TWISTED COMPLEX GEOMETRY

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Abstract

We introduce complex differential geometry twisted by a real line bundle. This provides a new approach to understand the various real objects that are associated with an anti-holomorphic involution. We also generalize a result of Greenleaf about real analytic sheaves from dimension 2 to higher dimensions.

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1. Introduction

Recently there has been considerable interest in gauge theory over complex manifolds with anti-holomorphic involutions. In physics, such complex manifolds are referred to as *orientifolds*. Ooguri and Vafa [9] propose a duality on open strings from a physicists' point of view, which suggests mathematical formulae counting holomorphic curves on a disc with boundary conditions in a Lagrangian submanifold of a Calabi-Yau threefold. When the Lagrangian submanifold is the real part of an anti-holomorphic involution, Katz and Liu [5] observe that these holomorphic curves on the disc can be identified with real holomorphic curves on the double $\mathbf{CP}^1$ of the disc. As a consequence, they are able to verify partially the mathematical formulae of [9]; see also Li and Song [6]. However when the real part of the anti-holomorphic involution is empty, the method of Katz and Liu does not work. Indeed, here one encounters closed strings rather than open ones, and Sinha and Vafa [10] propose a different duality that would suggest formulae for $\mathbf{RP}^2$-amplitudes.

In the case of complex surfaces with anti-holomorphic involutions, Welschinger [14] has given a definition and computation that enumerates real rational (holomorphic)

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curves in rational surfaces. For arbitrary complex surfaces with anti-holomorphic involutions, Tian and Wang [11] consider corresponding real Seiberg-Witten theory.

All the mathematical works cited above are for anti-holomorphic involutions that have non-empty real parts. The case of empty real parts requires different approaches which are yet to appear. In a simpler instance, the difference between the cases of empty and non-empty real parts is illustrated by the vanishing theorem of [12]. When the real part is non-empty, the adjunction formula can be applied to the real part which then implies the vanishing of the Seiberg-Witten invariants of the quotient manifold. If the real part is empty, then one has to use a different method as in [12], since the adjunction formula no longer applies.

The main purpose of this paper is to develop a new approach, based on which one can find methods suitable for the situations stated above that involve empty real parts. Our approach originates from Klein surfaces which were systematically investigated by Alling and Greenleaf [1]

Alling and Greenleaf studied Klein surfaces from the point of view of algebraic function fields. Greenleaf [3] compared the cohomology of analytic sheaves on such surfaces and that on the double covers. More recently, Klein surfaces have found applications in solitons, KdV equations, and conformal field theory, as seen in Natanzon's survey [8].

In this paper we generalize the concept of Klein surfaces to high dimensions. Furthermore, we investigate new differential geometry in the corresponding set-up. The key difference between [1] and our approach is that our emphasis is on the role played by a certain twisting line bundle, resulting in what we call twisted complex geometry. As an example, our twisted complex structures are a generalization of di-analytic structures in [1]. We will lay down the foundation of the twisted complex geometry by resolving various technical issues and also discuss a number of immediate applications.

In either Gromov-Witten or Seiberg-Witten theories over complex manifolds with anti-holomorphic involutions, one has to deal with the moduli spaces of real holomorphic curves or real Seiberg-Witten solutions. These moduli spaces appear as the real parts of the ordinary moduli spaces under the anti-holomorphic involutions in question. Twisted complex geometry allows the identification of real moduli spaces with the moduli spaces defined on twisted complex manifolds. Since the twisted complex geometry and the usual complex geometry enjoy many formal similarities, the advantage of the identification is that real moduli spaces can be treated formally like the ordinary complex moduli spaces in the twisted set-up. In a subsequent publication we intend to use this approach to study the real Gromov-Witten invariants suggested by [10] and the real Seiberg-Witten invariants not covered by [11].

The outline of the paper is as follows. In Section 2, we define basic concepts in twisted complex geometry, such as twisted functions, and twisted bundles. We

take a more global view in Section 3, where a twisted, almost complex structure is adapted to characterize the same set-up in an equivalent way. Section 4 gives the comparison between twisted complex objects on a manifold and real objects on the double covering manifold. In particular, we prove Greenleaf's theorem in any dimension. Section 5 continues with more applications towards gauge theory. After defining twisted stable bundles and Hermitian-Einstein connections on Klein surfaces, we prove the Narasimhan-Seshadri-Donaldson correspondence in this set-up, which extends our earlier result [13] where the correspondence was established in the special case of trivial topological twisted bundles. Section 6 concludes the paper with several further remarks and comments.

2. The L-twisted theory

We begin with basic concepts in the twisted complex geometry. The key idea is to twist the imaginary part with the help of a real line bundle L.

Let X be a smooth manifold of any dimension. Assume that X is closed without boundary. Fix any real line bundle $L \to X$, which we call a twisting bundle in this paper. This is so named since we will use L to twist various complex values. We are mainly interested in the case of a non-trivial L, hence $H^1(X, \mathbf{Z}_2)$ is typically non-trivial. In particular, X is allowed to be non-orientable and L may be taken as the orientation bundle of X.

Fix a fiber metric on L (the choice will be immaterial). Consider a system of orthonormal trivializations $\mathscr{U} = \{(U, s_U)\}$ of L. Then the transition function $s_{UU'}$ between two charts U, U' must be ± 1. For convenience, let $\tilde{U} = (U, s_U)$ denote the trivialization. Strictly speaking, one should also use $s_{\tilde{U}\tilde{U}'}$ rather than just $s_{UU'}$. However, to avoid excessive notation, we will use the latter. The same remark applies to similar situations in the future without additional explanation. Although not necessary, using a fiber metric on L will greatly facilitate the descriptions.

Consider a pair $(\mathscr{U}, f_{\mathscr{U}})$, where $f_{\mathscr{U}}$ is a family of complex functions $f_{\tilde{U}} : U \to \mathbf{C}$ associated to each $\tilde{U} \in \mathscr{U}$. The family $f_{\mathscr{U}}$ is subject to the following *twisted compatibility* condition on the intersection of two charts U, U':

$$(1) \qquad\qquad f_{\tilde{U}} = \begin{cases} f_{\tilde{U}'} & \text{if } s_{UU'} = 1, \\ \bar{f}_{\tilde{U}'} & \text{if } s_{UU'} = -1, \end{cases}$$

where the $\bar{f}_{\tilde{U}'}$ is the complex conjugate of $f_{\tilde{U}'}$. Let $(\mathscr{V}, g_{\mathscr{V}})$ be another pair similarly defined, where $\mathscr{V}$ is a second system of trivializations of L. We say that $(\mathscr{U}, f_{\mathscr{U}})$ is equivalent to $(\mathscr{V}, g_{\mathscr{V}})$ if, on the intersection of two charts $U \in \mathscr{U}$, $V \in \mathscr{V}$, we have $f_{\tilde{U}} = g_{\tilde{V}}$ or $f_{\tilde{U}} = \bar{g}_{\tilde{V}}$ depending on the whether transition s_{UV} is 1 or -1.

DEFINITION 2.1. An *L-twisted complex function* f is an equivalence class of pairs $[(\mathscr{U}, f_\mathscr{U})]$.

REMARK. (1) It is convenient to write $f = \{f_{\tilde{U}}\}$ and $f|_{\tilde{U}} = f_{\tilde{U}}$ (called a component of f).

(2) We emphasize that $f|_{\tilde{U}} = f_{U,s_U}$ depends on the choice of a trivialization s_U of L, which is a key feature of the twisting theory in this paper.

From the remark above, a twisted complex function f is not well-defined anywhere on X unless it is real valued. However its real part $\mathrm{Rm}\, f = \{\mathrm{Re}\, f_{\tilde{U}}\}$ is clearly a well defined real function on X. Also well defined is the function $f^+ : X \to \mathbf{C}^+$, $f^+ = \{\mathrm{Re}\, f_{\tilde{U}} + \mathbf{i} \,|\, \mathrm{Im}\, f_{\tilde{U}}|\}$, where $\mathbf{C}^+$ is the set of complex numbers of non-negative imaginary part. Note f^+ is only a continuous function on X if f is smooth, that is, each $f_{\tilde{U}}$ is smooth.

The set $\mathscr{C}_X = \mathscr{C}_{X,L}$ of smooth twisted complex functions forms an **R**-algebra. Moreover the conjugate $\bar{f}$, defined to be $(\mathscr{U}, \bar{f}_\mathscr{U})$, is also twisted complex.

Let (Y, K) be a second smooth manifold with a twisting bundle. We say a smooth function $h : X \to Y$ is (L, K)-*compatible* or *twist-preserving* if L is isomorphic to the pull-back h^*K. Equivalently, h can be lifted to a bundle homomorphism $\tilde{h} : L \to K$ which is fiberwise non-trivial (that is, fiberwise isomorphic) everywhere. Notice here one bundle homomorphism canonically determines the other. Of course any smooth function h may not be (L, K)-compatible. For example, any constant function is not so if L is non-trivial.

Take a lifting $\tilde{h}$ as above. Then any system of trivializations of K induces that of L. Furthermore, a K-twisted complex function on Y pulls back in an obvious way to an L-twisted complex function on X. The resulting function $\tilde{h}^* : \mathscr{C}_Y \to \mathscr{C}_X$ is an **R**-algebra homomorphism. Assuming X is connected, one sees that any two different liftings of h differ by a sign. Hence the induced maps from $\mathscr{C}_Y$ to $\mathscr{C}_X$ differ at most by conjugation.

As a special case, take any open subset $Y \subset X$ and $K = L|_Y$. The canonical inclusion $K \to L$ is a fiberwise non-trivial bundle morphism, which yields the restriction map $\mathscr{C}_X \to \mathscr{C}_Y$, though this is not the usual restriction map of ordinary functions. Under this restriction map, $\mathscr{C}_X$ forms a sheaf of **R**-algebras in the usual sense.

As another example, the lifting $(-1) : L \to L$ of the identity map $X \to X$ pulls back any twisted complex function $f \in \mathscr{C}_X$ to its conjugate $\bar{f} \in \mathscr{C}_X$.

