ON A SPECIAL CLASS OF FINITE 2-GROUPS

by MARIAN DEACONESCU

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1. Introduction. In the course of classifying those finite groups F which have exactly five maximal subgroups, R. W. van der Waall [**4**] proved that one encounters the following situation. One class of such groups F is described by $F = SP$, where $S = O_2(F) \in \mathrm{Syl}_2(F)$, $P \in \mathrm{Syl}_3(F)$, $S/\Phi(S) \cong Z_2 \times Z_2$, P is cyclic and P operates via conjugation on S as a group of order 3, because in this case $F/\Phi(F) \cong A_4$.

This raises the problem of determining the internal structure of those finite 2-groups with two generators which have an automorphism of order 3.

Apart from abelian groups with these properties—which turn out to be homocyclic—there are also nonabelian ones. A search in [**2**] through the non-abelian 2-groups of order at most 64 with two generators shows that there exist among these groups exactly three groups of the desired type, all of them of class two. These are numbered in [**2**] as 8/5, 32/18 and 64/181; 8/5 being the quaternion group Q_8.

This suggests that the 2-groups in question occur rather rarely among the 2-groups with two generators and gives some hope for a uniform description of them by a set of well-chosen parameters.

The aim of this note is to show that—at least for groups of class two—this is actually the case; we shall prove the following:

THEOREM. *Let G be a finite nonabelian 2-group with two generators which has an automorphism of order* 3.

(i) *If G has class two, then one of the following two cases occurs*:

(a) $G = \langle x, y \mid x^{2^k} = y^{2^k} = [x,y]^{2^b}, [x,y]^{2^d} = [x,[x,y]] = [y,[x,y]] = 1\rangle$, *where* $b \geq 0$ *and* $k = d = b + 1$.

(b) $G = \langle x, y \mid x^{2^k} = y^{2^k} = [x,y]^{2^d} = [x,[x,y]] = [y,[x,y]] = 1\rangle$, *where* $1 \leq d < k$.

(ii) *If G is a product of two cyclic subgroups, then* $G \cong Q_8$.

The "experimental" knowledge of such groups obtained from [**2**] might suggest that for a given s there exists at most one 2-group of order 2^s satisfying the restrictions of part (i) of the Theorem. However, the next result shows that even for such rather special 2-groups the intuition fails to offer the right answer:

COROLLARY. *The number of nonisomorphic types of* 2*-groups of order 2^s matching the description of* (b) *in the Theorem tends to infinity with s. On the other hand, a group of order 2^s and of type* (a) *exists if and only if* $s \equiv 0 \pmod 3$. *For each* $s \equiv 0 \pmod 3$ *there exists an unique group of type* (a) *and of order* 2^s.

All groups in discussion are finite. The notation is standard and follows that of [**1**].

2. Preliminary results. Throughout this section G will denote a 2-group which possesses an automorphism α of order 3. We shall denote $C_G(\alpha) := \{x \in G \mid x = \alpha(x)\}$ and $[G,\alpha] = \langle\{x^{-1}\alpha(x) \mid x \in G\}\rangle$. Note that $[G,\alpha]$ is a normal α-invariant subgroup of G.

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For the sake of convenience we shall list here some special cases of well-known results which appear in [**1**].

2.1 $G = C_G(\alpha)[G, \alpha]$. If, moreover, G is abelian, then $G = C_G(\alpha) \times [G, \alpha]$.

Proof. See Theorems 5.2.3 and 5.3.5 of [**1**].

2.2 If $C_G(\alpha) = 1$, then for every $g \in G$:
(i) $g\alpha(g)\alpha^2(g) = 1$
(ii) $g\alpha(g) = \alpha(g)g$.

Proof. See Th. 10.1.5 of [**1**].

2.3 If G is abelian and if α acts indecomposably on G, then G is homocyclic.

Proof. See Th. 5.2.2 of [**1**].

