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DELAY-DEPENDENT ROBUST H_∞ CONTROL FOR SINGULAR SYSTEMS WITH MULTIPLE DELAYS

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Abstract

This paper studies the problem of delay-dependent robust H_∞ control for singular systems with multiple delays. Based on a Lyapunov–Krasovskii functional approach, an improved delay-dependent bounded real lemma (BRL) for singular time-delay systems is established without using any of the model transformations and bounding techniques on the cross product terms. Then, by applying the obtained BRL, a delay-dependent condition for the existence of a robust state feedback controller, which guarantees that the closed-loop system is regular, impulse free, robustly stable and satisfies a prescribed H_∞ performance index, is proposed in terms of a nonlinear matrix inequality. The explicit expression for the H_∞ controller is designed by using linear matrix inequalities and the cone complementarity iterative linearization algorithm. Numerical examples are also given to illustrate the effectiveness of the proposed method.

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Keywords and phrases: singular time-delay system, multiple delays, delay-dependent robust H_∞ control, bounded real lemma, linear matrix inequality.

1. Introduction

The study of delay systems has received a lot of attention in recent years since time delays are often encountered in various engineering systems and are frequently the source of instability and poor performance. Many results on stability analysis and control of linear and nonlinear time-delay systems have been reported in the literature; see, for example, [1, 5, 7, 10, 11, 16, 17], and references therein. Singular time-delay systems, which are also referred to as implicit time-delay systems, descriptor

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time-delay systems or generalized differential-difference equations, often appear in engineering systems, including aircraft attitude control, flexible arm control of robots, large-scale electric network control, large-scale chemical engineering systems and lossless transmission lines. Since singular time-delay systems are matrix delay differential equations coupled with matrix difference equations, the study of such systems is much more complicated than that for standard state-space time-delay systems or singular systems. For results on stability of singular time-delay systems, readers are referred to [3, 4, 12, 18, 24, 26, 27, 29].

On the other hand, the problem of H_∞ control has been the subject of extensive research during the past few decades. The main objective of H_∞ control is to design a controller to stabilize the system, and to guarantee that the L_2-induced norm from the disturbance input to the controlled output of the closed loop satisfies a prescribed H_∞ performance level. Recently, H_∞ control for singular time-delay systems has attracted increasing attention in the research community. The existing results can be classified into two types: delay-independent ones [8, 9, 19, 23] and delay-dependent ones [6, 22, 25, 28]. Generally speaking, delay-dependent conditions are less conservative than the delay-independent ones, especially when the delay is small. Moreover, in engineering practice, information on the delay range is generally available. So recent effort has focused more on delay-dependent control.

It is now known that the conservativeness of the existing delay-dependent conditions stems from two causes: one is the model transformation used and the other is the inequality bounding technique employed for some cross product terms. In view of this, an equivalent model transformation, the descriptor system transformation method, was adopted in [6, 25], and another bounded real lemma (BRL) was obtained in [28] without resorting to any model transformations. However, conservativeness still remains as a result of the inequality bounding technique employed in [13, 14]. In the derivation of the BRL in [22], neither bounding techniques nor model transformations were involved, while only a single delay was considered. This motivates us to study further towards less conservative results on robust H_∞ control for singular systems with multiple delays.

In this paper, we shall consider the robust H_∞ control problem for singular systems with multiple delays. For simplicity, we discuss only the case of two delays, which are constant but unknown. First, we will establish, based on the Lyapunov–Krasovskii functional approach, a new delay-dependent BRL in terms of linear matrix inequalities (LMIs). It is noted that this BRL is obtained without using any model transformations and bounding techniques, and can be theoretically proved to be less conservative than that in [28]. Then we propose, by applying the obtained BRL, a delay-dependent condition for the existence of a robust state feedback controller, which guarantees that the closed-loop system is regular, impulse free, robustly stable and satisfies a prescribed H_∞ performance index, in terms of a nonlinear matrix inequality. It is shown that the design for the robust H_∞ controller can be converted to a cone complementarity problem subject to LMIs and consequently can be solved by using an iterative linearization algorithm.

2. Problem statement and preliminaries

Let us begin by introducing some notation for later use. We denote by R the set of real numbers, R^n the n-dimensional Euclidean space over the reals, and $R^{n\times m}$ the set of all $n \times m$ real matrices. For a real symmetric matrix X, the notation $X \geq 0$ ($X > 0$) means that the matrix X is positive semidefinite (positive definite). $C_{n,\tau} := C([-\tau, 0], R^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into R^n; $x_t := x(t+\theta)$, $\theta \in [-\tau, 0]$, $t \geq 0$ denotes the function family defined on $[-\tau, 0]$ which is generated by the n-dimensional real vector-valued continuous function $x(t)$, $t \in [-\tau, +\infty)$. Obviously, $x_t \in C_{n,\tau}$. The following norms will be used: $\|\cdot\|$ refers to the Euclidean vector norm or spectral matrix norm; $\|\phi\|_c := \sup_{-\tau\leq t\leq 0} \|\phi(t)\|$ stands for the norm of a function $\phi \in C_{n,\tau}$. The superscript T represents the transpose. The symbol $*$ will be used in some matrix expressions to induce a symmetric structure, for example,

$$\begin{bmatrix} X & Y \\ * & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}.$$

Consider the following uncertain singular system with two delays:

$$\begin{cases} E\dot{x}(t) = \sum_{i=0}^{2} A_{i\Delta} x(t-\tau_i) + B_{1\Delta} u(t) + B_{2\Delta} w(t), \\ z(t) = G_\Delta x(t) + D_\Delta u(t), \\ x(t) = \phi(t) \quad t \in [-\tau, 0], \end{cases} \tag{2.1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^q$ is the control input, $w(t) \in R^r$ is the disturbance input and $w(t) \in L_2[0, \infty)$, $z(t) \in R^s$ is the controlled output to be attenuated. $\tau_0 = 0$ and $\tau_i > 0$, $i = 1, 2$, are unknown constant delays and satisfy $\tau = \max\{\tau_1, \tau_2\} \leq \tau_m$. We take for simplicity two delays, but all the results are easily generalized to the case of any finite number of delays $\tau_1, \ldots, \tau_l$. $\phi(t) \in C_{n,\tau}$ is a compatible vector-valued initial function. The matrix $E \in R^{n\times n}$ is singular and we assume that $0 < \operatorname{rank} E = p < n$. The system matrices with norm-bounded uncertainties are assumed to be of the following forms:

$$\begin{cases} \begin{bmatrix} A_{0\Delta} & A_{1\Delta} & A_{2\Delta} & B_{1\Delta} & B_{2\Delta} \\ G_\Delta & \star & \star & D_\Delta & \star \end{bmatrix} = \begin{bmatrix} A_0 & A_1 & A_2 & B_1 & B_2 \\ G & \star & \star & D & \star \end{bmatrix} \\ \qquad\qquad + \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} F \begin{bmatrix} N_0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix} \\ F^T F \leq I, \quad F \in R^{i\times j} \end{cases}$$

where A_i ($i = 0, 1, 2$), B_i ($i = 1, 2$), G, D, M_i ($i = 1, 2$), N_i ($i = 0, 1, 2, 3, 4$) are known real constant matrices with appropriate dimensions, F is an uncertain real constant matrix and $\star$ are matrices which are not specified.

REMARK 1. Based on consideration of the robust H_∞ controller design for (2.1), to ensure the zero initial function $\phi(t) \equiv 0$, $t \in [-\tau, 0]$ is a compatible initial function, we limit the initial value of $w(t)$ as $w(0) = 0$. Generally speaking, the instantaneous value of disturbance will not influence the system, so the above-mentioned limit remains general.

Without loss of generality, we assume that

$$E = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix} \tag{2.2}$$

and denote

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad A_i = \begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{bmatrix} \tag{2.3}$$

with $x_i(t) \in R^p$, $A_{i1} \in R^{p \times p}$, $i = 0, 1, 2$.