In a similar way, an *L-twisted complex manifold structure* on X is an equivalence class of pairs $(\mathscr{U}, \varphi_\mathscr{U})$, where $\varphi_\mathscr{U}$ is an open cover of $\mathscr{U}$-holomorphic charts $\varphi_{\tilde{U}} : U \to \mathbf{C}^m$. The transition function $\varphi_{\tilde{U}\tilde{U}'} = \varphi_{\tilde{U}} \circ \varphi_{\tilde{U}'}^{-1}$ between two charts $\tilde{U}, \tilde{U}'$ should be subject to the twisted compatibility condition that it is either holomorphic or anti-holomorphic on a domain of $\mathbf{C}^m$, depending on whether $s_{UU'}$ is 1 or -1.

The equivalence relation has an obvious meaning here. For example, the pairs $(\mathcal{U}, \varphi_{\mathcal{U}})$ and $(-\mathcal{U}, \bar{\varphi}_{\mathcal{U}})$ are equivalent, where $-\mathcal{U}$ is the trivilization system $\{(U, -s_U)\}$, hence leading to the same twisted complex structure on X.

When X has real dimension two, the twisted complex structure is called a di-analytic structure in [1], and X is then called a Klein surface.

Any twisted complex manifold X has a conjugate $\overline{X}$. Moreover, an *L-twisted holomorphic function* f on X is by definition a twisted complex function $f = \{f_{\tilde{U}}\}$ such that every component function $f_{\tilde{U}} \circ \varphi_{\tilde{U}}^{-1}$ is holomorphic on $\mathrm{Im}\, \varphi_{\tilde{U}}$.

The set $\mathcal{O}_X$ of twisted holomorphic functions on X is again an **R**-algebra.

Next we define twisted holomorphic maps between two twisted complex manifolds. Let X and Y be twisted complex manifolds. A smooth map $h : X \to Y$ is (*twisted*) *holomorphic* if X and Y have representatives $(\mathcal{U}, \varphi_{\mathcal{U}})$ and $(\mathcal{V}, \psi_{\mathcal{V}})$ with the following property. For any point $x \in X$, there are holomorphic charts $(U, \varphi_{\tilde{U}}), (V, \psi_{\tilde{V}})$ containing $x, h(x)$ respectively such that the restriction of h is holomorphic, that is, $\psi_{\tilde{V}} \circ h \circ \varphi_{\tilde{U}}^{-1}$ is holomorphic.

REMARK. (1) Under this definition, a twisted complex manifold X is always isomorphic to its conjugate $\overline{X}$ through the identity map. This is because if X is represented by $(\mathcal{U}, \varphi_{\mathcal{U}})$, then $\overline{X}$ can be represented by $(-\mathcal{U}, \varphi_{\mathcal{U}})$ and the restriction of $\mathrm{Id} : X \to \overline{X}$ is clearly holomorphic under these charts.

(2) We point out that a twisted holomorphic function and a twisted holomorphic map have been defined with a subtle difference in nature. The former is a family of locally defined functions depending on trivializations of the twisting bundle, while the latter is a globally defined map. These differences are best explained in Section 4 where we consider the double cover picture.

The definition of a twisted holomorphic map h should ensure that h is (L, K)-compatible. Indeed the following result is a little stronger.

PROPOSITION 2.2. (1) *If* $h : (X, L) \to (Y, K)$ *is twisted holomorphic, then* h *is* (L, K)-*compatible and admits a canonical lifting* $\tilde{h} : L \to K$. *Consequently, there is a uniquely induced map* $h^* : \mathscr{C}_Y \to \mathscr{C}_X$.
(2) *By restriction,* $h^* : \mathcal{O}_Y \to \mathcal{O}_X$.

PROOF. (1) Let $\mathcal{U} = \{(U, s_U)\}$ and $\mathcal{V} = \{(V, t_V)\}$ be trivializations of L and K as in the definition of a twisted holomorphic map h. Then for any $x \in X$ and $y = h(x)$, there are charts (U, s_U) and (V, t_V) containing x, y such that the restriction of h is holomorphic. Using s_U, t_V, we define the unique lifting locally $\tilde{h}|_U : L|_U \to K|_V$ that is also fiberwise non-trivial. All this says is that local trivializations of L, K are matched in pairs; otherwise the restriction of h would be anti-holomorphic. Patching the charts $U \in \mathcal{U}$ together, we have the required lifting $\tilde{h} : L \to K$.

(2) It is clear from the definitions of $h^* = \bar{h}^*$ and $\tilde{h}$ from (1).   □

For example, the twisted bi-holomorphism Id: $X \to \bar{X}$ lifts to -1 on the twisting bundle L.

Having introduced twisted complex functions and manifolds, let us now consider bundles in the twisting theory. As before choose any trivialization system $\mathcal{U} = \{(U, s_U)\}$ of a twisting bundle $L \to X$. A rank r L-twisted complex vector bundle E is defined to be the quotient

$$\coprod_{(U, s_U) \in \mathcal{U}} U \times \mathbf{C}^r / \sim,$$

where the twisted gluing $\sim$ is defined as follows. Take two typical trivial products $U \times \mathbf{C}^r$, $U' \times \mathbf{C}^r$ associated with trivializations $\tilde{U} = (U, s_U)$, $\tilde{U}' = (U', s_{U'})$. Let $g_{\tilde{U}\tilde{U}'} \in \mathrm{GL}(r, \mathbf{C})$ denote the (twisted) transition function. For $(x, \xi) \in U \times \mathbf{C}^r$ and $(x, \xi') \in U' \times \mathbf{C}^r$, we have $(x, \xi) \sim (x, \xi')$ if and only if

$$(2) \qquad \xi' = \begin{cases} g_{\tilde{U}\tilde{U}'}(x)\xi & \text{with } s_{UU'} = 1, \\ g_{\tilde{U}\tilde{U}'}(x)\bar{\xi} & \text{with } s_{UU'} = -1. \end{cases}$$

The cocycle condition for the transition functions $g_{\tilde{U}\tilde{U}'}$ has the form

$$(3) \qquad g_{\tilde{U}\tilde{U}''} = \begin{cases} g_{\tilde{U}\tilde{U}'} g_{\tilde{U}'\tilde{U}''} & \text{when } s_{UU'} = 1, \\ g_{\tilde{U}\tilde{U}'} \bar{g}_{\tilde{U}'\tilde{U}''} & \text{when } s_{UU'} = -1. \end{cases}$$

To indicate that a local trivialization of E depends on that of L, we use the notation $E|_{\tilde{U}} := U \times \mathbf{C}^r$. This definition of twisted complex bundles coincides with that of d-complex vector bundles introduced in our earlier work [13].

Since the twisted gluings (2) still preserve the real linear structure, the underlying space of E carries a real vector bundle structure over X, which we denote by $E_{\mathbf{R}}$. It can be viewed as the realization of E. Conversely, E can be viewed as a twisted complex vector bundle structure on $E_{\mathbf{R}}$.

Obviously, E is a smooth manifold with projection onto X. So the set of all (twisted) sections, denoted by $\Gamma(E)$, is well-defined. Since E, $E_{\mathbf{R}}$ have the same total space, we have $\Gamma(E) = \Gamma(E_{\mathbf{R}})$, making $\Gamma(E)$ a vector space over $\mathbf{R}$. In terms of local trivializations, a section $s \in \Gamma(E)$ is a family of local sections $\{p_{\tilde{U}}\}_{\tilde{U} \in \mathcal{U}}$ glued together using the twisted compatibility condition. This shows that $\Gamma(E)$ is a $\mathscr{C}_X$-module.

EXAMPLE 1. The twisted trivial complex line bundle $\underline{\tilde{\mathbf{C}}} \to X$ is defined by the transition functions $g_{\tilde{U}\tilde{U}'} = 1$ under any trivialization system $\mathcal{U}$ of L. The sections are precisely the twisted complex functions, that is, $\Gamma(\underline{\tilde{\mathbf{C}}}) = \mathscr{C}_X$. Its realization $\underline{\tilde{\mathbf{C}}}_{\mathbf{R}}$ is isomorphic to $\underline{\mathbf{R}} \oplus L$.

Twisted complex bundle can be characterized in terms of sheaves as in the usual case of complex bundles.

PROPOSITION 2.3. *Through sections, there is a one-to-one correspondence between twisted complex bundles and locally free sheaves of $\mathscr{C}_X$-modules.*

PROOF. Let us focus on the rank 1 case; higher ranks can be shown in essentially the same way.

Suppose that $\mathscr{F}$ is a rank 1 locally free sheaf of $\mathscr{C}_X$-modules. Then for each small enough open set $U \subset X$, the restriction $\mathscr{F}|_U$ is sheaf-isomorphic to $\mathscr{C}_X|_U$. Choose any trivilization $\tilde{U} = (U, s_U)$ of L so that $\mathscr{C}_X|_U$ is sheaf-isomorphic to $\mathbf{C}_X^\infty|_U$, where $\mathbf{C}_X^\infty$ is the sheaf of smooth complex valued functions on X. Combining the two isomorphisms, we have a sheaf isomorphism $g_{\tilde{U}} : \mathscr{F}|_U \to \mathbf{C}_X^\infty|_U$. Since $\mathbf{C}_X^\infty|_U$ is a sheaf over $\mathbf{C}$, $\mathscr{F}|_U$ is over $\mathbf{C}$ also.

Take another open set $U' \subset X$ and a trivilization $\tilde{U}'$ of L. By the same argument, we have a sheaf isomorphism $g_{\tilde{U}'} : \mathscr{F}|_{U'} \to \mathbf{C}_X^\infty|_{U'}$. Since $g_{\tilde{U}}, g_{\tilde{U}'}$ are sheaf isomorphisms, by restriction to $U \cap U'$ we can introduce a sheaf isomorphism

$$g_{\tilde{U}\tilde{U}'} = g_{\tilde{U}} \circ g_{\tilde{U}'}^{-1} : \mathbf{C}_X^\infty|_{U \cap U'} \to \mathbf{C}_X^\infty|_{U \cap U'}.$$

In terms of isomorphisms of stalks, this induces equivalently

$$g_{\tilde{U}\tilde{U}'} : U \cap U' \to \mathrm{GL}(1, \mathbf{C}).$$

It is now straightforward to check that $g_{\tilde{U}\tilde{U}'}$ satisfies the twisted cocycle condition (3) and hence yields a twisted complex line bundle.

