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ARITHMETIC PROPERTIES OF INFINITE PRODUCTS OF CYCLOTOMIC POLYNOMIALS

PETER BUNDSCHUH✉ and KEIJO VÄÄNÄNEN

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Abstract

We study transcendence properties of certain infinite products of cyclotomic polynomials. In particular, we determine all cases in which the product is hypertranscendental. We then use various results from Mahler's transcendence method to obtain algebraic independence results on such functions and their values.

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1. Introduction and main results

For an integer $\ell > 1$, let $\Phi_\ell(x)$ denote the ℓth cyclotomic polynomial and put $\Phi_1(x) := 1 - x$ (which is minus the first cyclotomic polynomial). If $d \geq 2$ is an integer, the infinite product

$$F_{d,\ell}(z) := \prod_{j\geq 0} \Phi_\ell(z^{d^j}) \in \mathbb{Z}[[z]]$$

compactly converges on the open unit disk $\mathbb{D}$ and defines there a zero-free holomorphic function. Note that $F_{2,1}(z)$ is the generating function of the Thue–Morse sequence on $\{-1, 1\}$ beginning with 1 and $F_{2,3}(z)$ is the generating function of the Stern diatomic sequence.

Given the arithmetical characterisation of the Taylor coefficients of $F_{d,\ell}(z)$ about the origin, we deduce (using [10, pages 368–371] or [14, Aufgabe 167, page 143]) that $F_{d,\ell}(z)$ *is either a rational function or transcendental over* $\mathbb{C}(z)$. Carlson's famous work [6] on power series with integer coefficients gives the stronger statement: $F_{d,\ell}(z)$ *is either a rational function or has the unit circle as a natural boundary.*

Very recently, Duke and Nguyen [9] studied the analytic properties of the above infinite product for *primes* d and positive integers ℓ. They proved in [9, Theorem 1.1] that $F_{d,\ell}(z)$ is rational if and only if d divides ℓ, in which case

$$F_{d,\ell}(z) = 1/\Phi_{\ell/d^k}(z^{d^{k-1}}), \quad k := \mathrm{ord}_d \ell.$$

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These authors did not consider the case of *composite* d. This question is completely answered by our first main result, which we will formulate after recalling the following definition. An analytic function is said to be *hypertranscendental* if it satisfies no algebraic differential equation, that is, no finite collection of derivatives of the function is algebraically dependent over $\mathbb{C}(z)$.

THEOREM 1.1. *For given positive integers d, ℓ with $d \geq 2$, the following three statements are equivalent:*

(i) *d is composite or does not divide ℓ;*
(ii) *$F_{d,\ell}(z)$ is hypertranscendental;*
(iii) *$F_{d,\ell}(z)$ is not a rational function.*

The proof of the hardest part of this theorem, the implication (i) implies (ii), will be given in Section 2. Knowing (ii), it is trivial that $F_{d,\ell}(z)$ is transcendental over $\mathbb{C}(z)$ and hence also that $F_{d,\ell}(z)$ is not a rational function. But, then, from the result of Duke and Nguyen, statement (i) must be valid.

In Section 3, the implication (i) implies (ii) will be used to establish arithmetical applications, such as the following theorem.

THEOREM 1.2. *Suppose that d is composite or does not divide ℓ. Then, for any algebraic α with $0 < |\alpha| < 1$, the numbers $F_{d,\ell}(\alpha), F'_{d,\ell}(\alpha), F''_{d,\ell}(\alpha), \ldots$ are algebraically independent (over $\mathbb{Q}$).*

Note that the hypotheses on d and ℓ in this theorem just guarantee the hypertranscendence of the function $F_{d,\ell}(z)$.

In Section 4, we first study, for fixed d, the algebraic independence over $\mathbb{C}(z)$ of all functions $F_{d,\ell}(z)$, where ℓ satisfies $\gcd(d, \ell) = 1$. We denote this infinite set by S_d and remark that it is a subset of

$$T_d := \{\ell : (d, \ell) \text{ satisfies condition (i) of Theorem 1.1}\}.$$

More precisely, one can easily see that $T_d = S_d$ if d is a prime, and $T_d = \mathbb{N}$ if d is composite. By Theorem 1.1, for every $\ell \in T_d$, the function $F_{d,\ell}(z)$ is transcendental over $\mathbb{C}(z)$. Indeed, we can prove an algebraic independence result.

THEOREM 1.3. *For any fixed integer $d \geq 2$, the functions $F_{d,\ell}(z)$ with $\ell \in S_d$ are algebraically independent over $\mathbb{C}(z)$.*

PROBLEM 1.4. Is it possible to prove the same statement with T_d instead of S_d?

An arithmetical application of Theorem 1.3 is contained in the following result.

THEOREM 1.5. *Let $d \geq 2$ be fixed. Then, for any algebraic α with $0 < |\alpha| < 1$, all numbers $F_{d,\ell}(\alpha)$ with $\ell \in S_d$ are algebraically independent (over $\mathbb{Q}$).*

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In Section 5, we shall extend these results by simultaneously considering the functions $F_{d,\ell}(z)$ and another class of functions.

The main tool in most of our proofs in Sections 3–5 is Mahler's method for transcendence and algebraic independence. This method was created by Mahler as early as 1929–1930. It is mainly applicable if the functions involved satisfy certain functional equations, of which a particularly simple example is

$$F_{d,\ell}(z) = \Phi_\ell(z) F_{d,\ell}(z^d) \tag{1.1}$$

coming from the product definition of $F_{d,\ell}(z)$.