The purpose of this paper is to design a state feedback controller

$$u(t) = Kx(t) \tag{2.4}$$

such that for any constant time delay $\tau_i : 0 < \tau_i \le \tau_m$: $i = 1, 2$,

(i) the closed-loop system constructed by (2.1) and (2.4),

$$\begin{cases} E\dot{x}(t) = A_{0\Delta}x(t) + \sum_{i=1}^{2} A_{i\Delta}x(t - \tau_i) + B_{2\Delta}w(t), \\ z(t) = G_{k\Delta}x(t), \\ x(t) = \phi(t) \quad t \in [-\tau, 0], \end{cases} \tag{2.5}$$

with

$$A_{k\Delta} = A_k + M_1FN_k, \quad G_{k\Delta} = G_k + M_2FN_k,$$
$$A_k = A_0 + B_1K, \quad N_k = N_0 + N_3K, \quad G_k = G + DK,$$

is regular, impulse free and robustly internally stable (that is, the closed-loop system is robustly stable when $w(t) \equiv 0$);

(ii) the H_∞ performance index

$$J(w) = \int_0^\infty \left(z^T(t)z(t) - \gamma^2 w^T(t)w(t)\right) dt \le 0 \tag{2.6}$$

of the closed-loop system (2.5) is guaranteed for all nonzero disturbance $w(t) \in L_2[0, \infty)$ and a prescribed $\gamma > 0$ under zero initial condition.

In this case, System (2.1) is said to be robustly stabilizable with H_∞ performance γ and (2.4) is said to be a robust H_∞ controller for (2.1).

REMARK 2. Concerning the definition that the singular time-delay system is regular, impulse free and asymptotically stable, see [4, 18, 27, 29]. It is well known that if A_{04} in (2.3) is nonsingular, then System (2.1) is regular and impulse free.

Define the difference operator $\mathcal{D}$:

$$\mathcal{D}(x_{2t}) = x_2(t) + \sum_{i=1}^{2} A_{04}^{-1} A_{i4} x_2(t - \tau_i). \tag{2.7}$$

To get the main results of this paper, the following preliminary lemmas are needed.

LEMMA 2.1 ([4]). *If there exist matrices P_3, $Q_{13} > 0$, $Q_{23} > 0$ with appropriate dimensions that satisfy*

$$\begin{bmatrix} P_3 A_{04} + A_{04}^T P_3^T + \sum_{i=1}^{2} Q_{i3} & P_3 A_{14} & P_3 A_{24} \\ * & -Q_{13} & 0 \\ * & * & -Q_{23} \end{bmatrix} < 0, \tag{2.8}$$

then A_{04} is nonsingular and the difference operator $\mathcal{D}$ is stable for all $\tau_i > 0$, $i = 1, 2$ (that is, the equation $\mathcal{D}(x_{2t}) = 0$ is asymptotically stable).

LEMMA 2.2 ([4]). *Consider the unforced singular time-delay system*

$$E\dot{x}(t) = \sum_{i=0}^{2} A_i x(t - \tau_i) \tag{2.9}$$

with E, A_i, $i = 0, 1, 2$, given in (2.2) and (2.3). If the operator $\mathcal{D}$ is stable and there exist positive numbers α, β, γ and a continuous functional $V : C_{n,\tau} \to R$ such that

$$\beta \|x_1(t)\|^2 \le V(x_t) \le \gamma \|x_t\|_c^2, \qquad \dot{V}(x_t) \le -\alpha \|x(t)\|^2,$$

and the function $\overline{V}(t) = V(x_t)$ is absolutely continuous for x_t satisfying (2.9), then (2.9) is asymptotically stable.

LEMMA 2.3. *Given matrices $A_{11} < 0$, A_{12}, A_{13}, $A_{22} < 0$ and $A_{33} < 0$ with appropriate dimensions, then*

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ * & A_{22} & 0 \\ * & * & A_{33} \end{bmatrix} < 0$$

if and only if there exists a matrix $N_{11} > 0$ such that

$$\begin{bmatrix} A_{11} + N_{11} & A_{12} \\ * & A_{22} \end{bmatrix} < 0 \quad \text{and} \quad \begin{bmatrix} -N_{11} & A_{13} \\ * & A_{33} \end{bmatrix} < 0.$$

PROOF. See Appendix A. □

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3. Delay-dependent bounded real lemma

First of all, consider the unforced nominal system of (2.1):

$$\begin{cases} E\dot{x}(t) = \sum_{i=0}^{2} A_i x(t-\tau_i) + B_2 w(t), \\ z(t) = Gx(t), \\ x(t) = \phi(t) \quad t \in [-\tau, 0]. \end{cases} \tag{3.1}$$

In the following theorem, based on the Lyapunov–Krasovskii functional approach, we present a new delay-dependent bounded real lemma for (3.1), which will play a key role in solving the aforementioned robust H_∞ control problem.

THEOREM 3.1. *Suppose, for some prescribed $\gamma > 0$, that there exist matrices*

$$P = \begin{bmatrix} P_1 & P_2 \\ 0 & P_3 \end{bmatrix}, \quad P_1 > 0, \quad Q_i = \begin{bmatrix} Q_{i1} & Q_{i2} \\ * & Q_{i3} \end{bmatrix} > 0, \quad Z_i = \begin{bmatrix} Z_{i1} & Z_{i2} \\ * & Z_{i3} \end{bmatrix} > 0, \tag{3.2}$$

$$Y_i = \begin{bmatrix} Y_{i1} & 0 \end{bmatrix}, \quad W_i = \begin{bmatrix} W_{i1} & 0 \end{bmatrix}, \quad H_i = \begin{bmatrix} H_{i1} & 0 \end{bmatrix}, \quad i = 1, 2, \tag{3.3}$$

with appropriate dimensions and $P_1 \in R^{p\times p}$, $Q_{i1} \in R^{p\times p}$, $Z_{i1} \in R^{p\times p}$, $Y_{i1} \in R^{n\times p}$, $W_{i1} \in R^{n\times p}$, $H_{i1} \in R^{r\times p}$, $i = 1, 2$, satisfying the following LMI:

$$\begin{bmatrix} \Theta_{11} + G^T G & \Theta_{12} & \Theta_{13} & \Theta_{14} & -\tau_m Y_{11} & -\tau_m Y_{21} & \tau_m A_0^T Z_1 & \tau_m A_0^T Z_2 \\ * & \Theta_{22} & 0 & -H_1^T & -\tau_m W_{11} & 0 & \tau_m A_1^T Z_1 & \tau_m A_1^T Z_2 \\ * & * & \Theta_{33} & -H_2^T & 0 & -\tau_m W_{21} & \tau_m A_2^T Z_1 & \tau_m A_2^T Z_2 \\ * & * & * & -\gamma^2 I & -\tau_m H_{11} & -\tau_m H_{21} & \tau_m B_2^T Z_1 & \tau_m B_2^T Z_2 \\ * & * & * & * & -\tau_m Z_{11} & 0 & 0 & 0 \\ * & * & * & * & * & -\tau_m Z_{21} & 0 & 0 \\ * & * & * & * & * & * & -\tau_m Z_1 & 0 \\ * & * & * & * & * & * & * & -\tau_m Z_2 \end{bmatrix} < 0, \tag{3.4}$$

where

$$\Theta_{11} = PA_0 + A_0^T P^T + \sum_{i=1}^{2} \big(Y_i + Y_i^T + Q_i\big), \quad \Theta_{12} = PA_1 - Y_1 + W_1^T,$$

$$\Theta_{13} = PA_2 - Y_2 + W_2^T, \quad \Theta_{14} = PB_2 + \sum_{i=1}^{2} H_i^T,$$

$$\Theta_{22} = -W_1 - W_1^T - Q_1, \quad \Theta_{33} = -W_2 - W_2^T - Q_2.$$

Then the System (3.1) *is regular, impulse free, internally stable and satisfies* (2.6) *for all nonzero* $w(t) \in L_2[0, \infty)$ *and time delay* τ_i *satisfying* $0 < \tau_i \le \tau_m$, $i = 1, 2$.