Conversely, suppose that E is a twisted complex line bundle. We need to verify that the sections of E and their restrictions form a sheaf $\mathscr{F}(E)$ that is locally free and composed of $\mathscr{C}_X$-modules. First $\mathscr{F}(E) = \mathscr{F}(E_\mathbf{R})$ is certainly a sheaf. Moreover, for any small open set $U \subset X$ and trivilization $\tilde{U} = (U, s_U)$ of L, identify $\Gamma(E|_{\tilde{U}}) = \mathbf{C}_X^\infty(U)$. Since $\mathscr{C}_X(U) = \mathbf{C}_X^\infty(U)$ and both are subject to the same twisted compatibility condition, we see that $\mathscr{F}(E)|_U$ and $\mathscr{C}_X|_U$ are isomorphic sheaves, thus $\mathscr{F}(E)$ is locally free of $\mathscr{C}_X$-modules. □

From Proposition 2.3, it is possible to use the same formal language of ordinary sheaves in twisted bundle theory. However in practice it is more useful to employ the definition of local product decompositions, which is what we often do.

Given two L-twisted complex bundles E, F on X, a strong homomorphism $k : E \to F$ is a family of local complex linear homomorphisms subject to the twisted compatibility condition. More precisely, there is a common trivilization system $\mathscr{U}$ of L such that

$$E = \coprod_{\tilde{U} \in \mathscr{U}} U \times \mathbf{C}^r / \sim, \qquad F = \coprod_{\tilde{U} \in \mathscr{U}} U \times \mathbf{C}^s / \sim$$

and $k = \{k_{\tilde{U}}\}$, where $k_{\tilde{U}} : E|_{\tilde{U}} = U \times \mathbf{C}^r \to E|_{\tilde{U}} = U \times \mathbf{C}^s$ is fiberwise complex linear over U. The compatibility condition is straightforward to write down. Then we define $\mathrm{Hom}^\sim(E, F)$ to be the twisted bundle of all strong homomorphisms between E and F.

Similarly we can introduce other familiar operations such as $E \tilde{\otimes} F$, $E \tilde{\oplus} F$, $E \tilde{\wedge} F$, which are all L-twisted complex vector bundles, provided that the same trivilization system of L is used throughout the definition. Alternatively these operations can be formally defined in terms of the sheaf operations of the associated sheaves. In particular, $E^* = \mathrm{Hom}^\sim(E, \underline{\tilde{\mathbf{C}}})$ and $\det E$ are well-defined twisted complex vector bundles, and $E \tilde{\otimes} \underline{\tilde{\mathbf{C}}} = E$ clearly holds.

REMARK. For twisted complex bundles E, F on X, the associated real vector bundles $(E \tilde{\otimes} F)_{\mathbf{R}}$ and $E_{\mathbf{R}} \otimes F_{\mathbf{R}}$ have different ranks, hence can not be isomorphic each other. On the other hand, we have $(E \tilde{\oplus} F)_{\mathbf{R}} \approx E_{\mathbf{R}} \oplus F_{\mathbf{R}}$ naturally.

Similarly to ordinary complex bundle theory, one can define the rank rs twisted complex vector bundle $E \tilde{\otimes}_{\mathbf{R}} W$, where E and W are now a rank r twisted complex bundle and a rank s real vector bundle respectively. More precisely, start with the natural correspondence $\mathbf{C}^r \otimes_{\mathbf{R}} \mathbf{R}^s \approx \mathbf{C}^{rs}$ so that conjugation on the $\mathbf{C}^r$-factor corresponds to conjugation of $\mathbf{C}^{rs}$. Then define

$$(4) \qquad E \tilde{\otimes}_{\mathbf{R}} W = \coprod U \times \mathbf{C}^{rs}/\sim = \coprod U \times (\mathbf{C}^r \otimes_{\mathbf{R}} \mathbf{R}^s)/\sim$$

under the obvious twisted gluing $\sim$. Unlike $\tilde{\otimes}$, we do have a natural isomorphism

$$(E \tilde{\otimes}_{\mathbf{R}} W)_{\mathbf{R}} \approx E_{\mathbf{R}} \otimes_{\mathbf{R}} W.$$

As a special case, one has the (twisted) complexification of W, $W_{\tilde{c}} := \underline{\tilde{\mathbf{C}}} \tilde{\otimes}_{\mathbf{R}} W$. Conversely, $E \tilde{\otimes}_{\mathbf{R}} W = E \tilde{\otimes} W_{\tilde{c}}$. For example, $\underline{\tilde{\mathbf{C}}}$ is the complexification of the trivial real bundle $\underline{\mathbf{R}}$, while the complexification of L is a twisted complex bundle given by transition functions $g_{UU'} = s_{UU'}$.

When the context is clear, we will often drop the '$\sim$' sign in the various operations introduced above.

EXAMPLE 2. For any smooth manifold (X, L) with twisting bundle, consider the twisted complex bundle

$$\overset{r}{\tilde{\bigwedge}}_c T^*X := \underline{\tilde{\mathbf{C}}} \otimes \overset{r}{\bigwedge} T^*X.$$

Then the section space $\tilde{\Omega}^r_c(X)$ contains precisely the twisted complex valued r-forms on X, so for example $\tilde{\Omega}^0_c(X) = \mathscr{C}_X$. The exterior differential extends to $d : \tilde{\Omega}^r_c(X) \to \tilde{\Omega}^{r+1}_c(X)$ by differentiating the component forms.

Let X be an L-twisted complex manifold. Then an *L-twisted holomorphic bundle* $E \to X$ can easily be defined with little modification. The only requirement is that the transition functions must be either holomorphic or anti-holomorphic. Equivalently E is simply a locally free sheaf of $\mathscr{O}_X$-modules. A homomorphism between two twisted holomorphic bundles can be defined similarly as in the case of twisted complex bundles.

EXAMPLE 3. If X is a twisted complex manifold, then the tangent bundle TX admits a twisted holomorphic vector bundle structure. The resulting bundle will be denoted by $\tilde{T}X$.

There are two ways to recover the twisting bundle L from a twisted complex bundle, which is a useful fact on occasion.

PROPOSITION 2.4. (1) *If the rank of a twisted complex bundle E is odd, then* $\det(E_{\mathbf{R}})$ *can be identified with L canonically. If it is even, then* $\det(E_{\mathbf{R}})$ *is trivial canonically.*
(2) *With any rank,* $\det(\det E)_{\mathbf{R}}$ *is isomorphic to L canonically, where $\det E$ is the twisted determinant bundle.*

PROOF. (1) Take any trivilization system $\mathscr{U}$ of L so that $E|_{\tilde{U}} = U \times \mathbf{C}^r$. Then $E_{\mathbf{R}}|_{\tilde{U}} = U \times \mathbf{R}^{2r}$ and $\det(E_{\mathbf{R}})|_{\tilde{U}} = U \times \mathbf{R} = L|_{\tilde{U}}$ canonically. If $(U', s_{U'}) \in \mathscr{U}$ is a second chart with $s_{UU'} = -1$, then $E|_{\tilde{U}}$ is glued with $E|_{\tilde{U}'}$ under $\xi \mapsto \varphi_{\tilde{U}\tilde{U}'}\bar{\xi}$, where $\varphi_{\tilde{U}\tilde{U}'}$ is the complex transition function of E. This gluing (that is, the conjugation $\xi \mapsto \bar{\xi}$) preserves the natural fiberwise orientation on $\det(E_{\mathbf{R}})|_{\tilde{U}}$ if and only if r is even.

Hence when r is odd, $\det(E_{\mathbf{R}})|_{\tilde{U}}$ is glued with $\det(E_{\mathbf{R}})|_{\tilde{U}'}$ in the same way as $L|_{\tilde{U}}$ is glued with $L|_{\tilde{U}'}$. Therefore, $\det(E_{\mathbf{R}}) \approx L$. The isomorphism is canonical because it is independent of the choice of $\mathscr{U}$.

When r is even, the twisted gluing always preserves the complex orientation and $\det E_{\mathbf{R}}$ is surely trivial.

(2) Since $\det E$ is a twisted complex bundle of odd rank, by part (1) its realization has a determinant isomorphic to L canonically, that is, $\det(\det E)_{\mathbf{R}} = L$.   □

Clearly any strong homomorphism $k : E \to F$ induces a homomorphism $\det k : \det E \to \det F$, hence also $\check{k} : L \to L$ by part (2) of Proposition 2.4, which is in fact $+1$ by definition of k. Compare this with the remark below.

It is possible to generalize the concept of strong homomorphisms as defined above. Let $h : X \to Y$ be a smooth map and $E \to X$, $F \to Y$ be respectively L- and K-twisted complex bundles. We define a lifting *bundle homomorphism* $k : E \to F$ of h in the following familiar way: there are trivilization systems $\mathscr{U}$ of L and $\mathscr{V}$ of K,

such that for every point $x \in X$, there are $\tilde{U} \in \mathscr{U}$ containing x and $\tilde{V} \in \mathscr{V}$ containing $h(x)$ so that k restricts to a local complex linear homomorphism $k_{\tilde{U}\tilde{V}} : E|_{\tilde{U}} \to F|_{\tilde{V}}$. Clearly one has an induced bundle homomorphism $\det k : \det E \to \det F$ and real linear homomorphism $k_{\mathbf{R}} : E_{\mathbf{R}} \to F_{\mathbf{R}}$. Furthermore, by Proposition 2.4 above, $\det k$ induces a morphism $\check{k} : L \to K$ canonically, which means that $h : X \to Y$ is (L, K)-compatible with respect to the lifting $\check{k}$. Conversely, if $h : X \to Y$ is (L, K)-compatible with a lifting $\tilde{h} : L \to K$, then the pull-back $\tilde{h}^*F$ can be defined and there is a unique twisted bundle homomorphism $\tilde{h}^*F \to F$ which lifts h.