Finally, in Section 6, we will show how more recent transcendence criteria based on Schmidt's subspace theorem can also be used in the present realm. Such criteria have been systematically developed by Corvaja and Zannier [7]. As an example, we will give a full proof of a very particular case in Theorem 1.2.

THEOREM 1.6. *Suppose that d is composite or does not divide ℓ. Then, for any algebraic α with $0 < |\alpha| < 1$, the number $F_{d,\ell}(\alpha)$ is transcendental.*

2. Proof of the characterisation theorem (Theorem 1.1)

At the heart of the proof of the implication (i) implies (ii) in Theorem 1.1 are the following two lemmas. The first of these can be found in [4, Theorem 1] and recently has been generalised in [8, Proposition 3.1 and Remark 3.2].

LEMMA 2.1. *Let $P \in \mathbb{C}[z]$ be nonconstant with $P(0) = 1$. If the functional equation*

$$w(z) - dw(z^d) = \frac{zP'(z)}{P(z)}$$

has no solution $w \in \mathbb{C}(z)$, then the infinite product $\prod_{j\ge 0} P(z^{d^j})$ is hypertranscendental.

LEMMA 2.2. *Suppose that $d \ge 2$. If the functional equation*

$$w(z) - dw(z^d) = \frac{z\Phi'_\ell(z)}{\Phi_\ell(z)} \tag{2.1}$$

has a solution in $\mathbb{C}(z)$, then d is a prime dividing ℓ.

Having these two auxiliary results, we know from Lemma 2.2 that (i) implies that the functional equation (2.1) has no rational solution. But, then, according to Lemma 2.1, the product defining $F_{d,\ell}(z)$ is hypertranscendental, whence (ii) holds.

PROOF OF LEMMA 2.2. Suppose that (2.1) has a rational solution, which we can write as $w(z) = U(z)/V(z)$ with coprime $U, V \in \mathbb{Z}[z]\backslash\{0\}$. Then, from (2.1),

$$\Phi_\ell(z)(U(z)V(z^d) - dU(z^d)V(z)) = z\Phi'_\ell(z)V(z)V(z^d), \tag{2.2}$$

implying $\deg U = \deg V$. Moreover, using the coprimality of $U(z^d), V(z^d)$, we get the divisibility relation $V(z^d) \mid \Phi_\ell(z)V(z)$ (in $\mathbb{Q}[z]$). Thus, there exists $v \in \mathbb{Q}[z]\backslash\{0\}$ satisfying

$$V(z^d)v(z) = \Phi_\ell(z)V(z), \tag{2.3}$$

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leading to

$$\deg v = \varphi(\ell) - (d-1)\deg V \quad \text{and hence } \deg V \le \frac{\varphi(\ell)}{d-1},$$

where $\varphi(\,.\,)$ denotes Euler's totient function. If $\deg V = 0$, then $\deg U = 0$ and (2.2) gives a contradiction. Therefore, we have $\deg V > 0$ and hence $\deg v < \varphi(\ell)$, implying $\Phi_\ell(z) \nmid v(z)$. Hence, $\Phi_\ell(z) \mid V(z^d)$ and $V(\zeta_\ell^d) = 0$, where here and subsequently we write $\zeta_t := e^{2\pi i/t}$ for any positive integer t.

For the remainder of this section, we define $k \in \mathbb{N}_0, \tilde{\ell} \in \mathbb{N}$ by $\ell = d^k \tilde{\ell}$ with $d \nmid \tilde{\ell}$, and set $\ell_i := \ell/d^i$ $(i = 0, \ldots, k)$, so that $\ell_0 = \ell, \ell_k = \tilde{\ell}$.

If this k is positive, then $0 = V(\zeta_\ell^d) = V(\zeta_{\ell_1})$ and thus $\Phi_{\ell_1}(z) \mid V(z)$ and moreover $V(z) = \Phi_{\ell_1}(z)V_1(z)$ with some $V_1 \in \mathbb{Q}[z]\backslash\{0\}$. Then (2.3) leads to

$$\Phi_{\ell_1}(z^d)V_1(z^d)v(z) = \Phi_\ell(z)\Phi_{\ell_1}(z)V_1(z). \tag{2.4}$$

Suppose that $k \ge 2$. Since $\Phi_\ell(z) \mid \Phi_{\ell_1}(z^d)$ and $\varphi(\ell) = \varphi(d\ell_1) = d\varphi(\ell_1)$ (the last equation being valid if $k \ge 2$), we obtain $\Phi_\ell(z) = \Phi_{\ell_1}(z^d)$ and hence from (2.4)

$$V_1(z^d)v(z) = \Phi_{\ell_1}(z)V_1(z) \quad \text{giving } \deg v = \varphi(\ell_1) - (d-1)\deg V_1. \tag{2.5}$$

If $\deg V_1 = 0$, then $V(z) = c_1\Phi_{\ell_1}(z)$ with $c_1 \in \mathbb{Q}^\times$ and (2.2) leads, after cancellation by $c_1\Phi_\ell(z) = c_1\Phi_{\ell_1}(z^d)$, to

$$U(z)\Phi_\ell(z) - dU(z^d)\Phi_{\ell_1}(z) = c_1 z\Phi_\ell'(z)\Phi_{\ell_1}(z).$$