PROOF. Substituting (2.2), (2.3), (3.2) and (3.3) into (3.4) gives (2.8), which, by Lemma 2.1, implies that A_{04} is nonsingular. Therefore, System (3.1) is regular and impulse free. In addition, we also conclude that the operator $\mathcal{D}$ in (2.7) is stable.

Construct the Lyapunov–Krasovskii functional for (3.1) as:

$$V(x_t) = x^T(t)PEx(t) + \sum_{i=1}^{2}\left(\int_{t-\tau_i}^{t} x^T(s)Q_i x(s)\,ds + \int_{-\tau_i}^{0}\int_{t+\beta}^{t} \dot{x}_1^T(\alpha)Z_{i1}\dot{x}_1(\alpha)\,d\alpha\,d\beta\right).$$

Differentiating $V(x_t)$ along with the solution of (3.1) yields:

$$\begin{aligned}
\dot{V}(x_t)|_{(3.1)} &+ z^T(t)z(t) - \gamma^2 w^T(t)w(t) \\
&= 2x^T(t)P\begin{bmatrix}\dot{x}_1(t)\\ 0\end{bmatrix} + \sum_{i=1}^{2}\Big(x^T(t)Q_i x(t) - x^T(t-\tau_i)Q_i x(t-\tau_i) \\
&\quad + \tau_i \dot{x}^T(t)E^T Z_i E\dot{x}(t) - \int_{t-\tau_i}^{t}\dot{x}_1^T(\alpha)Z_{i1}\dot{x}_1(\alpha)\,d\alpha\Big) \\
&\quad + x^T(t)G^T Gx(t) - \gamma^2 w^T(t)w(t).
\end{aligned} \tag{3.5}$$

By the Newton–Leibniz formula $x_1(t) - x_1(t-\tau_i) = \int_{t-\tau_i}^{t}\dot{x}_1(\alpha)\,d\alpha$, $i = 1, 2$,

$$\begin{aligned}
2x^T(t)P&\begin{bmatrix}\dot{x}_1(t)\\ 0\end{bmatrix} \\
&= 2x^T(t)P\Bigg[\sum_{i=0}^{2}\left(\begin{bmatrix}A_{i1}\\ A_{i3}\end{bmatrix}x_1(t) + \begin{bmatrix}A_{i2}\\ A_{i4}\end{bmatrix}x_2(t-\tau_i)\right) \\
&\quad - \sum_{i=1}^{2}\begin{bmatrix}A_{i1}\\ A_{i3}\end{bmatrix}\int_{t-\tau_i}^{t}\dot{x}_1(\alpha)\,d\alpha + B_2 w(t)\Bigg] \\
&= 2x^T(t)P\left[\sum_{i=0}^{2}\left(\begin{bmatrix}A_{i1}\\ A_{i3}\end{bmatrix}x_1(t) + \begin{bmatrix}A_{i2}\\ A_{i4}\end{bmatrix}x_2(t-\tau_i)\right) + B_2 w(t)\right] \\
&\quad + \sum_{i=1}^{2}\Bigg[2x^T(t)\left(Y_{i1} - P\begin{bmatrix}A_{i1}\\ A_{i3}\end{bmatrix}\right)\int_{t-\tau_i}^{t}\dot{x}_1(\alpha)\,d\alpha \\
&\quad + 2x^T(t-\tau_i)W_{i1}\int_{t-\tau_i}^{t}\dot{x}_1(\alpha)\,d\alpha + 2w^T(t)H_{i1}\int_{t-\tau_i}^{t}\dot{x}_1(\alpha)\,d\alpha \\
&\quad - 2\big(x^T(t)Y_{i1} + x^T(t-\tau_i)W_{i1} + w^T(t)H_{i1}\big)\int_{t-\tau_i}^{t}\dot{x}_1(\alpha)\,d\alpha\Bigg]
\end{aligned}$$

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$$
\begin{aligned}
&= 2x^T(t)\left(P\begin{bmatrix}A_{01}\\A_{03}\end{bmatrix}x_1(t) + P\begin{bmatrix}A_{02}\\A_{04}\end{bmatrix}x_2(t) + \sum_{i=1}^{2} Y_{i1}x_1(t)\right) + 2x^T(t)PB_2w(t)\\
&\quad + \sum_{i=1}^{2}\left[2x^T(t)\left(P\begin{bmatrix}A_{i1}\\A_{i3}\end{bmatrix}x_1(t-\tau_i) + P\begin{bmatrix}A_{i2}\\A_{i4}\end{bmatrix}x_2(t-\tau_i) - Y_{i1}x_1(t-\tau_i)\right)\right.\\
&\quad + 2x^T(t-\tau_i)W_{i1}x_1(t) - 2x^T(t-\tau_i)W_{i1}x_1(t-\tau_i) + 2w^T(t)H_{i1}x_1(t)\\
&\quad - 2w^T(t)H_{i1}x_1(t-\tau_i) - 2\big(x^T(t)Y_{i1} + x^T(t-\tau_i)W_{i1} + w^T(t)H_{i1}\big)\\
&\quad \left.\times \int_{t-\tau_i}^{t} \dot{x}_1(\alpha)\,d\alpha\right]\\
&= 2x^T(t)\left(PA_0 + \sum_{i=1}^{2} Y_i\right)x(t) + 2x^T(t)PB_2w(t)\\
&\quad + \sum_{i=1}^{2}\left[2x^T(t)\big(PA_i - Y_i + W_i^T\big)x(t-\tau_i) - 2x^T(t-\tau_i)W_ix(t-\tau_i)\right.\\
&\quad + 2x^T(t)H_i^Tw(t) - 2x^T(t-\tau_i)H_i^Tw(t) - 2x^T(t)Y_{i1}\int_{t-\tau_i}^{t}\dot{x}_1(\alpha)\,d\alpha\\
&\quad \left. - 2x^T(t-\tau_i)W_{i1}\int_{t-\tau_i}^{t}\dot{x}_1(\alpha)\,d\alpha - 2w^T(t)H_{i1}\int_{t-\tau_i}^{t}\dot{x}_1(\alpha)\,d\alpha\right]. \qquad (3.6)
\end{aligned}
$$

Here and subsequently we define $\Phi(\mathcal{C}, \mathcal{M}) = \mathcal{C}\mathcal{M}\mathcal{C}^T$ whenever the product exists. More generally we shall also use $\Phi(\mathcal{C}, \mathcal{M}, \mathcal{E}) = \mathcal{C}\mathcal{M}\mathcal{E}^T$. Combining (3.6) and (3.5), we get

$$
\begin{aligned}
&\dot{V}(x_t)|_{(3.1)} + z^T(t)z(t) - \gamma^2w^T(t)w(t)\\
&\qquad = \frac{1}{\tau_1}\int_{t-\tau_1}^{t}\eta^T(t,\alpha)\Omega_1\eta(t,\alpha)\,d\alpha + \frac{1}{\tau_2}\int_{t-\tau_2}^{t}\eta^T(t,\alpha)\Omega_2\eta(t,\alpha)\,d\alpha, \qquad (3.7)
\end{aligned}
$$

where

$$
\eta(t,\alpha) = \begin{bmatrix}x^T(t) & x^T(t-\tau_1) & x^T(t-\tau_2) & w^T(t) & \dot{x}_1^T(\alpha)\end{bmatrix}^T,
$$

$$
\begin{aligned}
\Omega_1 &= \begin{bmatrix}
\Theta_{11} + G^TG + N_{11} & \Theta_{12} + N_{12} & \Theta_{13} + N_{13} & \Theta_{14} + N_{14} & -\tau_1 Y_{11}\\
* & \Theta_{22} + N_{22} & N_{23} & -H_1^T + N_{24} & -\tau_1 W_{11}\\
* & * & \Theta_{33} + N_{33} & -H_2^T + N_{34} & 0\\
* & * & * & -\gamma^2 I + N_{44} & -\tau_1 H_{11}\\
* & * & * & * & -\tau_1 Z_{11}
\end{bmatrix}\\
&\quad + \sum_{i=1}^{2}\Phi(\mathcal{C}_1, \tau_i Z_i),
\end{aligned}
$$

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and

$$\mathcal{C}_1 = \begin{bmatrix} A_0^T \\ A_1^T \\ A_2^T \\ B_2^T \\ 0 \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} -N_{11} & -N_{12} & -N_{13} & -N_{14} & -\tau_2 Y_{21} \\ * & -N_{22} & -N_{23} & -N_{24} & 0 \\ * & * & -N_{33} & -N_{34} & -\tau_2 W_{21} \\ * & * & * & -N_{44} & -\tau_2 H_{21} \\ * & * & * & * & -\tau_2 Z_{21} \end{bmatrix}$$

with $N_{ii} > 0$, $i = 1, 2, 3, 4$.

