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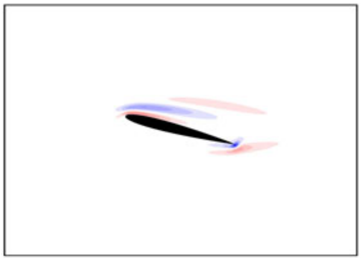
A basis for flow modelling



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Reduced-order models are often sought to efficiently represent key dynamical phenomena present among the broad range of temporal and spatial scales associated with unsteady and turbulent flow problems. Linear 'input–output' approaches and resolvent analyses reveal that important information about the most dangerous (most amplified) disturbances and the corresponding fluctuation response can be found with knowledge only of the base flow, or the turbulent mean field. In the work by Padovan *et al.* (*J. Fluid Mech.*, vol. 900, 2020, A14), an important advance is made with regards to flows which have a periodically time-varying base flow, for example during unsteady vortex shedding from a body. By forming a harmonic resolvent relative to this base flow, limitations associated with the traditional linear resolvent are overcome to determine efficient bases for modelling of limit cycle flows and reveal novel information about key triadic (resonant) interactions.

Key words: mathematical foundations, control theory

1. Introduction

Unsteady and turbulent flows have long provided intellectual challenge to the scientist and engineer alike. Physics-based models capable of replicating the key characteristics for practical applications at reasonable cost remain few and far between. One may ask, why is reduced-order flow modelling so hard? First, and most obviously, the Navier–Stokes equations (NSE) are nonlinear and support a multiscale energy cascade. Another issue is the non-normality of the Navier–Stokes operator when linearized relative to a base flow (Trefethen *et al.* 1993). Analysis of the eigenvalue spectrum does not guarantee the most efficient representation of the dynamics of a given flow because the eigenvectors of a non-normal system do not form an orthogonal basis. This foundational concept underlies transient disturbance growth in linearly stable systems, recognized as important for non-laminar as well as laminar flows, (e.g. Farrell & Ioannou 1993).

While there is a long history of studying the impact of non-normality on disturbance growth, the problem was reframed into a system-theoretic, 'input–output' or transfer function form for laminar flows in the seminal contribution of Jovanović & Bamieh (2005). This approach rewrites the NSE linearized relative to the laminar solution in terms of a transfer function between input disturbances giving rise to an output state response,

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admitting the possibility of a response driven by harmonic or other time-dependent, exogenous inputs. At the heart of the transfer function is the resolvent; its spectral properties reflect the solution, or base flow, about which the linearization was performed. Subsequently, McKeon & Sharma (2010) demonstrated that the resolvent formulated relative to the turbulent mean flow holds important information about disturbance amplification, an example where the base flow is not an equilibrium solution of the NSE. In this case, the term nonlinear in the fluctuations appears naturally as an endogenous forcing to the system, which may coexist relative to external disturbances or control signals, and the (academic, at least) possibility to provide nonlinear closure to the system. A pseudospectral analysis, or analysis of the linear operator perturbed by a nonlinear (or other) forcing is then more appropriate than eigenanalysis. A recent extension by Rigas, Sipp & Colonius (2020) considers a nonlinear input–output analysis, using the harmonic balance model to directly include a subset of frequency interactions in the resolvent operator. For broader reviews of resolvent and input–output analyses, especially as applied to unsteady and turbulent flows, the reader is directed to McKeon (2017) and Jovanovic (2020).

Remarkably, the resolvent typically displays low-rank characteristics preferentially at scales where physical mechanisms are active for a given flow. A consequence is that the resolvent can be efficiently approximated using a truncated expansion in terms of bases obtained from a singular value decomposition (SVD); the singular vectors identify efficient linear bases to represent the most amplified inputs and corresponding outputs, with the gains given by the singular values. The cost of performing the SVD can be dramatically less than that of a direct numerical simulation. However, the traditional approach is significantly less effective in the presence of a significant/dominant time scale, such as flows with vortex shedding or other limit cycle behaviours. The basis can be improved by modelling the nonlinear forcing (rather than approximating the linear resolvent), but at the cost of additional computation. In their recent paper, Padovan, Otto & Rowley (2020) provide an elegant approach to overcome this limitation on the linear analysis.

2. Overview: linear modelling in flows with dominant frequencies

The common (and most simple to obtain) choice for the base flow to enter the resolvent formulation is a temporally averaged mean field, either under a locally parallel assumption or reflecting, e.g., downstream spatial growth. Mean flows which reflect activity at a dominant spanwise spatial frequency have also been investigated (e.g. Rosenberg & McKeon 2019), while the resolvent can be modified to account for nonlinear interactions associated with spatially varying wall geometry (e.g. Chavarin & Luhar 2020). Scenarios with one or more dominant temporal frequencies in the base flow had remained relatively unexplored until the recent work of Padovan *et al.* (2020).