This equation implies $\Phi_{\ell_1}(z) \mid U(z)\Phi_\ell(z)$ and hence $\Phi_{\ell_1}(z) \mid U(z)$, and so $\Phi_{\ell_1}(z)$ is a common factor of $U(z), V(z)$, contradicting the coprimality of these polynomials. Thus, we conclude that $\deg V_1 > 0$ and hence $\deg v < \varphi(\ell_1)$ from (2.5), and therefore $\Phi_{\ell_1}(z) \nmid v(z)$. By (2.5), we have $V_1(\zeta_{\ell_1}^d) = V_1(\zeta_{\ell_2}) = 0$, whence $\Phi_{\ell_2}(z) \mid V_1(z)$, giving $V_1(z) = \Phi_{\ell_2}(z)V_2(z)$ with some $V_2 \in \mathbb{Q}[z]\backslash\{0\}$. Using (2.5) we find, as an analogue to (2.4), that

$$\Phi_{\ell_2}(z^d)V_2(z^d)v(z) = \Phi_{\ell_1}(z)\Phi_{\ell_2}(z)V_2(z).$$

If $k \ge 3$, then $\Phi_{\ell_2}(z^d) = \Phi_{\ell_1}(z)$ holds and therefore

$$V_2(z^d)v(z) = \Phi_{\ell_2}(z)V_2(z) \quad \text{implying } \deg v = \varphi(\ell_2) - (d-1)\deg V_2.$$

By repeating this procedure,

$$V_{k-1}(z^d)v(z) = \Phi_{\ell_{k-1}}(z)V_{k-1}(z), \quad \text{giving } \deg v = \varphi(\ell_{k-1}) - (d-1)\deg V_{k-1}. \tag{2.6}$$

If now $\deg V_{k-1} = 0$, then $V_{k-1}(z) = c_{k-1} \in \mathbb{Q}^\times$ and

$$V(z) = \Phi_{\ell_1}(z)V_1(z) = \Phi_{\ell_1}(z)\Phi_{\ell_2}(z)V_2(z) = \cdots = \Phi_{\ell_1}(z)\cdots\Phi_{\ell_{k-1}}(z)c_{k-1} \tag{2.7}$$

and substitution of this in (2.2) yields, after cancellation by $c_{k-1}\Phi_\ell(z)\cdots\Phi_{\ell_{k-2}}(z)$,

$$U(z)\Phi_\ell(z) - dU(z^d)\Phi_{\ell_{k-1}}(z) = c_{k-1}z\Phi_\ell'(z)\Phi_{\ell_1}(z)\cdots\Phi_{\ell_{k-1}}(z),$$

implying $\Phi_{\ell_{k-1}}(z) \mid U(z)$. By (2.7), $V(z)$ has the same divisor, which is a contradiction.

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Thus, we conclude that $\deg V_{k-1} > 0$ and hence $\deg v < \varphi(\ell_{k-1})$ by (2.6), and this implies $\Phi_{\ell_{k-1}}(z) \nmid v(z)$. But then, again by (2.6), we see that $V_{k-1}(\zeta_{\ell_{k-1}}^d) = V_{k-1}(\zeta_{\ell_k}) = 0$, whence $\Phi_{\ell_k}(z) \,|\, V_{k-1}(z)$, say $V_{k-1}(z) = \Phi_{\ell_k}(z)V_k(z)$ with some $V_k \in \mathbb{Q}[z]\backslash\{0\}$. This, together with (2.6), leads to

$$\Phi_{\ell_k}(z^d)V_k(z^d)v(z) = \Phi_{\ell_{k-1}}(z)\Phi_{\ell_k}(z)V_k(z), \tag{2.8}$$

implying

$$\deg v = \varphi(\ell_{k-1}) - (d-1)\varphi(\ell_k) - (d-1)\deg V_k. \tag{2.9}$$

To end our proof, we consider the two possible cases for $\delta := \gcd(d, \ell_k)$, where we introduce the coprime integers $\hat{d}, \hat{\ell}$ by $d = \delta\hat{d}, \ell_k = \delta\hat{\ell}$.

If first $\delta > 1$, then $\varphi(d\ell_k) < d\varphi(\ell_k)$ and, by (2.9), $\deg v < \varphi(\ell_k)$, whence $\Phi_{\ell_k}(z) \nmid v(z)$. Thus, $V_k(\zeta_{\ell_k}^d) = V_k(\zeta_{\hat{\ell}}^{\hat{d}}) = 0$ and $\Phi_{\hat{\ell}}(z) \,|\, V_k(z)$ and, by (2.9) again,

$$\varphi(\hat{\ell}) \le \deg V_k \le \frac{\varphi(\ell_k)}{d-1}.$$

The left-hand inequality also holds in case $k = 0$ from (2.3), since $V(\zeta_\ell^d) = V(\zeta_{\hat{\ell}}^{\hat{d}}) = 0$, implying $\Phi_{\hat{\ell}}(z) \,|\, V(z)$ and so $\varphi(\hat{\ell}) \le \deg V$. Now $\ell_k = \delta\hat{\ell}$ gives $\varphi(\ell_k) \le \delta\varphi(\hat{\ell})$ and hence $\varphi(\hat{\ell}) \le \delta\varphi(\hat{\ell})/(d-1)$, leading to $\delta \in \{d-1, d\}$. If $\delta = d-1$, then $d \ge 3$ and this δ cannot divide d, which is a contradiction. Thus, $\delta = d$, implying $d \,|\, \ell_k = \tilde{\ell}$, which is again a contradiction.

