Can. J. Math., Vol. XXIV, No. 6, 1972, pp. 1070–1078

# THE FRIEDBERG-MUCHNIK THEOREM RE-EXAMINED

ROBERT I. SOARE

In the well-known solution to Post's problem, Friedberg [**1**] and Muchnik [**13**] each constructed a pair of incomparable recursively enumerable (r.e.) degrees $\mathbf{a}$ and $\mathbf{b}$. Subsequently, Sacks [**15**, p. 81] constructed r.e. degrees $\mathbf{c}$ and $\mathbf{d}$ such that $\mathbf{c} \cup \mathbf{d} = \mathbf{0}'$ and $\mathbf{c}' = \mathbf{d}' = \mathbf{0}'$. Lachlan showed [**7**, p. 69] that such degrees $\mathbf{c}, \mathbf{d}$ could have no greatest lower bound in the upper semi-lattice of r.e. degrees. We show that the original Friedberg-Muchnik degrees $\mathbf{a}, \mathbf{b}$ automatically satisfy Sack's conditions and hence witness that the upper semi-lattice of r.e. degrees is not a lattice.

With this example in mind we consider a class of "finite injury" priority constructions of r.e. sets formulated by Sacks [**15**, § 4], and give sufficient conditions on the construction so that Sacks' "priority set" (the set constructed in the standard way) is automatically complete. We then show that any Sacks construction designed to produce an r.e. set with some property $P$ can be combined with a method of Yates to produce a set with property $P$ and recursive in a given nonrecursive r.e. set. It follows that a Sacks construction cannot be used to produce sets such as maximal sets whose degree $\mathbf{m}$ must satisfy $\mathbf{m}' = \mathbf{0}''$ by Martin [**10**].

Although not difficult to prove, these observations apply to many constructions in the literature. For example, as consequences, we derive results by Ladner [**8**] on non-mitotic r.e. sets, and affirmatively answer McLaughlin's question [**12**] of whether in every nonrecursive r.e. degree there is an r.e. set $A$ such that $\bar{A}$ is regressive but not retraceable.

Given a set $A$, $A'$ denotes the jump of $A$, and $K$ denotes $\emptyset'$. An r.e. set $A$ is *complete* if $K$ is recursive in $A$, denoted $K \leqq_T A$. A degree $\mathbf{a} \leqq \mathbf{0}'$ is *low* if $\mathbf{a}' = \mathbf{0}'$, *high* if $\mathbf{a}' = \mathbf{0}''$, and *intermediate* otherwise. Other definitions and notation can be found in Rogers [**14**], or will be explained when introduced.

**1. The Friedberg-Muchnik theorem re-examined.** For convenience we will refer to the proof of the Friedberg-Muchnik theorem given in Rogers [**14**, p. 163], although the same analysis applies to the original constructions [**1**], and [**13**]. It is assumed that the reader is familiar with Rogers' version

Received October 5, 1971 and in revised form, March 3, 1972. This research was partially supported by NSF Grant GP-19958, and by a senior postdoctoral fellowship at the University of Illinois at Chicago Circle. The results were presented to the American Mathematical Society [**17**] during a Special Session on Recursion Theory on November 27, 1971.

where r.e. sets $A$ and $B$ are constructed along with (nonrecursive) functions $f(x)$, $g(x)$ such that

(1) $(\forall x)[[f(x) \in A \Leftrightarrow f(x) \in W_x^B]$ and $[g(x) \in B \Leftrightarrow g(x) \in W_x^A]]$.

The function $f(x) = \lim_s f(x, s)$, where $f(x, s)$ is a recursive function denoting the integer occupied by the $x$th marker on the "$A$-list" at the end of stage $s$, and similarly for $g(x)$ and the "$B$-list".

THEOREM 1.
(i) $A \oplus B \equiv_T K$, *where* $A \oplus B$ *denotes "A join B"; and*
(ii) $A' \equiv_T B' \equiv_T K$.