If $\Omega_1 < 0$ and $\Omega_2 < 0$, which is equivalent, by using Lemma 2.3, to

$$\begin{bmatrix} \Theta_{11} + G^T G & \Theta_{12} & \Theta_{13} & \Theta_{14} & -\tau_1 Y_{11} & -\tau_2 Y_{21} \\ * & \Theta_{22} & 0 & -H_1^T & -\tau_1 W_{11} & 0 \\ * & * & \Theta_{33} & -H_2^T & 0 & -\tau_2 W_{21} \\ * & * & * & -\gamma^2 I & -\tau_1 H_{11} & -\tau_2 H_{21} \\ * & * & * & * & -\tau_1 Z_{11} & 0 \\ * & * & * & * & * & -\tau_2 Z_{21} \end{bmatrix} + \sum_{i=1}^{2} \Phi(\mathcal{C}_2, \tau_i Z_i),$$

$$\mathcal{C}_2 = \begin{bmatrix} A_0 & A_1 & A_2 & B_2 & 0 & 0 \end{bmatrix}^T, \tag{3.8}$$

then integrating from 0 to ∞ on both sides of (3.7) yields

$$\int_0^\infty \dot{V}(x_t)|_{(3.1)}\, dt + \int_0^\infty \big(z^T(t) z(t) - \gamma^2 w^T(t) w(t)\big)\, dt \le 0,$$

that is,

$$J(w) \le V(x_t)|_{t=0} - V(x_t)|_{t=\infty}.$$

Noticing that $V(x_t)|_{t=0} = 0$ and $V(x_t)|_{t=\infty} \ge 0$, we obtain $J(w) \le 0$, for all $w(t) \in L_2[0, \infty]$, $w(t) \ne 0$.

Now, applying the Schur complement equivalence to (3.4) leads to

$$\Theta + \Phi(\mathcal{C}_3, \mathcal{M}_1) + \sum_{i=1}^{2} \Phi(\mathcal{C}_4, \tau_m Z_i) < 0$$

with

$$\Theta = \begin{bmatrix} \Theta_{11} + G^T G & \Theta_{12} & \Theta_{13} & \Theta_{14} \\ * & \Theta_{22} & 0 & -H_1^T \\ * & * & \Theta_{33} & -H_2^T \\ * & * & * & -\gamma^2 I \end{bmatrix}, \quad \mathcal{C}_3 = \begin{bmatrix} -Y_{11} & -Y_{21} \\ -W_{11} & 0 \\ 0 & -W_{21} \\ -H_{11} & -H_{21} \end{bmatrix},$$

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$$\mathcal{C}_4 = \begin{bmatrix} A_0^T \\ A_1^T \\ A_2^T \\ B_2^T \end{bmatrix}, \quad \mathcal{M}_1 = \begin{bmatrix} \tau_m Z_{11}^{-1} & 0 \\ 0 & \tau_m Z_{21}^{-1} \end{bmatrix}.$$

Therefore, for any τ_i satisfying $0 < \tau_i \le \tau_m$, $i = 1, 2$, we have

$$\Theta + \Phi(\mathcal{C}_3, \mathcal{M}_2) + \sum_{i=1}^{2} \Phi(\mathcal{C}_4, \tau_i Z_i) \le \Theta + \Phi(\mathcal{C}_3, \mathcal{M}_1) + \sum_{i=1}^{2} \Phi(\mathcal{C}_4, \tau_m Z_i) < 0$$

where

$$\mathcal{M}_2 = \begin{bmatrix} \tau_1 Z_{11}^{-1} & 0 \\ 0 & \tau_2 Z_{21}^{-1} \end{bmatrix}$$

which, by Schur complement again, is equivalent to $\Omega_1 < 0$ and $\Omega_2 < 0$.

We now consider the internal stability of (3.1). In the case of $w(t) \equiv 0$, we have

$$\dot{V}(x_t)|_{(3.1)} = \frac{1}{\tau_1} \int_{t-\tau_1}^{t} \zeta^T(t, \alpha) \Lambda_1 \zeta(t, \alpha)\, d\alpha + \frac{1}{\tau_2} \int_{t-\tau_2}^{t} \zeta^T(t, \alpha) \Lambda_2 \zeta(t, \alpha)\, d\alpha,$$

where

$$\zeta(t, \alpha) = \begin{bmatrix} x^T(t) & x^T(t-\tau_1) & x^T(t-\tau_2) & \dot{x}_1^T(\alpha) \end{bmatrix}^T,$$

$$\Lambda_1 = \begin{bmatrix} \Theta_{11} + N_{11} & \Theta_{12} + N_{12} & \Theta_{13} + N_{13} & -\tau_1 Y_{11} \\ * & \Theta_{22} + N_{22} & N_{23} & -\tau_1 W_{11} \\ * & * & \Theta_{33} + N_{33} & 0 \\ * & * & * & -\tau_1 Z_{11} \end{bmatrix} + \sum_{i=1}^{2} \Phi(\mathcal{C}_5, \tau_i Z_i),$$

$$\mathcal{C}_5 = \begin{bmatrix} A_0^T \\ A_1^T \\ A_2^T \\ 0 \end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix} -N_{11} & -N_{12} & -N_{13} & -\tau_2 Y_{21} \\ * & -N_{22} & -N_{23} & 0 \\ * & * & -N_{33} & -\tau_2 W_{21} \\ * & * & * & -\tau_2 Z_{21} \end{bmatrix}$$

with $N_{ii} > 0$, $i = 1, 2, 3$. By Lemma 2.2, the internal stability of (3.1) is achieved from the stability of the operator $\mathcal{D}$ and $\Lambda_1 < 0$, $\Lambda_2 < 0$, which is equivalent, by using Lemma 2.3 again, to

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & -\tau_1 Y_{11} & -\tau_2 Y_{21} \\ * & \Theta_{22} & 0 & -\tau_1 W_{11} & 0 \\ * & * & \Theta_{33} & 0 & -\tau_2 W_{21} \\ * & * & * & -\tau_1 Z_{11} & 0 \\ * & * & * & * & -\tau_2 Z_{21} \end{bmatrix} + \sum_{i=1}^{2} \Phi(\mathcal{C}_6, \tau_i Z_i) < 0 \qquad (3.9)$$

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where $\mathcal{C}_6 = [\ A_0 \quad A_1 \quad A_2 \quad 0 \quad 0\ ]^T$. It is easy to see that (3.9) follows from (3.8) immediately. This completes the proof. □

In the case of single delay, System (3.1) becomes

$$\begin{cases} E\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + B_2 w(t), \\ z(t) = Gx(t), \\ x(t) = \phi(t) \quad t \in [-\tau, 0]. \end{cases} \tag{3.10}$$

By Theorem 3.1, the delay-dependent BRL for (3.10) is obtained as follows.

