(\{G_\alpha: \alpha \in I\})$ .*

**Corollary 3.4.** *Let  $\mathcal{C}$  be a class of connected solvable (nilpotent) Lie groups. Then any connected Lie group in  $V(\mathcal{C})$  is solvable (nilpotent).*

**Proof.** This follows from Theorems 3.2 and 2.3.

We now look at some examples.

Noting, (5), that the Lie algebras of the special unitary groups  $SU(n)$  are simple for  $n > 1$ , and that their dimension increases with  $n$ , Corollary 3.3 gives:  $SU(m) \in V(SU(n))$  if and only if  $m \leq n$ . Furthermore, noting that the unitary group  $U(n)$  is a quotient group of  $SU(n) \times T$  (cf. 29.48 of (6)), Theorem 2.5 shows that  $V(U(n)) = V(SU(n))$  for all  $n$ . Hence  $U(m) \in V(U(n))$  if and only if  $m \leq n$ .

The symplectic groups  $Sp(n)$  have simple Lie algebras, (5), and dimension an increasing function of  $n$ . Thus  $Sp(m) \in V(Sp(n))$  if and only if  $m \leq n$ .

The spin groups  $\text{spin}(n)$  have a slightly more curious behaviour. For  $n \neq 2, 4$  their Lie algebras are simple and the above argument again works. We can see that  $\text{spin}(2) \notin V(\text{spin}(1))$ , since  $\text{spin}(1)$  is of exponent 2 and  $\text{spin}(2)$  is not. Finally we observe that  $\text{spin}(4) \in V(\text{spin}(3))$ , since  $\text{spin}(4)$  is isomorphic to  $\text{spin}(3) \times \text{spin}(3)$ . Thus we have  $\text{spin}(m) \in V(\text{spin}(n))$  if and only if  $m \leq n$  or  $m = n + 1 = 4$ .

Since the orthogonal group  $O(n)$  is a quotient of the product of  $SO(n)$  and a suitable finite cyclic group (cf. 29.49 of (6)), we have  $V(O(n)) = V(SO(n))$ , for all  $n$ . Our usual argument leads to the result  $SO(m) \in V(SO(n))$ , if  $m > n$  and  $m \neq 4$ .

Non-compact Lie groups such as  $O(m, n)$ ,  $U(m, n)$  and  $Sp(m, n)$  can be treated similarly.

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### REFERENCES

- (1) S. BALCERZYK and JAN MYCIELSKI, On the existence of free subgroups in topological groups, *Fundamenta Mathematicae* **44** (1957), 303-308.
- (2) M. S. BROOKS, SIDNEY A. MORRIS and STEPHEN A. SAXON, Generating varieties of topological groups, *Proc. Edinburgh Math. Soc.* To appear.
- (3) JOSEPH DIESTEL, SIDNEY A. MORRIS and STEPHEN A. SAXON, Varieties of locally convex topological vector spaces, *Bull. Amer. Math. Soc.* **77** (1971), 799-803.
- (4) JOSEPH DIESTEL, SIDNEY A. MORRIS and STEPHEN A. SAXON, Varieties of linear topological spaces, *Trans. Amer. Math. Soc.* To appear.
- (5) S. HELGASON, *Differential Geometry and Symmetric Spaces* (Academic Press, New York, 1962).
- (6) E. HEWITT and K. A. ROSS, *Abstract Harmonic Analysis II* (Springer-Verlag, Berlin, Heidelberg, 1970).
- (7) G. HOCHSCHILD, *The Structure of Lie Groups* (Holden-Day Inc., San Francisco, London, Amsterdam, 1965).

(8) K. H. HOFMANN, *Introduction to the Theory of Compact Groups I* (Tulane University Lecture notes, 1968).

(9) D. MONTGOMERY and L. ZIPPIN, *Topological Transformation Groups* (Interscience tracts in pure and applied mathematics I, New York, 1955).

(10) SIDNEY A. MORRIS, Varieties of topological groups, *Bull. Australian Math. Soc.* 1 (1969), 145-160.

(11) SIDNEY A. MORRIS, Varieties of topological groups II, *Bull. Australian Math. Soc.* 2 (1970), 1-13.

(12) SIDNEY A. MORRIS, Varieties of topological groups III, *Bull. Australian Math. Soc.* 2 (1970), 165-178.

(13) SIDNEY A. MORRIS, Locally compact Abelian groups and the variety of topological groups generated by the reals, *Proc. Amer. Math. Soc.* To appear.

(14) HANNA NEUMANN, *Varieties of Groups* (Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 37, Springer-Verlag, Berlin, Heidelberg, 1967).

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