*Proof.* Both (i) and (ii) will follow immediately from:

$$A' \leqq_T A \oplus B, \quad \text{and} \quad B' \leqq_T A \oplus B. \tag{2}$$

To prove (2), let $h$ be a recursive function such that for any set $C$ and any index $x$,

$$W_{h(x)}{}^C = \begin{cases} N, & \text{if } x \in W_x{}^C \\ \emptyset, & \text{otherwise.} \end{cases} \tag{3}$$

Now by (1) and (3),

$$x \in W_x{}^B \Leftrightarrow f(h(x)) \in W_{h(x)}{}^B \Leftrightarrow f(h(x)) \in A,$$

and likewise for $B, A, g$ in place of $A, B, f$, respectively. It suffices to show that $f, g \leqq_T A \oplus B$, which follows easily from the observation that for all $s$ and $x$,

$$f(x, s+1) \neq f(x, s) \Rightarrow (\exists y)_{<x}[g(y, s) \in B^{s+1} - B^s], \tag{4}$$

and

$$g(x, s+1) \neq g(x, s) \Rightarrow (\exists y)_{\leqq x}[f(y, s) \in A^{s+1} - A^s]. \tag{5}$$

Simply compute $f(0), g(0), f(1), g(1), \ldots,$ in order using $A \oplus B$ as oracle.

For example, to compute $f(n)$, first compute from $A \oplus B$ all preceding values (if any) in the above list, $f(0), g(0), \ldots, f(n-1), g(n-1)$. Next find a sufficiently large stage $s_n$, such that for all $x \leqq$ any of these preceding values, $x \in A$ if and only if $x \in A^{s_n}$, and $x \in B$ if and only if $x \in B^{s_n}$. Since $A \oplus B$ is r.e., $s_n$ can be found recursively in $A \oplus B$ by the Modulus lemma [**16**, p. 23]. Clearly, $f(n, s_n) = f(n)$ by (4), and $g(n)$ is computed similarly using (5).

COROLLARY. *The Turing degrees of A and B have no greatest lower bound in the upper semi-lattice of r.e. degrees (or even of all degrees).*

The proof of Theorem 1 also applies to the Sacks' construction [**15**, p. 51] of a recursively independent sequence of r.e. sets $A_0, A_1, A_2, \ldots$, to show that the recursive join of the full sequence has *complete* r.e. degree while for any $n$, the recursive join of the deleted sequence $A_0, A_1, \ldots, A_{n-1}, A_{n+1}, \ldots$, has *low* r.e. degree, i.e., degree $a$ where $\mathbf{a}' = \mathbf{0}'$. The *recursive join* of an infinite sequence $B_0, B_1, \ldots$, is $\{p_m^{n+1} : n \in B_m\}$.

**2. Complete r.e. sets.** Frequently one constructs an r.e. set $T$ to have some property $P$ and later modifies the construction (usually by coding $K$ into $T$ on alternate stages) to produce a *complete* set $T$ with property $P$. We show that in many cases this is unnecessary because the original r.e. set constructed in the "standard" way is already complete. Another method for proving such sets complete was given by Martin [**11**] and Lachlan [**6**]. To formalize this principle we turn to a formulation by Sacks [**15**, § 4] of a broad class of finite-injury priority constructions where the requirements correspond to effectively open sets in $2^\omega$. In Sacks' scheme, there is a recursive function $t$ such that for every $s$, $t(s)$ is the index of a pair of disjoint finite sets $(F^s, H^s)$. Define $g(s) = (t(s))_2$ and *requirement*.

$$R_e = \{(F^s, H^s) : g(s) = e\}.$$

(We may always assume that $F^s \neq \emptyset$, by replacing $(\emptyset, H)$ with

$$\{(n, H) : n \notin H\}.)$$

Priorities are used to construct an r.e. set $T$, called the *priority set of $t$*, which, whenever possible, "meets" requirement $R_e$ by lying in the effectively open set,

$$\{X : (\exists s)[g(s) = e \ \& \ F^s \subseteq X \ \& \ H^s \cap X = \emptyset\}.$$