2.4 If H is a normal α-invariant subgroup of G, then α induces on $\bar{G} := G/H$ an automorphism $\bar{\alpha}$ defined by $\bar{\alpha}(gH) := \alpha(g)H$ such that $C_{\bar{G}}(\bar{\alpha}) = C_G(\alpha)H/H$.

Proof. See Th. 5.3.15 of [**1**].

3. Some lemmas. Throughout this section, G will denote a 2-group with two generators which has an automorphism α of order 3. Here is the place to make some notation which will be used freely in the sequel.

The Frattini subgroup $\Phi(G)$ of G will be denoted by Φ. The three maximal subgroups of G will be denoted by M_1, M_2 and M_3. One can suppose that $\alpha(M_1) = M_2$, $\alpha(M_2) = M_3$ and $\alpha(M_3) = M_1$.

If $\{x, y\}$ is a minimal set of generators of G, one can suppose that $x \in M_1$ and therefore that $\alpha(x) \in M_2 \backslash M_1$. Therefore there is no loss in supposing that $y = \alpha(x)$. From now on we shall denote $\alpha(x)$ by y.

3.1 If H is an α-invariant proper subgroup of G, then $H \leqslant \Phi$.

Proof. Suppose for instance that $H \leqslant M_1$. Then $H = \alpha(H) \leqslant M_1 \cap \alpha(M_1) = M_1 \cap M_2 = \Phi$.

3.2 $$[G, \alpha] = G.$$

Proof. Suppose by contrary that $[G, \alpha] \neq G$, so $[G, \alpha] \leqslant \Phi$ by 3.1. By 2.1, $G = C_G(\alpha)\Phi = C_G(\alpha)$, contradicting $|\alpha| = 3$.

3.3 $$C_G(\alpha) \lhd G \quad \text{iff} \quad C_G(\alpha) \leqslant Z(G).$$

Proof. If $C_G(\alpha) \leqslant Z(G)$ there is nothing to prove. If $C_G(\alpha) \lhd G$, let $f \in C_G(\alpha)$ and $g \in G$ be arbitrary. Then $\alpha(f^g) = f^{\alpha(g)} = f^g$, which means that $[G, \alpha] \leqslant C_G(C_G(\alpha))$. The result now follows by 3.2.

3.4 $$C_G(\alpha) = 1 \quad \text{iff} \quad G \text{ is abelian.}$$

Proof. If $C_G(\alpha) = 1$, then G is abelian because of 2.2(ii) and the fact that $G = \langle x, \alpha(x) \rangle$.

If G is abelian, then by 2.1 and 3.2 $G = C_G(\alpha) \times G$, thus $C_G(\alpha) = 1$.

3.5 If $C_G(\alpha) \leqslant H$ and if H is a normal α-invariant subgroup of G, then $G' \leqslant H$.

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Proof. A direct consequence of 2.4 and 3.4.

3.6 $$C_G(\alpha) \leq G'.$$

Proof. Let $\bar{G} := G/G'$ and define $\bar{\alpha}$ as in 2.4. Note that $\bar{G}$ has also two generators and that $|\bar{\alpha}| = 3$. Since $\bar{G}$ is abelian, it follows by 3.4 that $C_{\bar{G}}(\bar{\alpha}) = 1$. Therefore $C_G(\alpha) \leq G'$ by 2.4.

3.7 G' is the normal closure of $C_G(\alpha)$ in G.

Proof. Since the normal closure of $C_G(\alpha)$ in G is a normal α-invariant subgroup of G, the result follows by 3.5 and 3.6.

3.8 If k is the least positive integer for which $x^{2^k} \in G'$, then

$$G/G' \cong Z_{2^k} \times Z_{2^k}.$$

Proof. If $\bar{G}$ and $\bar{\alpha}$ are defined as in 2.4, where $\bar{G} = G/G'$, then by 3.2 $\bar{\alpha}$ acts indecomposably on $\bar{G}$. Therefore $\bar{G}$ is homocyclic by 2.3.

3.9 $$C_G(\alpha) \lhd G \quad \text{iff} \quad G' \leq Z(G).$$

Proof. This follows from 3.3, 3.6 and 3.7.