REMARK. (1) In the special case that $K = L$ and $h = \mathrm{Id} : X \to X$, the induced lifting $\check{k} : L \to L$ of Id must be ± 1. Only when $\check{k} = 1$, the homomorphism $k : E \to F$ is a strong homomorphism.

(2) In terms of these definitions, we now see that any twisted bundle E is isomorphic to its conjugate $\bar{E}$, but not strongly isomorphic. The importance of this distinction is seen in a later section.

Finally the case of trivial twisting bundles is what one may expect.

PROPOSITION 2.5. *If twisting bundles are trivial, then the twisted theory reduces to the usual complex geometry.*

PROOF. The theory includes twisted complex functions, twisted complex structures, twisted bundles, and twisting-preserving maps. We use the case of twisted complex functions as an illustration; the rest can be done similarly.

Let the bundle $L \to X$ be with a fixed global trivialization t of norm 1. Suppose $f = \{f_{\tilde{U}}\}$ is an L-twisted complex function associated with a system of local trivializations $\{(U, s_U)\}$ of L. We need to modify $f_{\tilde{U}}$ so that we can patch them together to form a global complex function $\tilde{f}$. In fact, let us define $\tilde{f}|_U = f_{\tilde{U}}$ if $t|_U = s_U$ and $\tilde{f}|_U = \bar{f}_{\tilde{U}}$ if $t|_U = -s_U$. Then it is easy to check that $\tilde{f}$ is a globally well-defined function. The procedure can be reversed. Hence we obtain a one-to-one correspondence between twisted complex functions and global complex functions. $\square$

We call a twisted complex structure *trivial* if it can be reduced to an ordinary complex structure. Note that if the global trivialization t is switched into $-t$, then we will get the conjugate function under the correspondence in the proof. In other words, the reduction into the ordinary theory stated in the proposition is not unique but up to conjugation, unless a trivialization of the twisting bundle is fixed.

COROLLARY 2.6. *A twisted complex manifold of even dimensions is always orientable as a smooth manifold. In the case of odd complex dimension, the manifold is orientable if and only if the twisted complex structure is trivial.*

PROOF. Recall that $\tilde{T}X$ denotes the tangent bundle of a twisted complex manifold X. The realization is the smooth tangent bundle $\tilde{T}_\mathbf{R}X = TX$. When the rank of $\tilde{T}X$ is even, $\det TX = \det(\tilde{T}_\mathbf{R}X)$ should be trivial by Proposition 2.4. Hence X is orientable.

When the rank of $\tilde{T}X$ is odd, $\det TX = \det(\tilde{T}_\mathbf{R}X)$ is the twisting line bundle L of X by Proposition 2.4. Hence X is orientable if and only if L is trivial, that is, if and only if the twisted complex structure is trivial, by Proposition 2.5. $\square$

This means that we have to consider non-orientable manifolds in the case of odd complex dimension in order to find interesting twisted complex structures.

3. Twisted almost complex structures

We have seen that a twisted complex vector bundle E carries a real vector bundle structure, namely $E_\mathbf{R}$. It is important to recognize that there is some kind of additional structure on $E_\mathbf{R}$; this is the main purpose of the section. We shall take a global and intrinsic approach here, rather than the approach of using local trivialization systems of the twisting bundle as done in the previous section.

First for any real line bundle L, $L^2 = L \otimes L$ is a trivial real line bundle. With a fixed fiber metric on L, L^2 has a canonical global trivialization given by patching together the constant trivializations $s_U \otimes s_U$, where $\mathscr{U} = \{(U, s_U)\}$ is any orthonormal trivialization system of L. Hence we have a canonical isomorphism $L^2 \approx \underline{\mathbf{R}}$.

DEFINITION 3.1. Let $E_\mathbf{R} \to X$ be a real vector bundle and $L \to X$ a real line bundle with fiber metric. An L-twisted (fiberwise) almost complex structure on $E_\mathbf{R}$ is a real bundle isomorphism $J : E_\mathbf{R} \to E_\mathbf{R} \otimes L$ such that the following composition is -1:

$$(5) \qquad\qquad E_\mathbf{R} \xrightarrow{J} E_\mathbf{R} \otimes L \xrightarrow{J \otimes 1} E_\mathbf{R} \otimes L^2 \longrightarrow E_\mathbf{R},$$

where the last homomorphism uses the canonical isomorphism $L^2 \approx \underline{\mathbf{R}}$.

The key difference from the usual almost complex structure is that this is only a globally meaningful concept and cannot be pointwisely defined, because of the involvement of L.

Recall an L-twisted complex vector bundle E has local complex trivializations built upon trivializations of L. From this perspective, we are led to the following.

THEOREM 3.2. *A real vector bundle $E_\mathbf{R} \to X$ admits an L-twisted complex vector bundle structure E if and only if $E_\mathbf{R}$ carries an L-twisted almost complex structure.*

PROOF. Suppose that E is a rank r twisted complex bundle. With respect to a trivialization system $\mathscr{U} = \{(U, s_U)\}$ of L, E is by definition $\coprod U \times \mathbf{C}^r/\sim$, subject to the twisted gluing rule $\sim$. Let us define $J : E_\mathbf{R} \to E_\mathbf{R} \otimes L$ first locally by $J_{\tilde{U}}(x, \xi) = (x, \mathbf{i}\xi \otimes s_U)$, where $\tilde{U} = (U, s_U) \in \mathscr{U}$ and $(x, \xi) \in U \times \mathbf{C}^r$. Certainly $J_{\tilde{U}}$ is real linear fiberwise. To see that different $J_{\tilde{U}}$'s fit together to yield J, let $\tilde{U}' = (U', s_{U'}) \in \mathscr{U}$ be another trivialization over U'. One needs to verify that $J_{\tilde{U}}$ and $J_{\tilde{U}'}$ commute with the gluings on $E_\mathbf{R}$ and $E_\mathbf{R} \otimes L$. On the $E_\mathbf{R}$ side, ξ is glued with $g\xi$ or $g\bar{\xi}$ according to $s_{UU'} = 1$ or -1, where $g = g_{UU'}$ is the transition function of E. On the $E_\mathbf{R} \otimes L$ side, $\mathbf{i}\xi \otimes s_U$ is glued with $g(\mathbf{i}\xi) \otimes s_{U'}$ or $g(\overline{\mathbf{i}\xi}) \otimes (-s_{U'})$ under the same conditions. Now

$$J_{\tilde{U}'}(x, g\xi) = (x, \mathbf{i}g\xi \otimes s_{U'}) = (x, g(\mathbf{i}\xi) \otimes s_{U'})$$

or

$$J_{\tilde{U}'}(x, g\bar{\xi}) = (x, \mathbf{i}g\bar{\xi} \otimes s_{U'}) = (x, g(\overline{\mathbf{i}\xi}) \otimes (-s_{U'})),$$

from which the aforementioned commutativity follows in either case. Thus J is well-defined. It is easy to check that J is a twisted almost complex structure, since $J_{\tilde{U}}$ satisfies the analogous property locally.

Conversely, suppose that $E_\mathbf{R}$ is a rank s real vector bundle with a twisted almost complex structure $J : E_\mathbf{R} \to E_\mathbf{R} \otimes L$. Take any trivialization system $\mathscr{U}$ of L as before. Suppose, after a common refinement if necessary, that $E_\mathbf{R}$ can be trivialized also over each U with transition functions $h_{UU'}$ so that $E_\mathbf{R}|_U \approx U \times \mathbf{R}^s$. Under these trivializations of $E_\mathbf{R}$ and L over U, write $J(x, \xi) = (x, J_{\tilde{U}}(\xi) \otimes s_U)$ for any $(x, \xi) \in U \times \mathbf{R}^s$. Then $J_{\tilde{U}} : \mathbf{R}^s \to \mathbf{R}^s$ is real linear and property (5) is equivalent to $J_{\tilde{U}}^2 = -1$. Hence the fiber $\mathbf{R}^s$ over x carries a complex structure given by $J_{\tilde{U}}$. If $-s_U$ is used as a trivialization of L over U, then the corresponding complex structure on $\mathbf{R}^s$ is the conjugate $-J_{\tilde{U}}$. In other words, the complex structure on a fiber of E is contingent on the choice of trivializations of L. Incorporating $J_{\tilde{U}}$ and $h_{UU'}$, one obtains twisted complex transition functions $g_{UU'}$ for an L-twisted complex bundle structure on $E_\mathbf{R}$. $\square$

Thus a twisted complex bundle E can be characterized as a pair $(E_\mathbf{R}, J)$. If $F \to Y$ is a K-twisted bundle over Y and $h : X \to Y$ is any smooth map, recall a lifting bundle morphism $k : E \to F$ of h associates to two real morphisms $k_\mathbf{R} : E_\mathbf{R} \to F_\mathbf{R}$ and $\check{k} : L \to K$. We can now reinterpret this in terms of twisted almost structures.