Sacks constructs $T$ as the limit of finite sets $T^s$, such that at certain stages $s + 1$, $T^{s+1} = T^s \cup F^{s+1}$. The requirement $R_e$ is *met at stage* $s + 1$ just if $e = g(s + 1)$ and $F^{s+1} \subseteq T^{s+1} - T^s$. (Note that at most one requirement is met at a given stage so that a requirement is never "accidentally" met.) The requirement $R_{g(s+1)}$ is later *injured at stage* $r + 1 > s + 1$ if $T^{r+1} \cap H^{s+1} \neq \emptyset$. Sacks' construction guarantees that each requirement is injured at most finitely often and that $T$ eventually meets every $t$-dense requirement, where $R_e$ is *$t$-dense* if for each finite set $L$, there exists $s$ such that $g(s + 1) = e$,

$$F^{s+1} \not\subseteq T^s, \ H^{s+1} \cap T^s = \emptyset \quad \text{and} \quad L \cap F^{s+1} = \emptyset.$$

THEOREM 2. *Let $t$ be a recursive function which enumerates requirements in the sense of Sacks. For the priority set $T$ of $t$ to be complete, it is sufficient that there exist recursive functions $f(e, s)$ and $h(e)$ such that for all $e$ and $s$,*

(i) *$R_e$ is met at stage $s + 1 \Rightarrow (\forall j)[f(j, s + 1) \in T^{s+1} - T^s \Leftrightarrow j = e]$,*

(ii) $f(e, s + 1) \neq f(e, s) \Rightarrow (\exists j)_{<e}[R_j$ *is met at stage* $s + 1]$, and

(iii) *$e \in K \Rightarrow R_{h(e)}$ is a $t$-dense requirement, and $e \notin K \Rightarrow R_{h(e)} = \emptyset$.*

https://doi.org/10.4153/CJM-1972-110-4 Published online by Cambridge University Press

*Remark.* Constructions satisfying (i) are frequently called "movable marker" constructions, where $f(e, s)$ denotes the integer occupied by the $e$th marker, $\Lambda_e$, at the end of stage $s$. Condition (i) asserts that marker $\Lambda_e$ is associated with an element of $T^{s+1} - T^s$ if and only if $R_e$ is met at stage $s + 1$. For most nontrivial constructions, (iii) can easily be satisfied either directly as in the proof of Theorem 1, or by use of the recursion theorem. The crucial condition for completeness of $T$ is (ii) which asserts that $\Lambda_e$ moves at $s + 1$ only if for some $j < e$, $R_j$ is met at $s + 1$, and hence by (i) contributes $f(j, s + 1) = f(j, s)$ as a "trace" in $T^{s+1} - T^s$. (For this purpose it would even suffice to have $x \in T^{s+1} - T^s$ for some $x \leqq f(j, s)$ in place of $x = f(j, s)$.)

*Proof of Theorem* 2. By (i), $f(e, s + 1) \in T^{s+1} - T^s$ only if $R_e$ is met at stage $s + 1$, which happens at most finitely often. Hence, by (ii), $f(e) = \lim_s f(e, s)$ exists and is recursive in $T$. To see that $K \leqq_T T$, fix $e$, and exactly as in Theorem 1 compute recursively in $T$,

$$s_e = (\mu s)(\forall t)_{>s}[f(h(e), s) = f(h(e), t)].$$

If $R_{h(e)}$ was met at some stage $s < s_e$, then $e \in K$ by (iii). Otherwise, we claim

$$e \in K \Leftrightarrow f(h(e)) \in T.$$

By (i) if $R_{h(e)}$ was not met at some stage $s < s_e$, then $f(h(e)) \notin T^{s_e}$. If $e \notin K$, then $R_{h(e)} = \emptyset$ by (iii) so $f(h(e))$ is never added to $T$ at $s > s_e$ by (i). If $e \in K$, then $R_{h(e)}$ is $t$-dense by (iii), and so is met at some stage $s + 1 \geqq s_e$ (since every $t$-dense requirement is eventually met) at which stage $f(h(e)) \in T^{s+1} - T^s$ by (i).