The distribution of information between the linear and nonlinear terms in a state space representation is dictated by the choice of base flow (e.g. Karban *et al.* 2020). Specifically, it may not be appropriate to treat fluctuations as small across all scales if the flow contains a natural, energetic, temporally periodic component. This has the effect of reducing the efficiency of the linear analysis and placing more weight on the details of the nonlinear forcing acting as input to the linear dynamics. This impacts the quality of flow reconstruction using the linear resolvent basis, (e.g. Symon *et al.* 2020), although further analysis of the nonlinear terms can improve the agreement between model and data (Rosenberg, Symon & McKeon 2019).

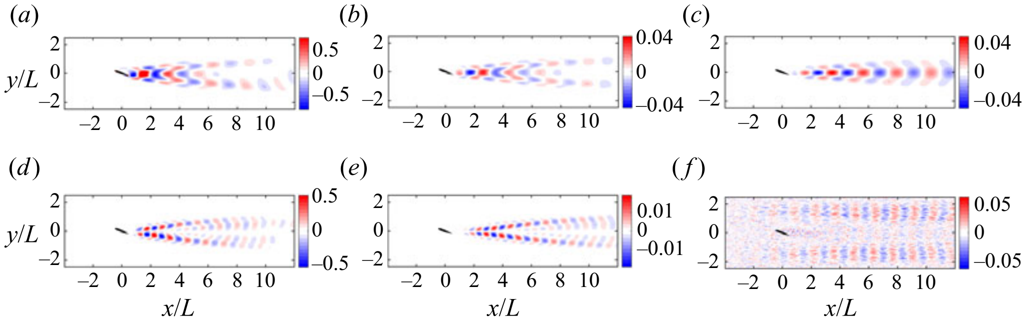

FIGURE 1. The harmonic resolvent (*b*,*e*) provides a much more accurate representation than the traditional resolvent (*c*,*f*) of the response to small amplitude input forcing at the frequency of unsteady vortex shedding for a two-dimensional NACA 0012 airfoil in incompressible flow at 20° angle of attack and chord Reynolds number $Re = 200$ (*a*,*d*). Panels (*a*–*c*) and (*d*–*f*) show responses at the input frequencies, ω, and 2ω, respectively. Compilation from figures 8–10 in Padovan *et al.* (2020).

For a flow containing a set of natural energetic temporal frequencies, Ω_b (perhaps a filtered representation of large scale structure), a triple decomposition of the instantaneous field into the sum of the (zero frequency) mean contribution, the dominant frequency (Ω_b) signal and the remaining fluctuations can be performed. Constructing an analysis using the sum of the mean and periodic components as the base flow leads to the formulation of the harmonic resolvent. Incorporating more information about the flow physics via this new, periodic base flow means that the linear analysis machinery identifies perturbations that are small relative to that base flow, and multi-frequency, since they can reflect scattering off the frequencies present in the base flow. Information on cross-frequency coupling is an integral part of this linear analysis; the forcing inputs corresponding to the most amplified triadically consistent, or resonant, interaction with each component of the periodic signal, Ω_b, can be identified. Padovan *et al.* (2020) consider only frequencies that are harmonics of the periodic signal, but note that the analysis is not *a priori* restricted to integer values of the frequencies contained within Ω_b.

The evidence is convincing that the harmonic resolvent provides an effective basis (figure 1) for the example of unsteady vortex shedding from a two-dimensional airfoil. The agreement is improved at the fundamental vortex shedding frequency, ω. However, disturbance amplification at 2ω is simply not captured by the resolvent based on the temporal mean. These interactions are confined to the nonlinear portion of the traditional analysis, but are encoded in the linear dynamics associated with the harmonic resolvent.

3. Future directions: modelling and control

Armed with the harmonic resolvent machinery, what can now be achieved? Clean partitioning of linear and nonlinear dynamics in the presence of a strong signal with non-zero frequency, such that effective input and (multi-frequency) output basis functions are obtained from the linear analysis, has important implications for model reduction techniques. Similarly, the sensitivity information revealed by the singular values, and especially the integral identification of important cross-frequency interactions, can be used to drive the design of effective control inputs, perhaps restricted in scale or variable via a masking operation applied to the harmonic resolvent. It will be interesting to see whether the resonant interactions revealed by harmonic analysis can be used for mean flow modification and suppression of the periodic behaviour itself.

An important next step will be an extension to consider the treatment of frequencies that may not be harmonics of the base flow frequency content. Such a scenario arises naturally with uncertainty in non-ideal flows and with real actuators. Related is the question of an extension of the approach to fully turbulent flows, where a careful selection of the appropriate frequencies to include in the base flow from the many energetic scales will be required. The mean or base flow is a required input for resolvent analysis; the harmonic resolvent also needs an *a priori* characterization of the dominant frequency activity. Data-driven methods may enable a bootstrapping that weights the most amplified response of the traditional resolvent analysis at Ω_b to provide an estimate of the periodic component of the harmonic resolvent analysis.

The work of Padovan *et al.* (2020) provides important insight into linear amplification mechanisms in periodically time-varying base flows, providing an exciting basis (or, rather, the most efficient linear bases) for flow modelling.

Declaration of interest

The author reports no conflict of interest.

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