COROLLARY 3.3. *Given twisted complex bundles* $(E_\mathbf{R}, J_E)$ *on* (X, L) *and* $(F_\mathbf{R}, J_F)$ *on* (Y, K), *a pair of real bundle homomorphisms* $k_\mathbf{R} : E_\mathbf{R} \to F_\mathbf{R}$ *and* $\check{k} : L \to K$ *yield a twisted complex bundle homomorphism if and only if the following diagram*

commutes:

$$
\begin{array}{ccc}
E_{\mathbf{R}} & \xrightarrow{\;k_{\mathbf{R}}\;} & F_{\mathbf{R}} \\
{\scriptstyle J_E}\downarrow & & \downarrow{\scriptstyle J_F} \\
E_{\mathbf{R}} \otimes L & \xrightarrow{\;k_{\mathbf{R}} \otimes \check{k}\;} & F_{\mathbf{R}} \otimes K.
\end{array}
$$

In particular, when $X = Y$ and $L = K$, $k_{\mathbf{R}}$ yields a strong twisted homomorphism if and only if the following diagram commutes:

$$
\begin{array}{ccc}
E_{\mathbf{R}} & \xrightarrow{\;k_{\mathbf{R}}\;} & F_{\mathbf{R}} \\
{\scriptstyle J_E}\downarrow & & \downarrow{\scriptstyle J_F} \\
E_{\mathbf{R}} \otimes L & \xrightarrow{\;k_{\mathbf{R}} \otimes 1\;} & F_{\mathbf{R}} \otimes L.
\end{array}
$$

PROOF. It is enough to show one direction, since the process will clearly be invertible. Suppose that k induces a twisted homomorphism on the corresponding twisted complex bundles E, F and that k covers a map $h : X \to Y$. By definition, for any $x \in X$, there are neighborhoods U, V of $x, h(x)$ respectively, such that we can trivialize $L|_U = U \times \mathbf{R}$ under a local frame s_U with respect to which $E|_{\tilde{U}} = U \times \mathbf{C}^r$ and we can trivialize $K|_V = V \times \mathbf{R}$ under a local frame t_V with respect to which $F|_{\tilde{V}} = V \times \mathbf{C}^s$. Furthermore, $k : E|_{\tilde{U}} \to F|_{\tilde{V}}$ is fiberwise complex linear and $\check{k} = 1 : L|_U \to K|_V$ under the trivilizations since $\check{k}$ must map s_U to t_V. Hence the following diagram commutes:

(6)

$$
\begin{array}{ccc}
E|_{\tilde{U}} & \xrightarrow{\;k\;} & F|_{\tilde{V}} \\
{\scriptstyle \mathbf{i}\otimes 1}\downarrow & & \downarrow{\scriptstyle \mathbf{i}\otimes 1} \\
E_{\mathbf{R}} \otimes L & \xrightarrow{\;k \otimes \check{k}\;} & F_{\mathbf{R}} \otimes L.
\end{array}
$$

From the proof of Theorem 3.2, locally $J_E = \mathbf{i} \otimes 1 : E|_{\tilde{U}} \to E|_{\tilde{U}} \otimes L|_U$ if we identify E with $E_{\mathbf{R}}$ as total spaces. Similarly, $J_F = \mathbf{i} \otimes 1 : F|_{\tilde{V}} \to F|_{\tilde{V}} \otimes K|_V$. Thus diagram (6) translates into the one that appears in Corollary 3.3. ☐

REMARK. The conjugate $\bar{E}$ is of course given by $(E_{\mathbf{R}}, -J)$. The twisted bundles $(E_{\mathbf{R}}, J)$ and $(E_{\mathbf{R}} \otimes L, J \otimes 1)$ are strongly isomorphic through $J : E_{\mathbf{R}} \to E_{\mathbf{R}} \otimes L$ by Corollary 3.3.

Recall that the section space $\Gamma(E) = \Gamma(E_{\mathbf{R}})$ for any twisted complex bundle E. Since a twisted almost structure is always a real bundle isomorphism, the following corollary is obvious.

COROLLARY 3.4. *Any real linear operator* $P : \Gamma(E_\mathbf{R}) \to \Gamma(F_\mathbf{R})$ *induces a unique real linear operator* $P' : \Gamma(E_\mathbf{R} \otimes L) \to \Gamma(F_\mathbf{R} \otimes L)$ *such that the following diagram commutes*:

$$
\begin{array}{ccc}
\Gamma(E_\mathbf{R}) & \xrightarrow{\;\;P\;\;} & \Gamma(F_\mathbf{R}) \\
\Big\downarrow{\scriptstyle J} & & \Big\downarrow{\scriptstyle J} \\
\Gamma(E_\mathbf{R} \otimes L) & \xrightarrow{\;\;P'\;\;} & \Gamma(F_\mathbf{R} \otimes L).
\end{array}
$$

In view of twisted almost structures, one should define a *twisted complex linear operator* to be a pair (P, P') as in the diagram above. Corollary 3.4 means that any real operator P determines uniquely a twisted complex operator pair, although, the choice of P' does depend on the twisted complex structures. We see a proper explanation of this in the next section.

Another basic concept can be generalized to the twisted set up.

DEFINITION 3.5. A *connection* on a twisted complex bundle E is a real linear operator $D : \Gamma(E) \to \Gamma(T^*X \otimes E)$ such that the Leibnitz formula

$$
D(fs) = df \otimes s + f \, Ds
$$

holds for any twisted complex function f and section $s \in \Gamma(E)$.

In this definition, the twisted complex information is built in the multiplication by the twisted complex function. In order to make sense of $f \, Ds$, one should view $T^*X \otimes E$ as the twisted complex bundle $E \, \tilde{\otimes}_\mathbf{R} \, T^*X$, that is, $E \otimes \tilde{T}_c^*X$ (see equation (4)).

Alternatively, it is possible to characterize D using a compatibility involving the twisted almost complex structure. First, with the chosen metric on L the transition functions $s_{UU'} = \pm 1$ are constant, so the exterior differential d on X extends to $d' : \Gamma(L) \to \Gamma(T^*X \otimes L)$ by differentiating sections locally. Then a real connection $D : \Gamma(E_\mathbf{R}) \to \Gamma(T^*X \otimes E_\mathbf{R})$ induces a connection on the twisted complex bundle E if and only if the following diagram commutes:

$$
\begin{array}{ccc}
\Gamma(E_\mathbf{R}) & \xrightarrow{\;\;D\;\;} & \Gamma(T^*X \otimes E_\mathbf{R}) \\
\Big\downarrow{\scriptstyle J} & & \Big\downarrow{\scriptstyle \mathrm{Id}_{T^*X} \otimes J} \\
\Gamma(E_\mathbf{R} \otimes L) & \xrightarrow{\;\;D \otimes d'\;\;} & \Gamma(T^*X \otimes L \otimes E_\mathbf{R}).
\end{array}
$$

Incidentally and for later use, we observe that d' extends to the differential complex

$$
\Gamma(L) \xrightarrow{\;d'\;} \Gamma(T^*X \otimes L) \xrightarrow{\;d'\;} \Gamma(\wedge^2 T^*X \otimes L) \xrightarrow{\;d'\;} \cdots
$$

from which we obtain the cohomology of X with twisted coefficients $H^k(X, L) := \mathrm{Ker}\, d' / \mathrm{Im}\, d'$.

4. Comparison with the real parts

In this section, we compare the twisted complex geometry on X with the equivariant geometry on a certain cover of X. Via such a comparison, we clarify a few issues that arose in the previous sections.

Let $L \to X$ be a non-trivial twisting line bundle with fiber metric. The associated $\mathbf{Z}_2$-principal bundle $\pi : \tilde{X} \to X$ is a non-trivial double cover of X. It is important to note that the pull-back bundle $\pi^*L \to \tilde{X}$ is canonically trivialized, which will be denoted as $\tilde{L}$. Likewise, as a convention, we will mark with the '~' sign other pull-back data on $\tilde{X}$. Then $\tilde{L}$ has a canonical global trivialization, that is, the fibers of $\tilde{L}$ are canonically oriented.

Use $\sigma : \tilde{X} \to \tilde{X}$ to denote the covering involution, so that $\pi\sigma = \pi$. We summarize the comparison of elementary results together.

THEOREM 4.1. (1) *Any L-twisted complex function f on X can be lifted to a unique complex function $\tilde{f} : \tilde{X} \to \mathbf{C}$, which is (σ, conj)-equivariant: $\tilde{f} \circ \sigma = \bar{\tilde{f}}$. The conjugation $\bar{f}$ lifts to that of $\tilde{f}$.*

(2) Given a second manifold with twisting bundle (Y, K) and its associated double cover $\tilde{Y}$, any twisting-preserving map $h : X \to Y$ can be lifted to a unique smooth map $\tilde{h} : \tilde{X} \to \tilde{Y}$ that is (σ, τ)-equivariant, where $\tau : \tilde{Y} \to \tilde{Y}$ is the covering transformation.

(3) An L-twisted complex bundle $E \to X$ can be lifted to a complex vector bundle $\tilde{E} \to \tilde{X}$, which admits a fiberwise anti-linear homomorphism $\tilde{\sigma} : \tilde{E} \to \tilde{E}$ covering σ. Furthermore, $\tilde{\sigma}$ is an involution.

(4) If $k : E \to F$ is a twisted bundle homomorphism covering $h : X \to Y$, then k itself can be lifted to a complex vector bundle homomorphism $\tilde{k} : \tilde{E} \to \tilde{F}$, which covers $\tilde{h}$ and is $(\tilde{\sigma}, \tilde{\tau})$-equivariant.

(5) If $k : E \to F$ is a strong twisted homomorphism (so $h = \text{Id} : X \to X$), then $\tilde{h} = \text{Id} : \tilde{X} \to \tilde{X}$ and $\tilde{k} : \tilde{E} \to \tilde{F}$ covers Id ($\tilde{h}$ would be σ if k is a non-strong homomorphism).

(6) If X is an L-twisted complex manifold, then $\tilde{X}$ has a canonical complex structure under which σ acts as anti-holomorphic involution. Furthermore, if f is twisted holomorphic on X, then its lift $\tilde{f}$ is holomorphic on $\tilde{X}$.

(7) If $h : X \to Y$ is a twisted holomorphic map, then its lifting $\tilde{h} : \tilde{X} \to \tilde{Y}$ is holomorphic and (σ, τ)-equivariant.

Moreover, all the statements above have appropriate converses, and as a result, we have a one-to-one correspondence between twisted complex objects on X and equivariant complex objects on $\tilde{X}$.

PROOF. (1) Since $\pi : \tilde{X} \to X$ is a $(\tilde{L}, L)$-twisting preserving map, the pull-back

$\tilde{f} = \pi^* f$ is an $\tilde{L}$-twisted complex function on $\tilde{X}$. Since $\tilde{L}$ is canonically trivial, it follows that $\tilde{f}$ can be identified with a well-defined complex function on $\tilde{X}$ by Proposition 2.5 and the remark following it. Alternatively, choose any trivialization system $\{(U, s_U)\}$ of L. Then $\tilde{X}$ is the union of open sets of the form $\tilde{U} = (U, s_U)$. If f is represented by $\{f_{\tilde{U}}\}$, then its pull-back $\tilde{f}$ is defined locally to be $\tilde{f}|_{\tilde{U}} = f_{\tilde{U}}$ on $\tilde{U}$. Either way, clearly $\tilde{f}$ is (σ, conj)-equivariant and the conjugation of f lifts to the conjugation of $\tilde{f}$.