Suppose secondly that $\delta = 1$. In case $k = 0$, (2.3) implies $\deg v < \varphi(\ell)$ since $\deg V > 0$. Therefore, $\Phi_\ell(z) \nmid v(z)$, whence $V(\zeta_\ell^d) = 0$, giving $\Phi_\ell(z) \,|\, V(z)$ and therefore $\varphi(\ell) \le \deg V \le \varphi(\ell)/(d-1)$, leading to $d = 2$ and $V(z) = c\Phi_\ell(z)$ with some $c \in \mathbb{Q}^\times$. Then (2.3) implies $\Phi_\ell(z^2)v(z) = \Phi_\ell(z)^2$ and hence $v(z) = 1$, and $\Phi_\ell(z^2) = \Phi_\ell(z)^2$ is contradictory. So, we are left with the case $k \ge 1$, where we use (2.8). The roots of $\Phi_{\ell_k}(z^d)$ are the roots of $z^d - \zeta_{\ell_k}^j$ with $\gcd(\ell_k, j) = 1$. For fixed j, these are

$$\zeta_{d\ell_k}^{j+h\ell_k} \quad (h = 0, \ldots, d-1),$$

where $\gcd(j + h\ell_k, \ell_k) = 1$. For each u modulo d, the congruence $j + h\ell_k \equiv u \pmod{d}$ has a unique solution $h \in \{0, \ldots, d-1\}$. Thus, the above roots contain one primitive ℓ_kth and $\varphi(d)$ primitive $(d\ell_k)$th roots of unity. Therefore, $\Phi_{\ell_{k-1}}(z)\Phi_{\ell_k}(z)$ divides $\Phi_{\ell_k}(z^d)$. Define

$$N := d\varphi(\ell_k) - \varphi(\ell_k) - \varphi(\ell_{k-1}) = (d-1)\varphi(\ell_k) - \varphi(d\ell_k) = (d-1-\varphi(d))\varphi(\ell_k)$$

(note here that $\delta = \gcd(d, \ell_k) = 1$). By comparing the degrees on both sides of (2.8),

$$N + (d-1)\deg V_k + \deg v = 0.$$

This shows that $N = 0$, $\deg V_k = 0$, $\deg v = 0$ and hence $\varphi(d) = d-1$ and $k \ge 1$. Consequently, d is a prime and $(k =)\ \mathrm{ord}_d \ell$ is a positive integer, proving our lemma. □

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3. Arithmetical consequences of hypertranscendence

Combination of the implication (i) implies (ii) from Theorem 1.1 and the following algebraic independence criterion for infinite products yields an immediate proof of Theorem 1.2. As usual, $\overline{\mathbb{Q}}$ denotes the field of all complex algebraic numbers.

Lemma 3.1 [4, Theorem 2]. *Let $P \in \overline{\mathbb{Q}}[z]$ be nonconstant and satisfy $P(0) = 1$. Suppose that the function $f_d(z) := \prod_{j\geq 0} P(z^{d^j})$ defined in $\mathbb{D}$ is hypertranscendental. Then, for any algebraic α with $0 < |\alpha| < 1$ and $P(\alpha^{d^j}) \neq 0$ for each $j \in \mathbb{N}_0$, the numbers $f_d(\alpha), f_d'(\alpha), f_d''(\alpha), \ldots$ are algebraically independent.*

So far, the statement of Theorem 1.2 concerns algebraic $\alpha \in \mathbb{D}\backslash\{0\}$, and one may ask what happens if $\alpha \in \mathbb{D}$ is transcendental. This question has first been treated by Philippon [13, Théorème 4], whose results lead to the following theorem.

Theorem 3.2. *Let d, ℓ satisfy the conditions of Theorem 1.2. Then, for any $m \in \mathbb{N}_0$ and for any nonzero $\alpha \in \mathbb{D}$, the following estimate holds for the transcendence degree:*

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(\alpha, F_{d,\ell}(\alpha), \ldots, F_{d,\ell}^{(m)}(\alpha)) \geq m + 1.$$

4. Algebraic independence of several functions $F_{d,\ell}(z)$

To prove Theorem 1.3, we give some preliminary results on cyclotomic polynomials collected in Lemmas 4.1 and 4.2. Having these, we shall apply Kubota's criterion for the algebraic independence of Mahler-type functions to be quoted as Lemma 4.3.

Lemma 4.1. *For coprime d, ℓ, the equation*

$$\Phi_\ell(z^d) = \Phi_\ell(z)\Psi_\ell(z)$$

holds with some $\Psi_\ell(z) \in \mathbb{Z}[z]$ satisfying $\Phi_\ell(z) \nmid \Psi_\ell(z)$.

Proof. Since $\{\zeta_\ell^j : \gcd(j, \ell) = 1\} = \{\zeta_\ell^{dj} : \gcd(j, \ell) = 1\}$,

$$\Phi_\ell(z^d) = \prod_{(j,\ell)=1} (z^d - \zeta_\ell^j) = \prod_{(j,\ell)=1} (z^d - \zeta_\ell^{dj}) = \Phi_\ell(z) \prod_{(j,\ell)=1} \frac{z^d - \zeta_\ell^{dj}}{z - \zeta_\ell^j} =: \Phi_\ell(z)\Psi_\ell(z).$$

The roots of $z^d - \zeta_\ell^j$ are just the $\zeta_{d\ell}^{j+k\ell}$ with $k \in \{0, \ldots, d-1\}$. Such a root is a primitive nth root of unity, where $\ell \,|\, n$, and $n = \ell$ holds if and only if $j + k\ell \equiv 0 \pmod{d}$. Since $\gcd(d, \ell) = 1$, this congruence has, for each j, a unique solution $k \in \{0, \ldots, d-1\}$. From this, the roots of $\Psi_\ell(z)$ are roots of unity of order greater than ℓ, whence $\Phi_\ell(z) \nmid \Psi_\ell(z)$. □