COROLLARY 3.2. *For some prescribed $\gamma > 0$, if there exist matrices*

$$P = \begin{bmatrix} P_1 & P_2 \\ 0 & P_3 \end{bmatrix}, \quad P_1 > 0,\ Q > 0, \quad Z = \begin{bmatrix} Z_1 & Z_2 \\ * & Z_3 \end{bmatrix} > 0, \quad Y = \begin{bmatrix} Y_1 & 0 \end{bmatrix},$$
$$W = \begin{bmatrix} W_1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 & 0 \end{bmatrix} \tag{3.11}$$

with $P_1 \in R^{p\times p}$, $Z_1 \in R^{p\times p}$, $Y_1 \in R^{n\times p}$, $W_1 \in R^{n\times p}$, $H_1 \in R^{r\times p}$, satisfying

$$\begin{bmatrix} \mathcal{F} & PA_1 - Y + W^T & PB_2 + H^T & -\tau_m Y_1 & \tau_m A_0^T Z \\ * & -W - W^T - Q & -H^T & -\tau_m W_1 & \tau_m A_1^T Z \\ * & * & -\gamma^2 I & -\tau_m H_1 & \tau_m B_2^T Z \\ * & * & * & -\tau_m Z_1 & 0 \\ * & * & * & * & -\tau_m Z \end{bmatrix} < 0, \tag{3.12}$$

where $F = PA_0 + A_0^T P^T + Y + Y^T + Q + G^T G$, then for any delay τ satisfying $0 < \tau \le \tau_m$, System (3.10) is regular, impulse free, internally stable, and satisfies (2.6) for all nonzero $w(t) \in L_2[0, \infty)$.

REMARK 3. From Corollary 3.2, it is easy to get the internal stability result for (3.10), which was recently provided in [29, Theorem 1] and is omitted here. It is worth noticing that only the stability problem was considered in [29], while in this paper, H_∞ performance index and controller design is discussed as well. Furthermore, only the case of single delay was considered in [29]. From the proof of Theorem 3.1 in this paper and that in [29], it can be seen that, to avoid using the bounding technique, $\dot{V}(x_t)$ is written in the integral form finally. However, it cannot be extended from the case of single delay to that of multiple delays directly. As shown by (3.7) and (3.8) in this paper, Lemma 2.3 is important in solving this problem successfully. So the main contribution of this paper is the improved BRL and the design of the robust H_∞ controller for singular systems with multiple delays. The stability result in [29] is only a by-product of Theorem 3.1.

REMARK 4. Compared with existing results, it is worth noting that there are mainly three advantages of our result. Firstly, the BRL in Theorem 3.1 (Corollary 3.2)

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is obtained without using any model transformations and bounding techniques on the related cross product terms [6, 25, 28], which leads to some conservatism in some sense. Secondly, the LMI (3.12) involves fewer variables. For example, for System (3.10) with $x(t) \in R^n$, $x_1(t) \in R^p$, $w(t) \in R^r$, $r \leq n$, the number of the variables to be determined in (3.12) is no more than $2n^2 + 2np + p^2/2 + n + p/2$, while the number of variables in [6] is $5n^2 + 5np + 11p^2/2 + 2n + 3p/2$ and in [25] is $12n^2 - 2np + p^2 + 4n + p$. Thirdly, unlike in [6], no decomposition of the system matrices is needed in this paper and thus the analysis procedure is relatively simple and reliable. Furthermore, it is easy to see that, following the method usually used to deal with the norm-bounded parametric uncertainties, Theorem 3.1 in our paper can be extended to deal with the delay-dependent robust H_∞ control problem for singular time-delay systems with norm-bounded parametric uncertainties, while with the BRL in [6] it is difficult to solve the aforementioned problem.

REMARK 5. In [22], another BRL was also obtained without resorting to any bounding techniques and model transformations. It can be seen that the method of introducing slack matrices in Theorem 3.1 is different from that in [22]. So in some cases, the number of the variables in the LMI (3.12) is less than that in [22] (see Example 1 below).

REMARK 6. When $E = I$, System (3.10) becomes a standard state-space system with single delay. It is easy to see that Corollary 3.2, with $H = 0$ in (3.12), coincides with [21, Theorem 1] with constant delay and without uncertainties. So Corollary 3.2 can be viewed as an extension of the BRL for a standard state-space time-delay system to the case of a singular time-delay system.

For (3.10), the delay-dependent BRL in [28] is obtained as follows.

LEMMA 3.3 ([28]). *For some prescribed $\gamma > 0$, if there exist matrices P, Q, Z and Y of* (3.11) *and $X \geq 0$ such that the LMIs*

$$\begin{bmatrix} \mathcal{F} + \tau_m X & PA_1 - Y & PB_2 & \tau_m A_0^T Z \\ * & -Q & 0 & \tau_m A_1^T Z \\ * & * & -\gamma^2 I & \tau_m B_2^T Z \\ * & * & * & -\tau_m Z \end{bmatrix} < 0, \tag{3.13}$$

$$\begin{bmatrix} X & Y_1 \\ * & Z_1 \end{bmatrix} \geq 0 \tag{3.14}$$

hold, then for any time delay τ satisfying $0 < \tau \leq \tau_m$, System (3.10) *is regular, impulse free, internally stable, and satisfies* (2.6) *for all nonzero $w(t) \in L_2[0, \infty)$.*

In the following theorem, we are in a position to show the relationship between Corollary 3.2 and Lemma 3.3.

THEOREM 3.4. *For given $\gamma > 0$, there exist matrices P, Q, Z, Y of* (3.11) *and $W = 0$, $H = 0$ satisfying* (3.12) *if and only if there exist matrices P, Q, Z, Y of* (3.11) *and $X \geq 0$ such that* (3.13) *and* (3.14) *hold.*

REMARK 7. The proof of Theorem 3.4 is similar to [20, Theorem 2] and is omitted. We can see from Theorem 3.4 that, since the number of the variables in Corollary 3.2 ($W = 0$, $H = 0$) is fewer than that in Lemma 3.3, redundant variables are introduced when inequality (3.14) is employed to derive the upper bounds of some cross product terms. When W and H are matrices to be determined, the BRL in Corollary 3.2 is improved compared with Lemma 3.3, which implies that the introduction of W and H reduces the conservatism of the BRL and W, H are not redundant matrices. The detailed example in Section 5 will illustrate the above results as well.

4. State feedback robust H_∞ control

In this section, based on Theorem 3.1, we are in a position to present the result on delay-dependent robust H_∞ control for System (2.1). The following lemma is needed.

LEMMA 4.1 ([15]). *Given matrices Ω, Γ and Ξ of appropriate dimensions with Ω symmetrical, then $\Omega + \Gamma F \Xi + (\Gamma F \Xi)^T < 0$ for all F satisfying $F F^T \leq I$, if and only if there exists a scalar $\epsilon > 0$ such that $\Omega + \epsilon \Gamma \Gamma^T + \epsilon^{-1} \Xi^T \Xi < 0$.*

THEOREM 4.2. *Suppose, for some prescribed $\gamma > 0$, that there exist matrices P, Q_i, Z_i, Y_i, W_i, H_i, $i = 1, 2$, of* (3.2) *and* (3.3), *L and a scalar $\epsilon > 0$ with P nonsingular, such that*

$$\Xi := [\mathcal{A}\ \mathcal{B}] < 0 \quad \textit{holds where}$$

$$\mathcal{A} = \begin{bmatrix} \Xi_{11} & \mathcal{K}_1 & \mathcal{K}_2 & B_2 + \sum_{i=1}^{2} H_i^T & -\tau_m Y_{11} & -\tau_m Y_{21} \\ * & \mathcal{L}_1 & 0 & -H_1^T & -\tau_m W_{11} & 0 \\ * & * & \mathcal{L}_2 & -H_2^T & 0 & -\tau_m W_{21} \\ * & * & * & -\gamma^2 I & -\tau_m H_{11} & -\tau_m H_{21} \\ * & * & * & * & \Xi_{55} & 0 \\ * & * & * & * & * & \Xi_{66} \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix},$$