In addition to the Friedberg-Muchnik construction of $A \oplus B$, Theorem 2 applies to many constructions in the literature including McLaughlin's construction [**12**] of an r.e. set $A$ such that $\bar{A}$ is regressive but not retraceable, and Lachlan's construction [**5**, Lemma 4] of a non-mitotic r.e. set. The following sketch is from Ladner [**8**, § 3] (with one minor change explained below) and gives an alternate proof of his result [**8**, Theorem 3] that there is a complete such set.

An r.e. set $A$ is *non-mitotic* if it cannot be decomposed as the disjoint union of r.e. sets $B$ and $C$ such that $B \equiv_T C \equiv_T A$. If $A$, $B$ and $C$ are r.e. sets and $\Theta$, $\Psi$ are partial recursive functionals, $(B, C, \Theta, \Psi)$ is a *mitotic splitting* of $A$ if $(B, C)$ is an r.e. splitting of $A$, and $\Theta(B) = A = \Psi(C)$. To insure that the $e$th such quadruple $(B_e, C_e, \Theta_e, \Psi_e)$ is not a mitotic splitting of $A$, we designate a certain element, say $f(e, s)$, which remains in $\bar{A}^s$ until some stage $s + 1$ where $(B_e, C_e, \Theta_e, \Psi_e)$ "threatens" to mitotically split $A$ because,

$$\Theta_e^s(B_e^s; x) = \Psi_e^s(C_e^s; x) = A^s(x), \text{ for } x = f(e, s), \tag{6}$$

$B_e{}^s \cap C_e{}^s = \emptyset$, and $B_e{}^s(x) \cup C_e{}^s(x) = A^s(x)$ for all $x \leqq k$, the maximum number in $B_e{}^s \cup C_e{}^s$ used in either computation in (6). One can then meet the $e$th requirement by enumerating $f(e, s) \in A^{s+1} - A^s$ and refraining from later enumerating in $A$ any $x \leqq k$. (In Sacks' formulation, one enumerates in requirement $R_e$ the pair $(\{f(e, s)\}, H)$, where $H = \{x : x \notin A^s \text{ and } x \leqq k\}$.) Since $f(e, s) \in A^{s+1} - A^s$ can be enumerated in only *one* of $B_e{}^{s+1}$, $C_e{}^{s+1}$, the other maintains the computation of (6) which $\neq A^{s+1}(f(e, s))$.

This construction clearly satisfies hypotheses (i) and (ii) of Theorem 2. To meet hypothesis (iii), use the recursion theorem to find a quadruple $(B_{h(e)}, C_{h(e)}, \Theta_{h(e)}, \Psi_{h(e)})$ which never threatens if $W_e = \emptyset$, and if $W_e \neq \emptyset$, continues to threaten until requirement $R_{h(e)}$ is met and not injured thereafter. (Ladner replaces "$x = f(e, s)$" by "all $x \leqq f(e, s)$" in (6), in which case we do not know whether (iii) applies.)

**3. Combining Sacks' constructions with Yates' permitting.** After trying to obtain a *complete* r.e. set with some property $P$, the recursion theorist frequently turns to *incomplete* r.e. degrees. In this section we show that for any nonrecursive r.e. set $C$, a Sacks construction which produces sets with property $P$ can always be combined with the "permitting" method of Yates [**18**] to produce an r.e. set $T \leqq_T C$ having property $P$. Corollaries include Ladner's result [**8**, Theorem 4] that there is a non-mitotic r.e. set below any nonrecursive r.e. degree, an answer to McLaughlin's question [**12**] concerning the degrees of r.e. sets $A$ such that $A$ is regressive and not retraceable, and a more natural proof of the relativized Friedberg theorem than Friedberg's original proof [**2**], which is repeated in Rogers [**14**, p. 175]. A further consequence is the fact that a Sacks construction cannot be used to produce an r.e. set with a property such as maximality which implies that $A$ has *high* r.e. degree. The following method has often appeared in the literature after being introduced by Yates [**18**].