3.10 If $G = \langle x \rangle \langle y \rangle$, then $\Phi = \langle x^2 \rangle \langle y^2 \rangle$.

Proof. Remember that $x \in M_1$ and $y \in M_2$, so $M_1 = \langle x \rangle \Phi$ and $M_2 = \langle y \rangle \Phi$. This shows that $\langle x \rangle \cap \Phi = \langle x^2 \rangle$ and $\langle y \rangle \cap \Phi = \langle y^2 \rangle$. Now $M_1 = M_1 \cap G = M_1 \cap \langle x \rangle \langle y \rangle = \langle x \rangle (M_1 \cap \langle y \rangle)$ and since $\langle y^2 \rangle = \langle y \rangle \cap \Phi \leq \langle y \rangle \cap M_1 < \langle y \rangle$ it follows that $M_1 \cap \langle y \rangle = \langle y^2 \rangle$. Finally, since $y^2 \in \Phi$, one obtains that $\Phi = \Phi \cap M_1 = \Phi \cap \langle x \rangle \langle y^2 \rangle = \langle y^2 \rangle (\Phi \cap \langle x \rangle) = \langle y^2 \rangle \langle x^2 \rangle = \langle x^2 \rangle \langle y^2 \rangle$.

4. Proofs of the results.

Proof of the Theorem. We start by proving (i). Since G is of class two, $G' \leq Z(G)$, so by 3.7 $C_G(\alpha) = G' \leq Z(G)$. But G has two generators, so $G' = \langle [x, y] \rangle$. Set $2^d = |G'|$ and let k be defined as in 3.8. Because $x^{2^k} \in G' = C_G(\alpha)$, one obtains that $y^{2^k} = \alpha(x^{2^k}) = x^{2^k}$. Moreover, since $x^{2^k} \in Z(G)$, one has $[x^{2^k}, y] = [x, y]^{2^k} = 1$, which shows that $d \mid k$.

Now $x^{2^k} = y^{2^k} \in G' = \langle [x, y] \rangle$, so there exists some b with $0 \leq b \leq d$ such that $x^{2^k} = y^{2^k} = [x, y]^{2^b}$.

From 3.4 and 2.2(i) we infer that $xy\alpha(y) = x\alpha(x)\alpha^2(x) \in G'$. Without any loss (one can change the set $\{x, y\}$ of generators of G if necessary), one can assume that $\alpha(y) = y^{-1}x^{-1}$. Then one verifies that $\alpha^3(x) = x$, $\alpha^3(y) = y$ and $\alpha([x, y]) = [x, y]$, provided that α defined by $\alpha(x) := y$ and $\alpha(y) := y^{-1}x^{-1}$ is indeed an automorphism of G. Note that the above defined α must also satisfy the relation $\alpha(y)^{2^k} = y^{2^k} = [x, y]^{2^b}$.

By taking into account that $|[x, y]| = 2^d$ divides 2^k and that $[x, y] \in Z(G)$, one obtains that $\alpha(y)^{2^k} = (y^{-1}x^{-1})^{2^k} = [x^{-1}, y^{-1}]^{2^{k-1}(2^k-1)} y^{-2^k} x^{-2k} = [x, y]^{-(2^{k-1}+2^{b+1})}$. From the equality $\alpha(y)^{2^k} = [x, y]^{2^b}$ it follows that $[x, y]^{2^{k-1}+3.2^b} = 1$. This implies that $2^d \mid 2^{k-1} + 3.2^b$. We distinguish two cases:

(a) If $b < d$, then $d - b \geq 1$ and $2 \mid 2^{k-b-1} + 3$, which forces $k = b + 1$. Thus $b + 1 \leq d \leq k = b + 1$, giving $k = d = b + 1$, $b \geq 0$.

(b) If $b = d$, then $2^d \mid 2^{k-1} + 3 \,.\, 2^d$, which imposes no other restrictions on b, d, k than $0 < b = d < k$. Here $b > 0$ because G is nonabelian.