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$$\mathcal{B} = \begin{bmatrix} \Xi_{17} & \Xi_{18} & 0 & PG^T + L^T D^T & \mathcal{M} & \mathcal{M} \\ \tau_m P A_1^T & \tau_m P A_1^T & 0 & 0 & P N_1^T & 0 \\ \tau_m P A_2^T & \tau_m P A_2^T & 0 & 0 & P N_2^T & 0 \\ \tau_m B_2^T & \tau_m B_2^T & 0 & 0 & N_4^T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\tau_m Z_1 & 0 & \epsilon \tau_m M_1 & 0 & 0 & 0 \\ * & -\tau_m Z_2 & \epsilon \tau_m M_1 & 0 & 0 & 0 \\ * & * & -\epsilon I & 0 & 0 & 0 \\ * & * & * & -I + \epsilon M_2 M_2^T & 0 & 0 \\ * & * & * & * & -\epsilon I & 0 \\ * & * & * & * & * & -\epsilon I \end{bmatrix}, \tag{4.1}$$

for $\mathcal{M} = P N_0^T + L^T N_3^T$, $\mathcal{K}_i = A_i P^T - Y_i + W_i^T$, $\mathcal{L}_i = -W_i - W_i^T - Q_i$, $(i = 1, 2)$ *and*

$$\Xi_{11} = A_0 P^T + P A_0^T + B_1 L + L^T B_1^T + \sum_{i=1}^{2} \big(Y_i + Y_i^T + Q_i\big) + \epsilon M_1 M_1^T,$$

$$\Xi_{17} = \Xi_{18} = \tau_m \big(P A_0^T + L^T B_1^T\big) + \epsilon \tau_m M_1 M_1^T,$$

$$\Xi_{55} = -\tau_m \begin{bmatrix} P_1 & 0 \end{bmatrix} Z_1^{-1} \begin{bmatrix} P_1 & 0 \end{bmatrix}^T, \quad \Xi_{66} = -\tau_m \begin{bmatrix} P_1 & 0 \end{bmatrix} Z_2^{-1} \begin{bmatrix} P_1 & 0 \end{bmatrix}^T.$$

Then System (2.1) is robustly stabilizable with H_∞ performance γ and the gain matrix of the H_∞ controller (2.4) is given by

$$K = L P^{-T}. \tag{4.2}$$

PROOF. Pre-multiplying by

$$\rho = \operatorname{diag}\big\{P^{-1}, P^{-1}, P^{-1}, I, P_1^{-1}, P_1^{-1}, Z_1^{-1}, Z_2^{-1}, I, I, I, I\big\}$$

and post-multiplying by ρ^T on both sides of (4.1), and noticing that $K = L P^{-T}$, $A_k = A_0 + B_1 K$, $G_k = G + DK$, $N_k = N_0 + N_3 K$, we get from the Schur complement that there exist $\bar{P}$, $\bar{Q}_i$, $\bar{Z}_i$, $\bar{Y}_i$, $\bar{W}_i$, $\bar{H}_i$, $i = 1, 2$, and ϵ satisfying

$$\Upsilon + \epsilon \Phi(\mathcal{C}_7, I) + \epsilon^{-1} \Phi(\mathcal{C}_8, I) < 0, \tag{4.3}$$

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where

$$\Upsilon = \begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} & \Upsilon_{14} & -\tau_m \bar{Y}_{11} & -\tau_m \bar{Y}_{21} & \tau_m A_k^T \bar{Z}_1 & \tau_m A_k^T \bar{Z}_2 & G_k^T \\ * & \Upsilon_{22} & 0 & -\bar{H}_1^T & -\tau_m \bar{W}_{11} & 0 & \tau_m A_1^T \bar{Z}_1 & \tau_m A_1^T \bar{Z}_2 & 0 \\ * & * & \Upsilon_{33} & -\bar{H}_2^T & 0 & -\tau_m \bar{W}_{21} & \tau_m A_2^T \bar{Z}_1 & \tau_m A_2^T \bar{Z}_2 & 0 \\ * & * & * & -\gamma^2 I & -\tau_m \bar{H}_{11} & -\tau_m \bar{H}_{21} & \tau_m B_2^T \bar{Z}_1 & \tau_m B_2^T \bar{Z}_2 & 0 \\ * & * & * & * & -\tau_m \bar{Z}_{11} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\tau_m \bar{Z}_{21} & 0 & 0 & 0 \\ * & * & * & * & * & * & -\tau_m \bar{Z}_1 & 0 & 0 \\ * & * & * & * & * & * & * & -\tau_m \bar{Z}_2 & 0 \\ * & * & * & * & * & * & * & * & -I \end{bmatrix},$$

$$\mathcal{C}_7 = \begin{bmatrix} \bar{P} M_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \tau_m \bar{Z}_1 M_1 & 0 \\ \tau_m \bar{Z}_2 M_1 & 0 \\ 0 & M_2 \end{bmatrix}, \quad \mathcal{C}_8 = \begin{bmatrix} N_k^T & N_k^T \\ N_1^T & 0 \\ N_2^T & 0 \\ N_4^T & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Upsilon_{11} = \bar{P} A_k + A_k^T \bar{P}^T + \sum_{i=1}^{2} (\bar{Y}_i + \bar{Y}_i^T + \bar{Q}_i), \quad \Upsilon_{12} = \bar{P} A_1 - \bar{Y}_1 + \bar{W}_1^T,$$

$$\Upsilon_{13} = \bar{P} A_2 - \bar{Y}_2 + \bar{W}_2^T, \quad \Upsilon_{14} = \bar{P} B_2 + \sum_{i=1}^{2} \bar{H}_i^T, \quad \Upsilon_{22} = -\bar{W}_1 - \bar{W}_1^T - \bar{Q}_1,$$

$$\Upsilon_{33} = -\bar{W}_2 - \bar{W}_2^T - \bar{Q}_2, \quad \bar{P} = P^{-1} := \begin{bmatrix} \bar{P}_1 & \bar{P}_2 \\ 0 & \bar{P}_3 \end{bmatrix}, \quad \bar{P}_1 = P_1^{-1} > 0$$

and, for $i = 1, 2$,

$$\bar{Q}_i = P^{-1} Q_i P^{-T} > 0, \quad \bar{Z}_i = Z_i^{-1} := \begin{bmatrix} \bar{Z}_{i1} & \bar{Z}_{i2} \\ * & \bar{Z}_{i3} \end{bmatrix} > 0,$$

$$\bar{Y}_i = P^{-1} Y_i P^{-T} = \bar{P} \begin{bmatrix} Y_{i1} & 0 \end{bmatrix} \begin{bmatrix} \bar{P}_1 & 0 \\ \bar{P}_2^T & \bar{P}_3^T \end{bmatrix} = \begin{bmatrix} \bar{P} Y_{i1} \bar{P}_1 & 0 \end{bmatrix} := \begin{bmatrix} \bar{Y}_{i1} & 0 \end{bmatrix},$$

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$$\bar{W}_i = P^{-1} W_i P^{-T} = \begin{bmatrix} \bar{P} W_{i1} \bar{P}_1 & 0 \end{bmatrix} := \begin{bmatrix} \bar{W}_{i1} & 0 \end{bmatrix},$$
$$\bar{H}_i = H_i P^{-T} = \begin{bmatrix} H_{i1} & 0 \end{bmatrix} \begin{bmatrix} \bar{P}_1 & 0 \\ \bar{P}_2^T & \bar{P}_3^T \end{bmatrix} = \begin{bmatrix} H_{i1} \bar{P}_1 & 0 \end{bmatrix} := \begin{bmatrix} \bar{H}_{i1} & 0 \end{bmatrix}.$$

Note that $A_{k\Delta} = A_k + M_1 F N_k$, $G_{k\Delta} = G_k + M_2 F N_k$, and from Lemma 4.1 and (4.3), we have that the above $\bar{P}$, $\bar{Q}_i$, $\bar{Z}_i$, $\bar{Y}_i$, $\bar{W}_i$, $\bar{H}_i$, $i = 1, 2$, such that

$$\bar{\Upsilon} = \Upsilon + \Phi(\mathcal{C}_7, \mathcal{M}_3, \mathcal{C}_8) + \Phi(\mathcal{C}_7, \mathcal{M}_3, \mathcal{C}_8)^\top < 0 \quad \text{with } \mathcal{M}_3 = \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}$$

holds for all uncertainties F satisfying $F^T F \le I$, where $\bar{\Upsilon}$ is the matrix obtained by replacing A_k, A_1, A_2, B_2 and G_k in Υ by $A_{k\Delta}$, $A_{1\Delta}$, $A_{2\Delta}$, $B_{2\Delta}$ and $G_{k\Delta}$, respectively. By Theorem 3.1, one gets that the closed-loop system (2.5) is regular, impulse free, robustly internally stable and satisfies (2.6) for all nonzero $w(t) \in L_2[0, \infty)$ and τ_i satisfying $0 < \tau_i \le \tau_m$, $i = 1, 2$. This completes the proof. □

REMARK 8. It is clear that the nonlinear terms Ξ_{55} and Ξ_{66} in (4.1) lead to a nonlinear matrix inequality. However, we can follow a similar line as in [28] to convert solving (4.1) to solving a sequence of convex optimization problems subject to LMIs.