Lemma 4.2. *If d, ℓ are coprime, then, for any $a(z) \in \mathbb{Z}[z]\backslash\{0\}$, the multiplicities of $\Phi_\ell(z)$ in $a(z)$ and in $a(z^d)$ are equal.*

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PROOF. Assume that $a(z) = \Phi_\ell(z)^k u(z)$ with $u(z) \in \mathbb{Z}[z]$, $\Phi_\ell(z) \nmid u(z)$. By Lemma 4.1,

$$a(z^d) = \Phi_\ell(z)^k \Psi_\ell(z)^k u(z^d) = \Phi_\ell(z)^{\hat{k}} U(z)$$

with $\Phi_\ell(z) \nmid U(z)$, whence the multiplicity $\hat{k}$ of $\Phi_\ell(z)$ in $a(z^d)$ is at least k.

Define $m := \varphi(\ell)$ and let $j_1, \ldots, j_m$ be those $j \in \{1, \ldots, \ell\}$ coprime to ℓ. Write $a(z) = c(z - \alpha_1) \cdots (z - \alpha_n)$ with $c, \alpha_1, \ldots, \alpha_n \in \mathbb{C}, c \neq 0$. If $\hat{k} > 0$, for each j_μ, there exists an i_μ such that $\alpha_{i_\mu} = \zeta_\ell^{d j_\mu}$. Moreover, $\alpha_{i_\nu} \neq \alpha_{i_\mu}$ for $\nu \neq \mu$ and, by Lemma 4.1,

$$\Phi_\ell(z) = \prod_{\mu=1}^{m} (z - \alpha_{i_\mu}) \quad \text{and} \quad \prod_{\mu=1}^{m} (z^d - \alpha_{i_\mu}) = \Phi_\ell(z)\Psi_\ell(z).$$

Without loss of generality, we may assume that $i_\mu = \mu$ for $\mu = 1, \ldots, m$, leading to

$$a(z) = \Phi_\ell(z) \prod_{i=m+1}^{n} (z - \alpha_i), \quad a(z^d) = \Phi_\ell(z)\Psi_\ell(z) \prod_{i=m+1}^{n} (z^d - \alpha_i).$$

If $\hat{k} > 1$, then we may repeat this procedure and, after $\hat{k}$ repetitions,

$$a(z) = \Phi_\ell(z)^{\hat{k}} \prod_{i=\hat{k}m+1}^{n} (z - \alpha_i), \quad a(z^d) = \Phi_\ell(z)^{\hat{k}} \Psi_\ell(z)^{\hat{k}} \prod_{i=\hat{k}m+1}^{n} (z^d - \alpha_i).$$

Since $\Phi_\ell(z) \nmid U(z)$, none of the α_i with $\hat{k}m < i \leq n$ equals $\zeta_\ell^{d j_\mu}$, whence $\hat{k} = k$. □

LEMMA 4.3 (Special case of [11]; see also [12, Theorem 3.5]). *Suppose that the series*

$$f_{i,j} \in \mathbb{C}[[z]] \;\; (1 \leq i \leq h, 1 \leq j \leq n(i)) \quad \textit{and} \quad f_i \in \mathbb{C}[[z]] \setminus \{0\} \;\; (h < i \leq k) \tag{4.1}$$

converge on $\mathbb{D}$ *and satisfy the functional equations*

$$f_{i,j}(z^d) = a_i(z) f_{i,j}(z) + a_{i,j}(z) \quad (1 \leq i \leq h, 1 \leq j \leq n(i)),$$
$$f_i(z^d) = b_i(z) f_i(z) \quad (h < i \leq k)$$

with $a_i, a_{i,j}, b_i \in \mathbb{C}(z) \setminus \{0\}$, *where none of the quotients* $a_i/a_{i'}$ $(1 \leq i < i' \leq h)$ *has the form* $s(z^d)/s(z)$ *for some* $s \in \mathbb{C}(z) \setminus \{0\}$. *Then the functions (4.1) are algebraically independent over* $\mathbb{C}(z)$ *if the* $a_i, a_{i,j}, b_i$ *satisfy the following additional conditions.*

(i) *For no* $i \in \{1, \ldots, h\}$ *is there a* $(c_{i,1}, \ldots, c_{i,n(i)}) \in \mathbb{C}^{n(i)} \setminus \{\underline{0}\}$ *such that the equation*

$$g(z^d) = a_i(z) g(z) - \sum_{j=1}^{n(i)} c_{i,j} a_{i,j}(z)$$

has a solution $g \in \mathbb{C}(z)$.

(ii) *If* $h < k$ *and* $(n_{h+1}, \ldots, n_k) \in \mathbb{Z}^{k-h} \setminus \{\underline{0}\}$, *the functional equation*

$$r(z^d) = r(z) \prod_{i=h+1}^{k} b_i(z)^{n_i}$$

has no solution $r \in \mathbb{C}(z) \setminus \{0\}$.

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Proof of Theorem 1.3. Assuming that $\{\ell_1, \ldots, \ell_m\}$ is any finite subset of S_d, we have to show that $F_{d,\ell_1}(z), \ldots, F_{d,\ell_m}(z)$ are algebraically independent over $\mathbb{C}(z)$. For this purpose, we try to apply Lemma 4.3. Since all functional equations (1.1) are homogeneous, we may take $h = 0, k = m$ and need only check condition (ii).