Given a nonrecursive r.e. set $C$, let $h$ be a $1:1$ recursive function which enumerates $C$. Following Yates, we will insure $T \leqq_T C$ by allowing $x \in T^{s+1} - T^s$ only if $x \geqq h(s + 1)$, in which case we say that $C$ *permits* $x \in T^{s+1} - T^s$. To meet Sacks' requirement $R_e$ we cannot immediately enumerate $F^r$ in $T$ when $(F^r, H^r) \in R_e$, but must wait for some stage $t$, where $h(t) \leqq \min F^r$. (For a finite set $S$, $\max S$ and $\min S$ denote the largest and smallest elements in $S$.) In the meantime we add $r$ to a set $K_e$ indicating that the elements of $F^r$ are "candidates" for meeting $R_e$, and we search for a new pair $(F^s, H^s) \in R_e$. Meanwhile we refrain from appointing any $x \leqq k_e = \max H^r$ as a candidate for any requirement $R_i$, $i \geqq e$. The modified construction yields an r.e. set $T$ which meets every $t$-dense requirement, because the least $t$-dense requirement not met would leave an infinite r.e. sequence of candidates witnessing corresponding initial segments of $C$ which have settled, thereby showing $C$ recursive contrary to hypothesis. In the following construction of $T$, we say that the requirement $R_e$ is *satisfied through stage s* if at some stage $r + 1 \leqq s$, $F \subseteq T^{r+1} - T^r$ for some $(F, H) \in R_e$ and $H \cap T^s = \emptyset$.

THEOREM 3. *Let t be a recursive function which enumerates requirements in the sense of Sacks. Let C be an arbitrary nonrecursive r.e. set. Then there is an r.e. set T such that*

(i) $T \leqq_T C$,

(ii) *for every t-dense requirement* $R_e$, *there exists s, such that* $R_e$ *is satisfied through all stages* $r \geqq s$.

*Proof.* At stage $s$ in the construction of $T$ we will define for each $e$, a finite set $K_e^s$ and a number $k(e, s)$, which correspond to the $K_e$ and $k_e$ intuitively described above. Recall that at stage $r$, $t$ enumerates $(F^r, H^r)$ in $R_e$, where $e = g(r)$. In the following construction we appoint new candidates at odd stages, and permit elements to enter $T$ at even stages.

*Stage* $s + 1$ *where* $s + 1$ *is odd.* Let $T^{s+1} = T^s$. Let $g(r) = e$, where $s + 1 = 2r + 1$. Define $K_j^{s+1} = K_j^s$, and $k(j, s + 1) = k(j, s)$ for all $j$ unless:

(i) $R_e$ is not satisfied through $s$,

(ii) $\min F^r > k(i, s)$, for all $i \leqq e$, and

(iii) $F^r \nsubseteq T^s$ and $H^r \cap T^s = \emptyset$,

in which case define $K_e^{s+1} = K_e^s \cup \{r\}$, and

$$k(e, s + 1) = \max(F^r \cup H^r \cup \{k(e, s)\}),$$

$$K_j^{s+1} = \emptyset, \qquad k(j, s + 1) = 0 \text{ for all } j > e, \text{ and}$$

$$K_j^{s+1} = K_j^s, \quad k(j, s + 1) = k(j, s) \text{ for all } j < e.$$

*Stage* $s + 1$ *where* $s + 1$ *is even.* Let $T^{s+1} = T^s$, $K_j^{s+1} = K_j^s$ and $k(j, s + 1) = k(j, s)$ for all $j$ unless there exists $r < s/2$ such that:

(i) $r \in K_j^s$, where $j = g(r)$, and

(ii) $h((s + 1)/2) \leqq \min F^r$,

in which case let $e$ be the least such $j$ and $u$ the least $r$ corresponding to $e$. Define $T^{s+1} = T^s \cup F^u$, $K_e^{s+1} = \emptyset$, and $k(e, s + 1) = k(e, s)$. For all $j \neq e$, define $K_j^{s+1}$ and $k(j, s + 1)$ as above. Let $T = \bigcup_{s=0}^{\infty} T^s$.

*Definition.* If the second alternative holds at stage $s + 1$, we say that requirement $R_e$ *receives attention at stage* $s + 1$.