First, introducing new matrices

$$U_i = \begin{bmatrix} U_{i1} & U_{i2} \\ * & U_{i3} \end{bmatrix} > 0, \quad V_1 > 0, \quad T_{i1} > 0, \quad S_{i1} > 0, \tag{4.4}$$

with $U_i \in R^{n\times n}$, $U_{i1} \in R^{p\times p}$, $V_1 \in R^{p\times p}$, $T_{i1} \in R^{p\times p}$, $S_{i1} \in R^{p\times p}$, $i = 1, 2$, we can see that if there are matrices P, Q_i, Z_i, Y_i, W_i, H_i, $i = 1, 2$, of (3.2) and (3.3), U_i, V_1, T_{i1}, S_{i1}, $i = 1, 2$, of (4.4), L and $\epsilon > 0$ such that

$$\bar{\Xi} < 0, \tag{4.5}$$
$$\begin{bmatrix} U_{i1} & V_1 \\ * & S_{i1} \end{bmatrix} \ge 0, \quad i = 1, 2 \tag{4.6}$$

and

$$U_i Z_i = I_n, \quad P_1 V_1 = I_p, \quad T_{i1} S_{i1} = I_p, \quad i = 1, 2,$$

then (4.1) holds for the above P, Q_i, Z_i, Y_i, W_i, H_i, $i = 1, 2$, L and ϵ, where $\bar{\Xi}$ is the matrix obtained by replacing Ξ_{55} and Ξ_{66} of Ξ in (4.1) by $-\tau_m T_{11}$ and $-\tau_m T_{21}$, respectively. Thus the H_∞ control problem can be considered as a complementary problem subject to LMIs:

$$\begin{aligned} &\text{minimize} && \{\operatorname{tr}(U_1 Z_1 + U_2 Z_2) + \operatorname{tr}(P_1 V_1 + T_{11} S_{11} + T_{21} S_{21})\} \\ &\text{subject to LMIs:} && (3.2),\ (3.3),\ (4.4)\text{–}(4.6) \quad \text{and, for } i = 1, 2, \\ & && \begin{bmatrix} U_i & I_n \\ * & Z_i \end{bmatrix} \ge 0, \quad \begin{bmatrix} P_1 & I_p \\ * & V_1 \end{bmatrix} \ge 0, \quad \begin{bmatrix} T_{i1} & I_p \\ * & S_{i1} \end{bmatrix} \ge 0. \end{aligned}$$

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TABLE 1. Comparison of maximum τ_m allowed for Example 1.

γ	4	20	40
[6]	0.8563	1.0883	1.1208
Lemma 3.3	0.3721	0.5500	0.5641
[22]	0.9079	1.1151	1.1348
Corollary 3.2	0.9079	1.1151	1.1348

TABLE 2. Comparison of minimum γ allowed for Example 1.

τ_m	0.1	1	1.12
[6]	2.2451	8.1293	39.0732
Lemma 3.3	3.0276	–	–
[22]	3.0045	5.2026	22.8635
Corollary 3.2	3.0045	5.2026	22.8635

The above nonconvex optimization problem can be solved by using the linearization iterative algorithm proposed in [2, 13, 25, 28].

5. Numerical examples

EXAMPLE 1. Consider the unforced singular time-delay system in [6] of form (3.10) and

$$A_0 = \begin{bmatrix} 0.5 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix},$$
$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0.5 & 1 \end{bmatrix}.$$

When $w(t) = 0$, applying the stability conditions in [29] and [6], we can obtain that the system is internally stable for $\tau \leq 1.1547$ in both cases. When $w(t) \neq 0$, the comparison results on maximum τ_m allowed for different values of γ and minimum γ allowed for different values of τ_m are listed in Tables 1 and 2. It can be seen that the result in Theorem 3.1 is less conservative than those in [6] and Lemma 3.3. And though the computational result by Corollary 3.2 is identical to that by [22], the LMI in Corollary 3.2 involves fewer variables in this example since the number of variables is 14 in Corollary 3.2 and 17 in [22], respectively.

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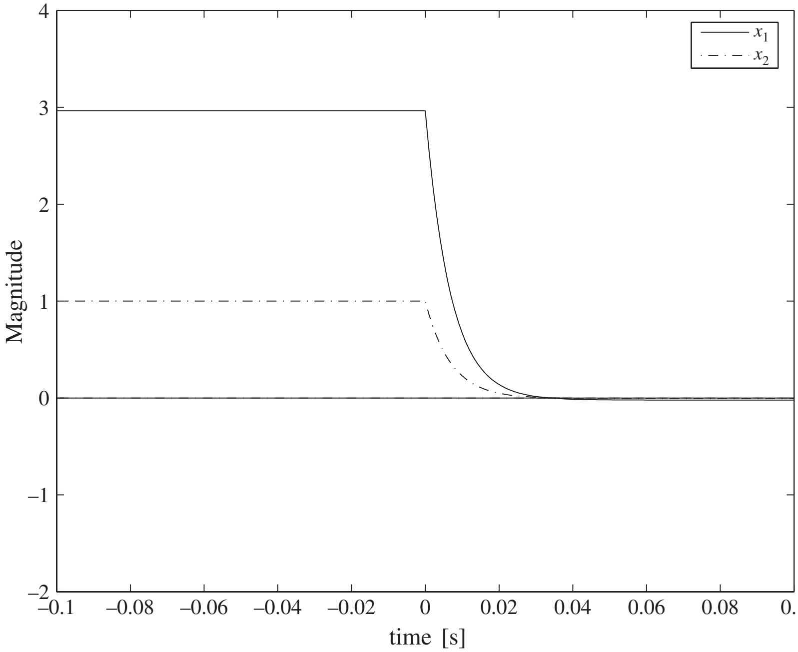

FIGURE 1. State response of x_1 and x_2 of the closed-loop system when $w(t) = 0$.

EXAMPLE 2. Consider the singular time-delay system in [6] of form (2.1) with single delay and

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad D = 0.1, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0.2 \end{bmatrix}.$$

When $\tau_m = 1.2$, the authors of [6] obtained a near minimum value of $\gamma = 21$ and the corresponding controller gain matrix was $K = [175.62 \; -430\,680]$. In [25], the minimum γ allowed was 4 after 51 iterations and the corresponding controller gain matrix was $K = [1401 \; -25\,036]$. And in [22], the minimum $\gamma = 15.0268$ was calculated and $K = [0.0651 \; -1.3454]$. However, by Theorem 4.2, for the same $\tau_m = 1.2$, the minimum γ allowed is 3.4349 after ten iterations, and in the case of $\gamma = 21$ and $\gamma = 4$, the corresponding gain matrix is $K = [272.225 \; -809.4105]$ and $K = [238.6 \; -3249.5]$, respectively. It was also shown that in [25], when $\gamma = 21$, the maximum τ_m allowed was 1.8 after 22 iterations and the corresponding gain matrix was $K = [1097 \; -13\,461]$. Now, for the same $\gamma = 21$, using Theorem 4.2, the maximum τ_m allowed is 6.0168 after ten iterations. And when $\tau_m = 1.8$, $\gamma = 21$, the corresponding controller gain matrix computed by Theorem 4.2 is $K = [-46.7047 \; -994.0828]$.