Conversely, any (σ, conj)-equivariant complex function $\tilde{f}$ on $\tilde{X}$ can be pushed down to an L-twisted complex function f on X by reversing the argument. This proves part (1) and its converse.

Parts (2)–(5) can be proved analogously and details are omitted.

(6) Using π, $\tilde{X}$ has a pull back $\tilde{L}$-twisted complex structure. Since $\tilde{L}$ is canonically trivialized, by Proposition 2.5, $\tilde{X}$ inherits a unique complex structure. The rest is straightforward.

(7) Similar to (6).   □

REMARK. In part (7), the twisted holomorphic map h is obtained from $\tilde{h}$ by quotienting both $\tilde{X}$ and $\tilde{Y}$ (under σ and τ). If we were to do this for a holomorphic function $\tilde{f}$ on $\tilde{X}$, then we would have to quotient $\mathbf{C}$ under conjugation, but $\mathbf{C}/\text{conj}$ is not so convenient to use because of the boundary. To avoid this, as in part (6), we only quotient the $\tilde{X}$ side. However, as a result, our twisted holomorphic function f is defined by a family of functions depending on trivializations of L. This explains the subtle difference between the definitions of a twisted holomorphic function and a twisted holomorphic map that was alluded to in the remark before Proposition 2.2.

In the next theorem, we do not distinguish between a twisted complex bundle E and its realization $E_{\mathbf{R}}$.

THEOREM 4.2. *For a twisted complex bundle E, the twisted almost complex structure $J : E \to E \otimes L$ lifts to the almost complex structure $\tilde{J}$ on $\tilde{E}$.*

PROOF. Since $\tilde{\sigma} : \tilde{E} \to \tilde{E}$ is anti-complex linear fiberwise, the almost complex structure $\tilde{J} : \tilde{E} \to \tilde{E} \otimes \underline{\mathbf{R}}$ is equivariant under the maps $\tilde{\sigma}$ and $(\tilde{\sigma}, -1)$. Quotienting the bundles $\tilde{E}$, $\tilde{E} \otimes \underline{\mathbf{R}}$ and noting that σ is free, we have the induced map $J : E \to E \otimes L$. In other words, $\tilde{J}$ is the lift of J.   □

COROLLARY 4.3. *Any section $s \in \Gamma(E)$ lifts to a $\tilde{\sigma}^*$-equivariant section $\tilde{s}$ on $\tilde{E}$, where the action is $\tilde{\sigma}^* \tilde{s} = \tilde{\sigma}^{-1} \circ \tilde{s} \circ \sigma$. In fact, under the pull-back map π^*, more is true:*

$$\Gamma(E) = \text{Fix}\,\tilde{\sigma}^*, \quad \Gamma(E \otimes L) = \text{Fix}^-\,\tilde{\sigma}^* := \{t \in \Gamma(\tilde{E}) \mid \tilde{\sigma}^* t = -t\}.$$

PROOF. To verify that $\Gamma(E \otimes L) = \text{Fix}^- \tilde{\sigma}^*$, note that $\Gamma(E \otimes L)$ contains precisely those sections in $\tilde{E} \otimes \mathbf{R}$ which are fixed under $(\tilde{\sigma}, -1)$. These in turn are precisely in the set $\{t \in \Gamma(\tilde{E}) \mid \tilde{\sigma}^* t = -t\}$. The rest of the corollary is clear. $\square$

The corollary implies the decompositions

(7) $\Gamma(\tilde{E}) = \text{Fix } \tilde{\sigma}^* \oplus \text{Fix}^- \tilde{\sigma}^* = \text{Fix } \tilde{\sigma}^* \oplus \tilde{J}(\text{Fix } \tilde{\sigma}^*).$

Consider a real linear operator $P : \Gamma(E) \to \Gamma(E)$. Pull it back to the covers so one has $\tilde{P} : \text{Fix } \tilde{\sigma}^* \to \text{Fix } \tilde{\sigma}^*$ by Corollary 4.3. In order to extend this to an equivariant complex linear operator on $\Gamma(\tilde{E})$, it is necessary and sufficient to define $\tilde{P}'(t) = \tilde{J} \tilde{P}(t)$ for $t \in \text{Fix}^- \tilde{\sigma}^*$, in view of (7). Quotienting $\tilde{E} \otimes \mathbf{R}$ by $(\tilde{\sigma}, -1)$ and by Corollary 4.3, one has a linear operator $P' : \Gamma(E \otimes L) \to \Gamma(E \otimes L)$ that is uniquely determined by P and J. Compare with the discussion after Corollary 3.4 (here for simplicity, the same bundle E is taken to replace F).

REMARK. (1) One can verify easily that a connection D on E corresponds uniquely to a connection $\tilde{D}$ on $\tilde{E}$ that is $\tilde{\sigma}$-equivariant. Here the action is defined by $(\tilde{\sigma} \tilde{D})(s) = \tilde{\sigma}^{*-1}(\tilde{D}(\tilde{\sigma}^* s))$.

(2) Similar to sections, we have over the field $\mathbf{R}$ that

$$H^k(X) = \{\alpha \in H^k(\tilde{X}) \mid \sigma^* \alpha = \alpha\}$$

and

$$H^k(X, L) = \{\alpha \in H^k(\tilde{X}) \mid \sigma^* \alpha = -\alpha\}.$$

REMARK. In parts (3) and (6) of Theorem 4.1, $\tilde{\sigma}$ and σ are usually referred as *real structures* on $\tilde{E}$ and $\tilde{X}$ respectively. The K-theory of such virtual bundles with real structures is the well-known KR group of $\tilde{X}$ defined by Atiyah in his early work. Of course, the main point of our approach has been to work on the quotient X. Theorem 4.1 and its corollaries tell that the real parts of objects defined on $\tilde{X}$ can be identified naturally with the twisted objects on X. In particular, $\text{KR}(\tilde{X})$ can be identified with the K-group of twisted bundles on X.

Another result in a similar spirit is Greenleaf's formula [3] for analytic sheave cohomology on Klein surfaces, which we now generalize to higher dimensions.

Recall for a twisted complex manifold X, the structure sheaf $\mathcal{O} = \mathcal{O}_X$ is by definition the sheaf of twisted holomorphic functions. This is a sheaf of $\mathbf{R}$-algebras. On the other hand, for the double $\tilde{X}$ with the canonical complex structure, the structure sheaf of holomorphic functions $\tilde{\mathcal{O}}$ is of course defined over $\mathbf{C}$. Theorem 4.1 implies that π induces an injective cohomomorphism $\pi^* : \mathcal{O} \rightsquigarrow \tilde{\mathcal{O}}$, that is, on the space of sections, $\pi^* : \mathcal{O}(U) \to \tilde{\mathcal{O}}(\tilde{U})$ is injective for any open set $U \subset X$, where $\tilde{U} = \pi^{-1}(U)$ as before.

If F is a sheaf on X and $h : Y \to X$ is a smooth map, then the inverse image sheaf $h^* F$ on Y is defined by the direct limit

$$h^* F(V) = \varinjlim \{ F(U) \mid U \supseteq h(V) \text{ is open in } X \}$$

for each open set $V \subset Y$. In particular, if h is onto, then $h^* F(h^{-1} U) = F(U)$ for any open set $U \subset X$.

Now take any (twisted) analytic sheaf F on X, that is, a sheaf of $\mathscr{O}$-modules on X. Its inverse image $\tilde{F} = \pi^* F$ on $\tilde{X}$ is only a sheaf of $\mathscr{O}$-modules. To get an analytic sheaf on $\tilde{X}$, a suitable tensor product is required.

DEFINITION 4.4. The *lifted analytic sheaf* of F is defined to be $F' = \tilde{\mathscr{O}} \otimes_{\mathscr{O}} \tilde{F}$, where we use the cohomomorphism $\pi^* : \mathscr{O} \rightsquigarrow \tilde{\mathscr{O}}$ in the tensor product.

The sheaf cohomology $H^p(X, F)$ is a real vector space, while $H^p(\tilde{X}, F')$ is a complex vector space. Since F' is coherent, $H^p(\tilde{X}, F')$ is finite dimensional, and so is $H^p(X, F)$ as a consequence of the theorem below.

THEOREM 4.5. *For any analytic sheaf F on X and its analytic lift F' on $\tilde{X}$, there is a canonical isomorphism $\mathbf{C} \otimes_{\mathbf{R}} H^p(X, F) \approx H^p(\tilde{X}, F')$.*

PROOF. First we show that for any open set $U \subset X$, there is a canonical isomorphism between the spaces of sections, $\mathbf{C} \otimes_{\mathbf{R}} F(U) \approx F'(\tilde{U})$, or equivalently,

(8) $$8\mathbf{C} \otimes_{\mathbf{R}} F(U) \approx \tilde{\mathscr{O}}(\tilde{U}) \otimes_{\mathscr{O}(U)} F(U)$$

by noting that $\tilde{F}(\tilde{U}) = F(U)$ canonically since π is onto.

In fact, for any holomorphic function $g \in \tilde{\mathscr{O}}(\tilde{U})$, decompose $g = g' + \mathbf{i}g''$, where $g' = (g + \hat{g})/2$, $g'' = \mathbf{i}(\hat{g} - g)/2$ and $\hat{g}(x) = \bar{g}(\sigma(x))$. Since $\hat{g}' = g'$, $\hat{g}'' = g''$, by Theorem 4.1 g', g'' are both in the image of $\pi^* : \mathscr{O}(U) \to \tilde{\mathscr{O}}(\tilde{U})$. Thus the natural map $\mathbf{C} \otimes_{\mathbf{R}} \mathscr{O}(U) \to \tilde{\mathscr{O}}(\tilde{U})$ is surjective, and hence an isomorphism since it is injective by Theorem 4.1. Therefore, $\mathbf{C} \otimes_{\mathbf{R}} F(U) = (\mathbf{C} \otimes_{\mathscr{O}} (U)) \otimes_{\mathscr{O}(U)} F(U)$ is naturally isomorphic to $\tilde{\mathscr{O}}(\tilde{U}) \otimes_{\mathscr{O}(U)} F(U)$, which verifies (8).