Suppose that $(n_1, \ldots, n_m) \in \mathbb{Z}^m$ and assume that the functional equation

$$r(z^d) = r(z) \prod_{i=1}^{m} \Phi_{\ell_i}(z)^{-n_i} \tag{4.2}$$

has a rational solution $r \neq 0$. Then it also has a solution $\tilde{r} \in \mathbb{Q}(z)\backslash\{0\}$, which we may write as $\tilde{r}(z) = a(z)/b(z)$ with $a, b \in \mathbb{Z}[z]\backslash\{0\}$. On rewriting (4.2) for $\tilde{r}$ instead of r, we are led to the new polynomial equation

$$a(z^d)b(z) \prod_{i \in U_1} \Phi_{\ell_i}(z)^{n_i} = a(z)b(z^d) \prod_{i \in U_2} \Phi_{\ell_i}(z)^{-n_i},$$

where $U_1 \cup U_2 = \{1, \ldots, m\}$ and $n_i \geq 0$ exactly for $i \in U_1$. By Lemma 4.2, $\Phi_{\ell_i}(z)$ has the same multiplicity in $a(z), a(z^d)$ (and similarly in $b(z), b(z^d)$) for each $i = 1, \ldots, m$. So, $n_1 = \cdots = n_m = 0$, and Theorem 1.3 is proved. □

Remark 4.4 (Remark on the Open Problem 1.4). As we saw above, the proof of Theorem 1.3 is a rather direct consequence of Lemmas 4.1 and 4.2. Both may not hold if d and ℓ are not coprime (for example, if $d = 6$ and $\ell = 3$). Therefore, some new ideas seem to be necessary to attack the problem.

Proof of Theorem 1.5. Theorem 1.3 combined with [12, Theorem 4.2.1] immediately yields Theorem 1.5, since all $\Phi_\ell(z)$ are zero-free in $\mathbb{D}$. □

Combining Theorem 1.3 with [13, Théorème 4] gives a more general result.

Corollary 4.5. *Let $\ell_1, \ldots, \ell_m \in S_d$ be distinct. Then, for any nonzero $\alpha \in \mathbb{D}$, the following inequality holds:*

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(\alpha, \Phi_{\ell_1}(\alpha), \ldots, \Phi_{\ell_m}(\alpha)) \geq m.$$

Of course, for algebraic α, this inequality is contained in Theorem 1.5. For transcendental α, it was already proved in [1, Theorem 3] in case $m = 2$ (see also the remark after [12, Theorem 4.5.1]).

5. The functions $F_{d,\ell}$ and another class of functions

In this section, we simultaneously study, for fixed d, the functions $F_{d,\ell}(z)$ and the functions $G_{d,j}(z)$ $(j \in \mathbb{N}_0)$, defined in $\mathbb{D}$ by

$$G_{d,j}(z) := \sum_{n=0}^{\infty} \frac{z^{d^n}}{1 - z^{d^{n+j}}},$$

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with respect to algebraic independence over $\mathbb{C}(z)$. Note that each $G_{d,j}$ satisfies the functional equation

$$G_{d,j}(z^d) = G_{d,j}(z) - \frac{z}{1 - z^{d^j}}\,. \tag{5.1}$$

Observe that $G_{2,1}(z) = z/(1 - z)$ in $\mathbb{D}$, whereas all $G_{d,j}(z)$ with $(d, j) \neq (2, 1)$ are transcendental over $\mathbb{C}(z)$ (and have the unit circle as natural boundary). This transcendence statement is a consequence of combining [12, Theorem 1.3] with the following lemma from [15, Lemmas 1 and 2], which excludes the rationality of the function under consideration.

Lemma 5.1. *Let* $(c_0, \ldots, c_m) \in \mathbb{C}^{m+1}\backslash\{\underline{0}\}$ *with* $m \in \mathbb{N}$. *If the functional equation*

$$g(z^d) = g(z) - \sum_{j=0}^{m} \frac{c_j z}{1 - z^{d^j}}$$

has a solution $g \in \mathbb{C}(z)$, *then* $d = 2$ *and* $c_1 \neq 0$ *hold.*

Lemma 5.1 is not only used *en passant* to prove the functional transcendence of all $G_{d,j}$ with $(d, j) \neq (2, 1)$. Much more essential is its application in the following proof of the algebraic independence over $\mathbb{C}(z)$ of all these functions (for fixed d).

Theorem 5.2. *For any fixed integer* $d \geq 3$, *the functions* $G_{d,j}(z)$ $(j \in \mathbb{N}_0)$ *and* $F_{d,\ell}(z)$ $(\ell \in S_d)$ *are algebraically independent over* $\mathbb{C}(z)$.

Proof. We apply Lemma 4.3 to the $2m + 1$ functions $G_{d,0}(z), \ldots, G_{d,m}(z)$ and $F_{d,\ell}(z)$ with $\ell \in \{\ell_1, \ldots, \ell_m\} \subset S_d$ for some $m \in \mathbb{N}$. Thus, take $h = 1, n(1) = m + 1, k = m + 1$ such that the $f_{1,j}(z)$ are the functions $G_{d,0}(z), \ldots, G_{d,m}(z)$, and $f_{1+i}(z) := F_{d,\ell_i}(z)$ for $i = 1, \ldots, m$. We only have to settle conditions (i) and (ii) in Lemma 4.3: for (i), we use Lemma 5.1 with $d \geq 3$, whereas the validity of (ii) follows (independently of $d \geq 3$ or $d = 2$) along the lines of our proof of Theorem 1.3. □

As we already know, a direct analogue of Theorem 5.2 for $d = 2$ cannot hold without removing the rational function $G_{2,1}(z)$. But, instead, we may include the so-called twisted version $B(z)$ of $F_{2,3}(z)$ introduced in [2] and studied arithmetically by the present authors [5]. Notice that $B(z)$ satisfies the functional equation

$$B(z) = 2 - \Phi_3(z)B(z^2) \tag{5.2}$$

that leads to the power series solution about the origin.