In order to prove (ii), we may assume that $G=\langle x\rangle\langle y\rangle$, so by 3.10 $\Phi=\langle x^2\rangle\langle y^2\rangle$. If k is defined as in 3.8, then by 3.8 and repeated use of 3.10 one obtains $G'=\langle x^{2^k}\rangle\langle y^{2^k}\rangle$.

We claim that $C_G(\alpha)=G'$.

Assume the contrary; then one can suppose that $x^{2^k}\notin C_G(\alpha)$. Indeed, otherwise $y^{2^k}=x^{2^k}$, so $G'=\langle x^{2^k}\rangle$ is cyclic. Since $C_G(\alpha)\leqslant\Phi(G')$, one contradicts 3.7. Now since $x^{2^k}\notin C_G(\alpha)$, then $y^{2^k}\neq x^{2^k}$ and this together with $G'=\langle x^{2^k}\rangle\langle y^{2^k}\rangle$ shows that G' has also two generators.

If $\beta:=\alpha|_{G'}$, then $C_{G'}(\beta)=C_G(\alpha)<G'$, which shows that β has order 3. By 3.2, $C_G(\alpha)\leqslant\Phi(G')$, contradicting again 3.7. This proves the claim.

Now by 3.3 $C_G(\alpha)=G'\leqslant Z(G)$, which means that G has class two. Therefore $G'=\langle x^{2^k}\rangle=\langle[x,y]\rangle$ and one can assume $x^{2^k}=[x,y]$. If (b,d,k) has the same meaning as in case (i) of the Theorem, we obtain in this case that $(b,d,k)=(0,1,1)$. But then $G=\langle x,y\mid x^2=y^2=[x,y],\ \ [x,[x,y]]=[y,[x,y]]=[x,y]^2=1\rangle\cong Q_8$, which ends the proof.

Proof of the Corollary. Note that if the presentation of G is described by a list (b,d,k), then $|G|=|G:G'|\,.\,|G'|=2^{2k+d}$. This shows at once that if s is given, there exists a group G of order 2^s with a presentation of type (a) if and only if $s=3d\equiv 0\,(\mathrm{mod}\,3)$. Moreover, s determines uniquely the presentation of G in this case because $b=d-1$, $k=d=s/3$. This proves the second assertion of the Corollary.

Suppose now that s is given and that we want to produce a group of order 2^s admitting a presentation of type (b). Then such a presentation is described by a list (b,d,k) where $1\leqslant b=d<k$ and $2k+d=s$ (although b does not appear explicitly in the description of a group of type (b)), we use this notation for the sake of uniformity).

This shows that the number of the lists (b,d,k) verifying these conditions tends to infinity with s and proves the Corollary.

Remarks. (i) It is interesting to relate the outcome of our analysis to a result of U. Martin [**3**]. She proved that if p is a prime, if $a(d,n)$ is the number of d-generators p-groups of Frattini class n and if $e(d,n)$ is the number of these groups having no coprime automorphisms, then $\lim\limits_{d\to\infty} a(d,n)/e(d,n)=1$.

In particular this shows that if we keep the Frattini class of a 2-group fixed, the number of 2-groups of this Frattini class and having no coprime automorphisms tends to infinity with their number of generators.

Our Corollary may be regarded as a counterpart of this observation: it shows that the number of the nonisomorphic types of class two 2-groups with two generators and which have coprime automorphisms also tends to infinity with their Frattini length.

(ii) It is easy to prove that for a given s the number $n(s)$ of the 2-groups of order 2^s, of class two, with two generators, which have an automorphism of order 3 (of course, we refer here to nonisomorphic such groups) is given by the following formulas:

$$n(s)=\left[\frac{s-1}{2}\right]-\frac{s}{3}+1$$

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if $s \equiv 0 \pmod 3$ and

$$n(s) = \left[\frac{s-1}{2}\right] - \left[\frac{s}{3}\right]$$

if $s \not\equiv 0 \pmod 3$.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TIMIŞOARA
ROMANIA