Let $\mathscr{C}_F^0$, $\mathscr{C}_{F'}^0$ denote the sheaves of discontinuous sections of F, F'. Then the canonical isomorphism in (8) induces one between $\mathbf{C} \otimes \mathscr{C}_F^0(U)$ and $\mathscr{C}_{F'}^0(\tilde{U})$, and hence between their quotients $\mathbf{C} \otimes \mathscr{Z}_F^1(U) = \mathbf{C} \otimes \mathscr{C}_F^0(U)/\mathbf{C} \otimes \mathscr{O}(U)$ and $\mathscr{Z}_{F'}^1(\tilde{U}) = \mathscr{C}_{F'}^0(\tilde{U})/\tilde{\mathscr{O}}(\tilde{U})$. This in turn induces a canonical isomorphism on their discontinuous section spaces $\mathbf{C} \otimes \mathscr{C}_F^1(U) \to \mathscr{C}_{F'}^1(\tilde{U})$. By induction, we have the following diagram concerning the canonical resolutions of F and F' by discontinuous sections:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{C} \otimes F(U) & \longrightarrow & \mathbf{C} \otimes \mathscr{C}_F^0(U) & \longrightarrow & \mathbf{C} \otimes \mathscr{C}_F^1(U) & \longrightarrow & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F'(\tilde{U}) & \longrightarrow & \mathscr{C}_{F'}^0(\tilde{U}) & \longrightarrow & \mathscr{C}_{F'}^1(\tilde{U}) & \longrightarrow & \cdots
\end{array}
$$

in which the vertical arrows are all isomorphisms. The commutativity follows because all isomorphisms are canonical.

Finally, setting $U = X$ so $\tilde{U} = \tilde{X}$ in the last diagram, we obtain the diagram for the global sections:

$$
\begin{array}{ccccccc}
0 \longrightarrow & \mathbf{C} \otimes \Gamma(F) & \longrightarrow & \mathbf{C} \otimes \Gamma(\mathscr{C}_F^0) & \longrightarrow & \mathbf{C} \otimes \Gamma(\mathscr{C}_F^1) & \longrightarrow & \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & \Gamma(F') & \longrightarrow & \Gamma(\mathscr{C}_{F'}^0) & \longrightarrow & \Gamma(\mathscr{C}_{F'}^1) & \longrightarrow & \cdots
\end{array}
$$

which yields the desired isomorphisms $\mathbf{C} \otimes H^p(X, F) \to H^p(\tilde{X}, F')$ for $p = 0, 1, \ldots$. $\qquad\square$

Consider a twisted holomorphic bundle E on X and its pull back $\tilde{E}$. Let $\mathscr{E}, \mathscr{E}'$ denote their structure sheaves of sections. By Theorem 4.1, the anti-holomorphic lifting $\tilde{\sigma}$ on $\tilde{E}$ induces a complex anti-linear sheaf isomorphism $\tilde{\sigma} : \mathscr{E}' \to \mathscr{E}'$, hence an anti-linear involution $\tilde{\sigma} : H^p(\tilde{X}, \mathscr{E}') \to H^p(\tilde{X}, \mathscr{E}')$.

COROLLARY 4.6. *The real part* $H^p(\tilde{X}, \mathscr{E}')_{\mathbf{R}} := \mathrm{Fix}\, \tilde{\sigma}$ *is naturally isomorphic to* $H^p(X, \mathscr{E})$.

The corollary follows from Theorem 4.5 since the lift analytic sheaf of $\mathscr{E}$ is $\mathscr{E}'$. This result can be used in describing the deformation of the moduli space of real stable bundles.

5. Applications

We continue using the same notation as in previous sections. So for example, $E \to X$ denotes an L-twisted complex vector bundle, $\mathscr{U} = \{\tilde{U}\}$ a unity trivialization system of L, and $\tilde{E} \to \tilde{X}$ the lifting complex vector bundle.

A *Hermitian metric* h on E is a family of Hermitian metrics $h_{\tilde{U}}$ on $E|_{\tilde{U}}$ subject to the twisted compatibility that over $U \cap U'$, $h_{\tilde{U}}$ equals $h_{\tilde{U}'}$ or $\bar{h}_{\tilde{U}'}$ depending on whether $s_{UU'} = 1$ or -1. Equivalently, h is a *Riemannian metric* on $E_{\mathbf{R}}$ where the almost complex structure $J : E_{\mathbf{R}} \to E_{\mathbf{R}} \otimes L$ is an isometry, $E_{\mathbf{R}} \otimes L$ has the product metric. Of course, in either case, h lifts to a unique Hermitian metric $\tilde{h}$ on $\tilde{E}$ that is $\tilde{\sigma}$-equivariant, which means

$$
\tilde{h}(\tilde{\sigma} u, \tilde{\sigma} v) = \overline{\tilde{h}(u, v)},
$$

since $\tilde{\sigma} : \tilde{E} \to \tilde{E}$ is complex anti-linear fiberwise.

In particular, if T^*X has a twisted complex structure $\tilde{T}^*X$, then a Hermitian metric on X is a Riemannian metric g on T^*X such that $J : T^*X \to T^*X \otimes L$ is isometric.

Further, let us extend g to the twisted complexification $\tilde{T}^*_c X$ in the standard way, and define the associated fundamental 2-form *with values in L*:

$$\omega(V, W) = g(V, JW)$$

for vector fields V, W on X. Naturally we say X is *twisted Kähler* if ω is a closed form. Then the Kähler class $[\omega] \in H^2(X, L)$.

We defined the concept of a connection D on E in Section 3. Extend D in the usual way to $D : \Gamma(T^*X \otimes E) \to \Gamma(\wedge^2 T^*X \otimes E)$ so that the curvature F can be produced. Note that $F \in \Gamma(\wedge^2 T^*X \otimes \widetilde{\text{End}} E)$ should be viewed as a strong endomorphism of E. We need to examine F more carefully in order to define the Chern classes for E via a Chern-Weil type formula.

A typical local component $F_{\tilde{U}}$ of F consists of $r \times r$-complex matrix valued 2-forms, where r is the complex rank of E. Define

$$\frac{\mathbf{i}}{2\pi} F_{\tilde{U}} = \left(\Omega^i_j(\tilde{U}) \right)_{1 \leq i \leq r, 1 \leq j \leq r}.$$

Then the twisted compatibility for F implies that the complex 2-forms $\Omega^i_j(\tilde{U})$, $\Omega^i_j(\tilde{U}')$ must satisfy the condition that on $U \cap U'$,

(9)
$$\Omega^i_j(\tilde{U}) = \begin{cases} \Omega^i_j(\tilde{U}') & \text{if } s_{UU'} = 1, \\ -\overline{\Omega^i_j(\tilde{U}')} & \text{if } s_{UU'} = -1. \end{cases}$$

Expand the determinant and set

$$\det\left(I + \frac{\mathbf{i}}{2\pi} F_{\tilde{U}}\right) = 1 + \gamma_1(\tilde{U}) + \gamma_2(\tilde{U}) + \cdots + \gamma_r(\tilde{U}).$$

Apply the standard determinant formula and (9) so that on $U \cap U'$ we find

$$
\begin{aligned}
\gamma_k(\tilde{U}) &= \frac{1}{k!} \sum \delta^{j_1 \cdots j_k}_{i_1 \cdots i_k} \Omega^{i_1}_{j_1}(\tilde{U}) \wedge \cdots \wedge \Omega^{i_k}_{j_k}(\tilde{U}) \\
&= \begin{cases} \frac{1}{k!} \sum \delta^{j_1 \cdots j_k}_{i_1 \cdots i_k} \Omega^{i_1}_{j_1}(\tilde{U}') \wedge \cdots \wedge \Omega^{i_k}_{j_k}(\tilde{U}') & \text{if } s_{UU'} = 1, \\ \frac{(-1)^k}{k!} \sum \delta^{j_1 \cdots j_k}_{i_1 \cdots i_k} \overline{\Omega^{i_1}_{j_1}(\tilde{U}')} \wedge \cdots \wedge \overline{\Omega^{i_k}_{j_k}(\tilde{U}')} & \text{if } s_{UU'} = -1 \end{cases} \\
&= \begin{cases} \gamma_k(\tilde{U}') & \text{if } s_{UU'} = 1, \\ (-1)^k \gamma_k(\tilde{U}') & \text{if } s_{UU'} = -1. \end{cases}
\end{aligned}
$$

The last step uses the fact that the $2k$-forms $\gamma_k(\tilde{U})$, $\gamma_k(\tilde{U}')$ are real valued. Thus if $k = 2l$, $\gamma_k(\tilde{U})$, $\gamma_k(\tilde{U}')$ can be patched together to yield a global $4l$ form on X, which results in our Chern class $c_{2l}(E) \in H^{4l}(X)$. If $k = 2l + 1$, $\gamma_k(\tilde{U})$, $\gamma_k(\tilde{U}')$ are glued

the same way as L, and hence lead to a global form with values in L. This produces the Chern class $c_{2l+1}(E) \in H^{4l+2}(X, L)$. As a special case, $c_1(E)$ is represented by $(2\pi)^{-1}J \operatorname{tr} F$, with values in L since $J \operatorname{tr} F \in \Gamma(\wedge^2 T^*X \otimes L)$.

Although less instructive, it is possible to define $c_k(E)$ using the equivariant Chern classes $c_k(\tilde{E})$. Since $\tilde{\sigma} : \tilde{E} \to \tilde{E}$ is complex anti-linear, one sees clearly that $\sigma^* c_k(\tilde{E}) = (-1)^k c_k(\tilde{E})$. Hence when k is even, $c_k(\tilde{E})$ is equivariant, descending to a class $c_k(E)$ in $H^k(X)$. When k is odd, $c_k(\tilde{E})$ is anti-equivariant, descending to a class $c_k(E)$ in $H^k(X, L)$.