LEMMA 1. *If* $F^u \subseteq T^{s+1} - T^s$ *and* $u \in K^s_{g(u)}$, *then* $T^{s+1} \cap H^u = \emptyset$, *so that* $R_{g(u)}$ *is satisfied through* $s + 1$.

*Proof.* Suppose $F^u \subseteq T^{s+1} - T^s$ and $u \in K_e^s$, where $e = g(u)$. Let $v$ be the least $r$ such that $u \in K_e^r$ for all $r$, $v \leqq r \leqq s$. Then $k(e, v) \geqq \max H^u$, but for all $x \leqq k(e, v)$, $x \in A^s$ if and only if $x \in A^v$, since otherwise for some $i \leqq e$ requirement $R_i$ received attention at some even stage $r$, $v \leqq r < s$, in which case $K_e^{r+1} = \emptyset$.

https://doi.org/10.4153/CJM-1972-110-4 Published online by Cambridge University Press

LEMMA 2. *Each requirement receives attention at most finitely often.*

*Proof.* Suppose $e$ is the least $i$, such that requirement $R_i$ receives attention infinitely often. Choose $s_0$ so that no $R_i$, $i < e$, receives attention at any stage $s + 1 \geqq s_0$. Now $R_e$ never receives attention at an even stage $s + 1 \geqq s_0$, else by Lemma 1, $R_e$ is satisfied through $s + 1$, and clearly remains satisfied thereafter. Hence, $K_e^{s+1} \supseteq K_e^s$ for all $s \geqq s_0$, and $R_e$ receives attention at infinitely many odd stages $s + 1 > s_0$, at which some $u \in K_e^{s+1} - K_e^s$. For such an $s$,

$$C^{s/2} \cap [0, \min F^u] = C \cap [0, \min F^u] \tag{7}$$

else $R_e$ receives attention at some even stage $r > s + 1$. Furthermore if $u, v \in K_e^s$ and $u \neq v$, then $F^u \cap F^v = \emptyset$ by the definition of $k(e, s + 1)$ and condition (ii) for $s + 1$ odd. Hence,

$$\{\min F^u : (\exists\, s)_{\geqq s_0}[u \in K_e^{s+1} - K_e^s]\}$$

is an infinite r.e. set which by (7) implies that $C$ is recursive, contrary to hypothesis.

LEMMA 3. *For every t-dense requirement $R_e$, there exists $s$, such that $R_e$ is satisfied through all stages $r \geqq s$.*

*Proof.* Let $R_e$ be $t$-dense. By Lemma 2, choose $s$ such that all $R_i$, $i \leqq e$, have ceased to receive attention at any stage $r \geqq s$. Then $R_e$ is satisfied through all $r \geqq s$, because otherwise $R_e$ receives attention at some odd $r \geqq s$ by the construction and the definition of $T$ being $t$-dense, where

$$L = \{n : n \leqq \max_{i \leqq e} k(i, s)\}.$$

**4. Further comments.** It is natural to ask under what conditions a "maximum degree principle" like Theorem 2 holds for the Yates permitting constructions of Theorem 3. For example, if a construction satisfying the hypotheses of Theorem 2 is combined with the method of Theorem 3 to produce $T \leqq_T C$, does $C \leqq_T T$ automatically follow? In general the answer is no. For example, it fails for the Friedberg construction in Theorem 1, contrary to our announcement [**17**, Theorem 2], although the counterexample is fairly complicated.

The principle does hold, however, for certain special constructions where there is no negative restraint, that is, where a Sacks' requirement $R_e$ contains only pairs of the form $(F, \emptyset)$. In this case the construction of Theorem 3 may be simplified by not cancelling candidates $K_i^s$ of lower priority than $R_e$, $e < i$, whenever $R_e$ receives attention. For example, Yates [**18**, Theorems 1 and 2] used this method to produce a simple set $S$ and semicreative set $A$ recursive in $C$, an arbitrary nonrecursive r.e. set. He then took joins of $S$ and $A$ with sets of degree $C$ to produce such sets of the *same* degree as $C$. Jockusch and Soare [**3**, Lemma 5.1] have shown the latter to be unnecessary since Yates' *original* sets $S$ and $A$ automatically satisfy $C \leqq_T S$, $C \leqq_T A$. Jockusch modified the method to give a simple proof of Martin's theorem [**10**] that there is a maximal set in any *high* r.e. degree.