Theorem 5.3. *The functions* $G_{2,j}(z)$ $(j \in \mathbb{N}_0\backslash\{1\}), B(z)$ *and* $F_{2,\ell}(z)$ $(\ell \in S_2)$ *are algebraically independent over* $\mathbb{C}(z)$.

Proof. Once again, we use Kubota's Lemma 4.3, which we apply to the $2m + 1$ functions $G_{2,0}(z), G_{2,2}(z), \ldots, G_{2,m}(z), B(z)$ and $F_{2,\ell}(z)$ $(\ell \in \{\ell_1, \ldots, \ell_m\} \subset S_2)$ for some $m \in \mathbb{N}$. This time, we take $h = 2, n(1) = m, n(2) = 1, k = 2 + m$. The functions $f_{1,j}(z)$ are the functions $G_{2,0}(z), G_{2,2}(z), \ldots, G_{2,m}(z)$, $f_{2,1}(z) := B(z)$ and $f_{2+i}(z) := F_{2,\ell_i}(z)$ for $i = 1, \ldots, m$. By (5.1) and (5.2), the two $a_i(z)$ appearing here are $a_1(z) := 1$,

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$a_2(z) := -1/\Phi_3(z)$. Their quotient $-\Phi_3(z)$ cannot be of the form $s(z^2)/s(z)$ with some $s \in \mathbb{C}(z)\backslash\{0\}$, as is easily seen from Lemma 4.2.

Concerning conditions (i) and (ii) in Lemma 4.3, the second one was already checked when proving Theorem 5.2. To establish the first, we have to ensure that neither of the two equations

$$g(z^2) = g(z) - \sum_{\substack{j=0 \\ j \neq 1}}^{m} \frac{c_{1,j}z}{1 - z^{2^j}} \quad \text{and} \quad g(z^2) = -\frac{g(z)}{\Phi_3(z)} - \frac{2c_{2,1}}{\Phi_3(z)}$$

has a rational solution g for $(c_{1,0}, c_{1,2}, \ldots, c_{1,m}) \neq (\underline{0})$ and $c_{2,1} \neq 0$, respectively. The first assertion follows from Lemma 5.1. For the second, the rational insolubility was proved in [5, Section 2] for $c_{2,1} = -1$ (which is equivalent to the general case $c_{2,1} \neq 0$). □

Analogously to the end of Sections 3 and 4, Theorems 5.2 and 5.3 combined with [12, Theorem 4.2.1] (or with [12, Theorem 4.5.1]) immediately lead to the following arithmetical applications, which we formulate this time only for algebraic points.

Corollary 5.4. *Let $d \geq 3$ be fixed. Then, for any nonzero algebraic $\alpha \in \mathbb{D}$, the numbers $G_{d,j}(\alpha)$ $(j \in \mathbb{N}_0)$ and $F_{d,\ell}(\alpha)$ $(\ell \in S_d)$ are algebraically independent. The same holds verbatim in case $d = 2$ if $G_{2,1}(\alpha)$ is replaced by $B(\alpha)$.*

The case $d = 2$ of this corollary gives the algebraic independence of some well-known numbers. Using the notation of [3], we define

$$f_{TMM}(z) := \sum_{n=0}^{\infty} t_n z^n, \quad f_{RPF}(z) := \sum_{n=0}^{\infty} u_n z^n,$$

where (t_n) is the Thue–Morse sequence ($t_0 = 0$, $t_{2n} = t_n$, $t_{2n+1} = 1 - t_n$ $(n \geq 0)$) and (u_n) is the regular paper-folding sequence ($u_{4n} = 1$, $u_{4n+2} = 0$, $u_{2n+1} = u_n$ $(n \geq 0)$). Then

$$F_{2,1}(z) = \frac{1}{1-z} - 2f_{\mathrm{TMM}}(z), \quad G_{2,2}(z) = zf_{\mathrm{RPF}}(z).$$

As was noted earlier, $F_{2,3}(z)$ is the generating function of the Stern diatomic sequence and $B(z)$ its twisted version. The following result is a special case of Corollary 5.4.

Corollary 5.5. *For every $b \in \mathbb{Z}\backslash\{0, \pm 1\}$, the five numbers*

$$f_{\mathrm{TMM}}\Big(\frac{1}{b}\Big), \ f_{\mathrm{RPF}}\Big(\frac{1}{b}\Big), \ F_{2,3}\Big(\frac{1}{b}\Big), \ B\Big(\frac{1}{b}\Big), \ \sum_{n=0}^{\infty} \frac{1}{b^{2^n} + 1}$$

are algebraically independent.

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6. Transcendence via Schmidt's subspace theorem

We first make some notational remarks following the presentation in [7]. Let K be a number field and $M(K)$ be the set of normalised absolute values of K. For every $v \in M(K)$, let $|\,.\,|_v$ denote a continuation of it to $\overline{\mathbb{Q}}$, to be normalised with respect to K: according to this normalisation, the absolute logarithmic Weil height of $x \in K^\times$ is

$$h(x) := \sum_{v \in M(K)} \log^+ |x|_v \tag{6.1}$$

(where $\log^+ t := \max(0, \log t)$ for $t \in \mathbb{R}_+$) and the product formula $\prod_v |x|_v = 1$ holds.