If $E \to X$ is a twisted complex bundle, then $\underline{\tilde{\mathbf{C}}} \otimes E_{\mathbf{R}} = \tilde{E} \oplus E$. In particular, if X is a compact twisted Kähler manifold which we assume for the rest of the section, then $\tilde{T}^*_{\mathbf{R}} X \otimes \tilde{\mathbf{C}} = \tilde{T}^{1,0}_X \oplus \tilde{T}^{0,1}_X$. Although $\tilde{T}^{1,0}_X, \tilde{T}^{0,1}_X$ are isomorphic each other, only $\tilde{T}^{0,1}_X$ is strongly isomorphic to $\tilde{T}^*X$. Let $\wedge^{p,q}_X = (\wedge^p \tilde{T}^{1,0}_X) \wedge (\wedge^q \tilde{T}^{0,1}_X)$.

Since X is a twisted Kähler manifold, by Proposition 2.4 the orientation bundle $\xi = \det T^*X$ of X is canonically trivial, that is, X is canonically oriented, if $m = \dim_c X$ is even; or canonically isomorphic to L if $m = \dim_c X$ is odd. As a consequence, the integral $\int_X \alpha$ is well-defined for any $2m$-form α with values in the line bundle L^m (recall L^2 is canonically trivialized). For example, any twisted complex bundle E has a *degree* defined by

$$\deg E = \int_X c_1(E) \wedge \omega^{m-1},$$

where the Kähler form $\omega \in \Gamma(\wedge^{1,1}_X \otimes L) \subset \Gamma(\wedge^2_X \otimes L)$.

Using the orientation bundle, we have the Hodge star operator

$$* : \wedge^k_X \to \wedge^{2m-k}_X(\xi),$$

see for example [13]. This extends naturally to

$$* : \wedge^{p,q}_X \to \wedge^{m-q,m-p}_X(\xi)$$

after twisted complexification.

The Kähler form ω as usual defines the map

$$W : \Gamma(\wedge^{p,q}_X) \to \Gamma(\wedge^{p+1,q+1}_X \otimes L), \quad \alpha \mapsto \omega \wedge \alpha.$$

The adjoint of W yields the contraction operator

$$\Lambda = *^{-1} \circ W \circ * : \Gamma(\wedge^{p,q}_X) \to \Gamma(\wedge^{p-1,q-1}_X \otimes L).$$

The bundle ξ is canceled after both $*$ and $*^{-1}$.

Consider a twisted holomorphic bundle $E \to X$. Any Hermitian metric h on E induces a unique compatible connection D. Its curvature is $F \in \Gamma(\wedge^{1,1}_X \otimes \operatorname{End}^{\sim} E)$, so applying the contraction operator gives a section ΛF on $\operatorname{End}^{\sim} E \otimes L$.

DEFINITION 5.1. The metric h is called *Hermitian-Einstein* if the curvature of the associated connection satisfies $\wedge F = cJ(I)$, where c is some constant, I is the identity homomorphism on E, and $J(I) \in \Gamma(\mathrm{End}^\sim E \otimes L)$ using the twisted almost complex structure J on $\mathrm{End}^\sim E$.

It can be verified easily that the constant

$$c = -\deg E/(\mathrm{rank}\, E \cdot \mathrm{vol}\, X) = -\mu(E)/\mathrm{vol}\, X,$$

where $\mu(E) := \deg E / \mathrm{rank}\, E$ is usually referred to as the *slope* of E.

An analytic sheaf $\mathscr{F}$ on X is called *coherent* if any point $x \in X$ has an open neighborhood U such that there is an exact sequence for some integers p, q

$$\mathscr{O}^p|_U \to \mathscr{O}^q|_U \to \mathscr{F}|_U \to 0,$$

where $\mathscr{O}$ is the structure sheaf of twisted holomorphic functions over X.

DEFINITION 5.2. A twisted holomorphic bundle E is called *stable* if for every proper coherent sub-sheaf $\mathscr{F}$ of $\mathscr{F}_E$, we have $\mu(\mathscr{F}) < \mu(\mathscr{F}_E)$.

We now state the essential part of the Narasimhan-Seshadri-Donaldson correspondence. Recall a Klein surface is a twisted complex manifold of real dimension 2.

THEOREM 5.3. *Any stable twisted holomorphic bundle E over a Klein surface X with Kähler metric admits a Hermitian-Einstein fiber metric h.*

In the special case that E is a topologically trivial rank-2 twisted bundle, the theorem was proved in Wang [13]. This extra assumption was made there because the correspondence was stated as between Yang-Mills connections on an ordinary $SU(2)$ bundle and twisted stable bundles. However, essentially the same argument in [13] can be carried over to the more general case in this paper. Basically this amounts to an adaptation of Donaldson's analytic proof [2] to the local twisted picture. The details are omitted.

We have not checked the higher dimensional version of the theorem, although we believe it is plausible to go through the arguments of Donaldson or Uhlenbeck-Yau to establish the same correspondence between stable bundles and Hermitian-Einstein metrics for the case of $\dim X > 2$.

6. Some further remarks

Consider a twisted Kähler manifold X and its Kähler double cover $\tilde{X}$. There are many interesting problems about the moduli space M_X of twisted stable bundles on X

that are of a fixed rank and degree. For example, one is tempted to determine the orientability and cohomology of M_X, especially in the case of a Klein surface X. Perhaps there is some perfect Morse-Bott function in the picture.

Let $M_{\tilde{X}}$ denote the moduli space of ordinary stable bundles on $\tilde{X}$. The anti-holomorphic involution $\sigma : \tilde{X} \to \tilde{X}$ induces an involution

$$\hat{\sigma} : M_{\tilde{X}} \to M_{\tilde{X}}, \quad [\tilde{E}] \mapsto \overline{[\sigma^*\tilde{E}]}.$$

The *real moduli space* on $\tilde{X}$ is by definition $\tilde{M}_{\mathbf{R}} = \mathrm{Fix}\,\hat{\sigma}$. If $[\tilde{E}] \in \tilde{M}_{\mathbf{R}}$, then $\tilde{E}$ admits a lifting $\tau : \tilde{E} \to \tilde{E}$ of σ that has a finite even order (the order depends on $\tilde{E}$ only following from the fact that stable bundles must be simple). The most familiar cases are order two and four, the so-called real or quaternionic types. Only those stable bundles with real type liftings can descend to twisted stable bundles on X. Thus by focusing on twisted bundles, we actually only single out part of the real moduli space on $\tilde{X}$. This is why we expect to get stronger results in the twisted set-up. Moreover, we expect the complement of the twisted moduli space in the real moduli space to have co-dimension 1, thus comprising a relatively small part.

We have been restricted to closed smooth twisted manifolds; thus the involution σ is free of fixed points so the real part $\tilde{X}_{\mathbf{R}} = \emptyset$. We have done so partly because this is important in applications alluded in the introduction. In the dimension 2 case, one can allow a non-free involution, provided one is willing to consider surfaces with boundary. Indeed, Klein surfaces include those with boundary. The higher dimension case appears more subtle, since the twisted manifolds are bound to be singular (except dimension 4).

It is interesting to compare with twisted vector bundles over a gerbe, as defined in [7]. A *gerbe*, in the sense of Hitchin, consists of a family of complex line bundles defined on the intersections of pairs of charts from an atlas of the underlying manifold. A twisted bundle is given by a family of vector bundles on the charts, such that on a pairwise intersection, the local bundles become the same after one is twisted by the complex line bundle in the given gerbe. Our twisted complex bundles are twisted by global real line bundles which are at a lower hierarchy than gerbes, hence may be viewed as a prototype of gerbe-twisted complex bundles. Another kind of twisting appears in the thesis of Gualtieri [4], in which the integrability of a generalized almost complex structure on a manifold X can be characterized using the twisting by the maximal isotropic complex sub-bundle of $(TX \oplus T^*X) \otimes \mathbf{C}$ of the generalized almost complex structure (see [4, Section 4.4]).

Acknowledgment

References

[1] N. L. Alling and N. Greenleaf, *Foundations of the theory of Klein surfaces*, Lecture Notes in Math. 219 (Springer, Berlin, 1971).

[2] S. K. Donaldson, 'A new proof of a theorem of Narasimhan and Seshadri', *J. Differential Geom.* **18** (1983), 269–278.

[3] N. Greenleaf, 'Analytic sheaves on Klein surfaces', *Pacific J. Math.* **37** (1971), 671–675.

[4] M. Gualtieri, *Generalized complex geometry* (Ph.D. Thesis, Oxford University, 2003).

[5] S. Katz and C. Liu, 'Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disc', *Adv. Theor. Math. Phys.* **5** (2001), 1–49.

[6] J. Li and Y. Song, 'Open string instantons and relative stable morphisms', *Adv. Theor. Math. Phys.* **5** (2001), 67–91.

[7] M. Mackaay, 'A note on the holonomy of connections in twisted bundles', *Can. Topol. Géom. Différ. Catég.* **44** (2003), 39–62.

[8] S. M. Natanzon, 'Klein surfaces', *Russian Math. Surveys* **45** (1990), 53–108.

[9] H. Ooguri and C. Vafa, 'Knot invariants and topological strings', *Nuclear Phys. B* **577** (2000), 69–108.

[10] S. Sinha and C. Vafa, 'SO and Sp Chern-Simons at large N', preprint, hep-th/0012136.

[11] G. Tian and S. Wang, 'Orientability and real Seiberg-Witten invariants', in preparation.

[12] S. Wang, 'A vanishing theorem for Seiberg-Witten invariants', *Math. Res. Lett.* **2** (1995), 305–310.

[13] ——, 'A Narasimhan-Seshadri-Donaldson correspondence over non-orientable surfaces', *Forum Math.* **8** (1996), 461–474.

[14] J.-Y. Welschinger, 'Invariants of real rational symplectic 4-manifolds and lower bounds in real enumerative geometry', *C.R. Math. Acad. Sci. Paris* **336** (2003), 341–344.

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