In a permitting construction of $T \leqq_T C$, if the recursion theorist cannot achieve $C \leqq_T T$ automatically (or by later taking a join) he may want to code $C$ into $T$ *during* the construction. In general this is impossible even if the unmodified construction produces a *complete* r.e. set. For example, Ladner [**9**] has produced a nonrecursive r.e. degree which contains no non-mitotic r.e. set. However, if the requirements admit a bound on negative restraint (for example, if there is a recursive function $h$ such that for all $e$, $(F, H) \in R_e$ implies that $H$ has $\leqq h(e)$ members) then frequently $C$ *can* be coded into $T$ during the permitting construction of $T \leqq_T C$. By such a method we have shown that for every non-recursive r.e. set $C$, there exists an r.e. set $A \equiv_T C$, such that $\bar{A}$ is regressive but not retraceable, answering a question of McLaughlin [**11**].

*Acknowledgement.* The author is grateful to M. Lerman for posing the question answered in Theorem 1(i), and particularly to C. G. Jockusch Jr. for several valuable suggestions and corrections.

## References

**1.** R. M. Friedberg, *Two recursively enumerable sets of incomparable degrees of unsolvability*, Proc. Nat. Acad. Sci. U.S.A. *43* (1957), 236–238.

**2.** ——— *The fine structure of degrees of unsolvability of recursively enumerable sets*, Summaries of Cornell University Summer Institute for Symbolic Logic (1957), 404–406.

**3.** C. G. Jockusch Jr. and R. I. Soare, *Degrees of members of $\pi_1^0$ classes*, Pacific J. Math. *40* (1972), 605–616.

**4.** A. H. Lachlan, *Lower bounds for pairs of recursively enumerable degrees*, Proc. London Math. Soc. *16* (1966), 537–569.

**5.** ——— *The priority method. I*, Zeitschr. f. math. Logik und Grundlagen Math. *13* (1967), 1–10.

**6.** ——— *Complete recursively enumerable sets*, Proc. Amer. Math. Soc. *19* (1968), 99–102.

**7.** ——— *The priority method for the construction of recursively enumerable sets* (to appear).

**8.** R. E. Ladner, *Mitotic recursively enumerable sets* (to appear).

**9.** ——— *A completely mitotic nonrecursive recursively enumerable degree* (to appear).

**10.** D. A. Martin, *Classes of r.e. sets and degrees of unsolvability*, Zeitschr. f. math. Logik und Grundlagen Math. *12* (1966), 295–310.

**11.** ——— *Completeness, the recursion theorem, and effectively simple sets*, Proc. Amer. Math. Soc. *17* (1966), 838–842.

**12.** T. G. McLaughlin, *Retraceable sets and recursive permutations*, Proc. Amer. Math. Soc. *17* (1966), 427–429.

**13.** A. A. Muchnik, *Negative answer to the problem of reducibility of the theory of algorithms* (Russian), Dokl. Akad. Nauk SSSR *108* (1956), 194–197.

**14.** H. Rogers Jr., *Theory of recursive functions and effective computability* (McGraw-Hill, New York, 1967).

**15.** G. E. Sacks, *Degrees of unsolvability*, Ann. of Math. Studies No. 55 (Princeton University Press, Princeton, New Jersey, 1963).

**16.** J. R. Shoenfield, *Degrees of unsolvability* (North-Holland Publishing Company, Amsterdam, 1971).

**17.** R. I. Soare, *The Friedberg-Muchnik theorem re-examined* (abstracts), Notices Amer. Math. Soc. *18* (1971), 381–382, and 1086.

**18.** C. E. M. Yates, *Three theorems on the degrees of recursively enumerable sets*, Duke Math. J. *32* (1965), 461–468.

*University of Illinois at Chicago Circle,*
*Chicago, Illinois*