Furthermore, let v be a fixed absolute value of K and let $\mathbb{C}_v$ denote a completion of an algebraic closure of K_v. The notion of convergence, as used in the following lemma (to be found in [7, Corollary 1]), refers to $\mathbb{C}_v$.

Lemma 6.1. *Let $f(z) := \sum_{j \geq 0} b_j z^j$ be a nonpolynomial power series with algebraic $b_j \in \mathbb{C}_v$ and converging in $|z|_v < 1$. Let $\alpha \in \mathbb{C}_v^\times$ be algebraic with $|\alpha|_v < 1$. Suppose that there exist an infinite set $\mathcal{N} \subset \mathbb{N}$ and a finite set $S \subset M(K)$ containing all archimedean absolute values of K such that $f(\alpha^n) \in K^\times$ and $|f(\alpha^n)|_v \leq 1$ hold for any $n \in \mathcal{N}$ and $v \in M(K) \backslash S$. Then*

$$\lim_{n \to \infty, n \in \mathcal{N}} \frac{1}{n} h(f(\alpha^n)) = \infty.$$

Proof of Theorem 1.6. Consider $f(z) := 1/F_{d,\ell}(z) \in \mathbb{Z}[[z]]$ satisfying

$$f(z^{d^t}) = f(z) \prod_{\tau=1}^{t} \Phi_\ell(z^{d^{\tau-1}}) \tag{6.2}$$

for any $t \in \mathbb{N}_0$. Since $F_{d,\ell}(z)$ is not a rational function, we know that f is not a polynomial. Let us assume, contrary to Theorem 1.5, that there is a nonzero algebraic $\alpha \in \mathbb{D}$ such that $f(\alpha)$ is algebraic, and apply Lemma 6.1 to $K := \mathbb{Q}(\alpha, f(\alpha))$. For any $\beta \in K^\times$, the set $\{v \in M(K) \colon |\beta|_v \neq 1\}$ is finite, so the set $\{v \in M(K) \colon |\alpha|_v = 1, f(\alpha)|_v \leq 1\}$ is infinite. Since the set of (normalised) archimedean absolute values of K is finite, we may choose a finite set S of places of K containing all archimedean ones such that

$$|\alpha|_v = 1 \quad \text{and} \quad |f(\alpha)|_v \leq 1 \tag{6.3}$$

for every $v \in M(K) \backslash S$. From (6.2) and (6.3), $f(\alpha^{d^t}) \in K^\times$ and $|f(\alpha^{d^t})|_v \leq 1$ for every $t \in \mathbb{N}_0$ and $v \in M(K) \backslash S$ (since all these v are nonarchimedean). By Lemma 6.1,

$$\lim_{t \to \infty} \frac{h(f(\alpha^{d^t}))}{d^t} = \infty. \tag{6.4}$$

To show that (6.4) is wrong, that is, that we have the desired contradiction, we state the following properties of $h(\,.\,)$. The case $m = 2$ can be found in [16, Property 3.3], whereas the cases $m \geq 3$ are easily settled by induction. Note that, in contrast to (6.1), the absolute logarithmic Weil height in [16] is defined for *any* $x \in K$ by $\sum_v \log \max(1, |x|_v)$, coinciding with the sum in (6.1) if $x \neq 0$.

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Lemma 6.2. *If $\gamma_1, \dots, \gamma_m$ are $m \geq 1$ nonzero algebraic numbers, then*

$$h\Big(\prod_{\mu=1}^{m} \gamma_\mu\Big) \leq \sum_{\mu=1}^{m} h(\gamma_\mu) \quad \text{and} \quad h\Big(\sum_{\mu=1}^{m} \gamma_\mu\Big) \leq (m-1)\log 2 + \sum_{\mu=1}^{m} h(\gamma_\mu).$$

To conclude the proof interrupted after (6.4), we estimate $h(\Phi_\ell(\beta))$ for certain powers β of α. To this end, note that $\Phi_\ell(\beta)$ is a sum of at most $1 + \varphi(\ell)$ terms of the shape $a_\lambda \beta^\lambda$, where the a_λ are just the nonzero rational integer coefficients of the polynomial $\Phi_\ell(x)$. Then we deduce from Lemma 6.2 that

$$h(\Phi_\ell(\beta)) \leq \varphi(\ell)\log 2 + \sum_{\substack{\lambda=0 \\ a_\lambda \neq 0}}^{\varphi(\ell)} (h(a_\lambda) + \lambda h(\beta)) \leq c_0(\ell) + c_1(\ell)h(\beta)$$

with positive constants c_0, c_1 depending only on ℓ. Thus, we conclude from (6.2) that

$$h(f(\alpha^{d^t})) \leq h(f(\alpha)) + \sum_{\tau=1}^{t} h(\Phi_\ell(\alpha^{d^{\tau-1}})) \leq h(f(\alpha)) + c_0(\ell)t + c_1(\ell)h(\alpha)\sum_{\tau=0}^{t-1} d^\tau \leq c_2\, d^t$$

for any $t \in \mathbb{N}_0$, where $c_2 > 0$ is independent of t. The last chain of inequalities shows that (6.4) is not valid. □

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PETER BUNDSCHUH, Mathematisches Institut,
Universität zu Köln, Weyertal 86–90,
50931 Köln, Germany
e-mail: pb@math.uni-koeln.de

KEIJO VÄÄNÄNEN, Department of Mathematical Sciences,
University of Oulu, PO Box 3000,
90014 Oulu, Finland
e-mail: keijo.vaananen@oulu.fi

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