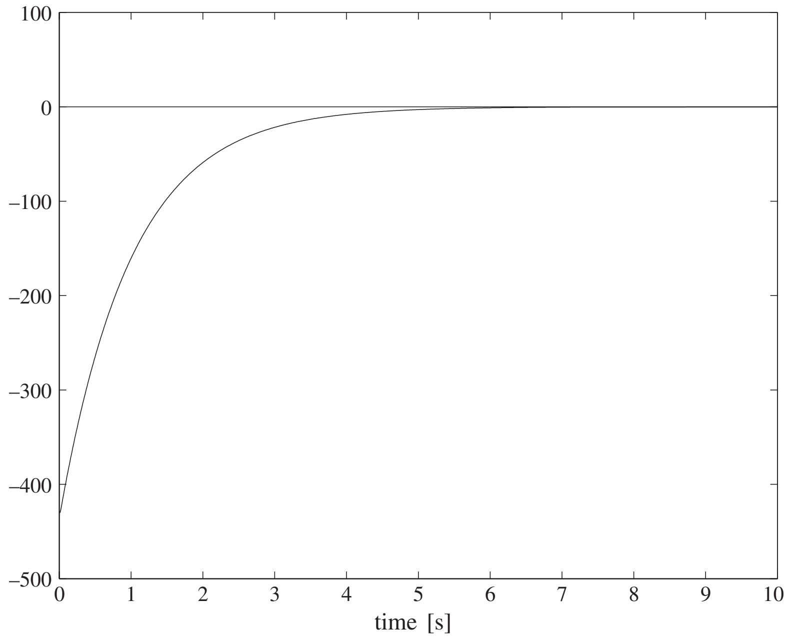

FIGURE 2. Response of $z^T(t)z(t) - \gamma^2 w^T(t)w(t)$ of the closed-loop system when $w(t) = e^{-0.5t}$.

The simulation results of the state of the closed loop when $w(t) = 0$ are shown in Figure 1, from which we can see that the closed loop is internally stable. And the simulation results of $z^T(t)z(t) - \gamma^2 w^T(t)w(t)$ are given in Figure 2, which shows that the designed H_∞ controller satisfied the performance index (2.6). Here $\tau_m = 1.2$, $\gamma = 21$, $\phi(t) = [2.966\ 1]^T$, $t \in [-1.2, 0]$ and $K = [272.225\ \ -809.4105]$.

EXAMPLE 3. Consider the uncertain singular time-delay system of form (2.1) with

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & -0.5 \\ 0 & -1 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad D = 0.1, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$
$$G = \begin{bmatrix} 1 & 0.2 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix},$$
$$N_0 = N_1 = N_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \quad N_4 = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}.$$

When $\tau_m = 1$, $\gamma = 21$, by Theorem 4.2, the corresponding controller gain matrix computed is $K = [-580.6\ \ -1166.6]$ after ten iterations.

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6. Conclusions

We have presented in this paper an improved delay-dependent BRL for singular systems with multiple delays, which is established without using any of the model transformations and bounding techniques on the cross product terms. Based on this BRL, we have solved the robust H_∞ control problem in terms of a nonlinear matrix inequality. To get the H_∞ controller, both a linear matrix inequalities technique and the cone complementarity method have been employed. Three numerical examples have been proposed to illustrate the effectiveness and less conservativeness of the proposed method.

Appendix A. Proof of Lemma 2.3

We begin with a proof of sufficiency. Suppose that there exists a matrix N_{11} such that

$$\begin{bmatrix} A_{11}+N_{11} & A_{12} \\ * & A_{22} \end{bmatrix} < 0 \quad \text{and} \quad \begin{bmatrix} -N_{11} & A_{13} \\ * & A_{33} \end{bmatrix} < 0. \tag{A1}$$

According to the Schur complement, we can deduce that

$$A_{11}+N_{11}-A_{12}A_{22}^{-1}A_{12}^T < 0 \quad \text{and} \quad -N_{11}-A_{13}A_{33}^{-1}A_{13}^T < 0,$$

which implies that

$$A_{11}-A_{12}A_{22}^{-1}A_{12}^T-A_{13}A_{33}^{-1}A_{13}^T < 0. \tag{A2}$$

By the Schur complement again, (A2) is equivalent to

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ * & A_{22} & 0 \\ * & * & A_{33} \end{bmatrix} < 0. \tag{A3}$$

This completes the proof of sufficiency.

Turning to necessity, if (A3) holds, which is equivalent to (A2), then it is clear that there exists a scalar $\epsilon > 0$ which is sufficiently small, such that

$$A_{11}-A_{12}A_{22}^{-1}A_{12}^T-A_{13}A_{33}^{-1}A_{13}^T+\epsilon I < 0.$$

Let $N_{11} = -A_{13}A_{33}^{-1}A_{13}^T+\epsilon I > 0$. Then

$$\begin{aligned} A_{11}+N_{11}-A_{12}A_{22}^{-1}A_{12}^T &= A_{11}-A_{12}A_{22}^{-1}A_{12}^T-A_{13}A_{33}^{-1}A_{13}^T+\epsilon I < 0, \\ -N_{11}-A_{13}A_{33}^{-1}A_{13}^T &= A_{13}A_{33}^{-1}A_{13}^T-\epsilon I-A_{13}A_{33}^{-1}A_{13}^T = -\epsilon I < 0, \end{aligned}$$

which by the Schur complement, is equivalent to (A1). This completes the proof of necessity.

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References

[1] M. V. Basin and A. E. Rodkina, "On delay-dependent stability for a class of nonlinear stochastic delay-difference equations", *Dyn. Contin. Discrete Impuls. Syst.* **12A** (2005) 663–675.

[2] L. El Ghaoui, F. Oustry and M. Ait Rami, "A cone complementarity linearization algorithm for static output-feedback and related problems", *IEEE Trans. Automat. Control* **42** (1997) 1171–1176.

[3] J. Feng, S. Zhu and Z. Cheng, "Guaranteed cost control of linear uncertain singular time-delay systems", in *Proceedings of 41st IEEE Conference on Decision and Control*, 2002, 1802–1807.

[4] E. Fridman, "Stability of linear descriptor systems with delay: a Lyapunov-based approach", *J. Math. Anal. Appl.* **273** (2002) 24–44.

[5] E. Fridman and U. Shaked, "A descriptor system approach to H_∞ control of linear time-delay systems", *IEEE Trans. Automat. Control* **47** (2002) 253–270.

[6] E. Fridman and U. Shaked, "H_∞-control of linear state-delay descriptor systems: an LMI approach", *Linear Algebra Appl.* **351–352** (2002) 271–302.

[7] J. K. Hale and S. M. Verduyn Lunel, *Introduction to functional differential equations* (Springer, New York, 1993).

[8] J. K. Kim, "New design method on memoryless H_∞ control for singular systems with delayed state and control using LMI", *J. Franklin Inst.* **342** (2005) 321–327.

[9] J. K. Kim, J. H. Lee and H. B. Park, Robust H_∞ control of singular systems with time delays and uncertainties via LMI approach, in *Proceedings of American Control Conference*, 2002, 620–621.

[10] V. B. Kolmanovskii and V. R. Nosov, *Stability of functional differential equations* (Academic Press, New York, 1986).

[11] X. Li and C. E. de Souza, "Criteria for robust stability and stabilization of uncertain linear systems with state-delay", *Automatica* **33** (1997) 1657–1662.

[12] H. Logemann, "Destabilizing effects of small time delays on feedback-controlled descriptor systems", *Linear Algebra Appl.* **272** (1998) 131–153.

[13] Y. S. Moon, P. Park, W. H. Kwon and Y. S. Lee, "Delay-dependent robust stabilization of uncertain state-delayed systems", *Internat. J. Control* **74** (2001) 1447–1455.

[14] P. Park, "A delay-dependent stability criterion for systems with uncertain time-invariant delays", *IEEE Trans. Automat. Control* **44** (1999) 876–877.

[15] I. R. Petersen, "A stabilization algorithm for a class of uncertain linear systems", *Syst. Control Lett.* **8** (1987) 351–357.

[16] A. E. Rodkina and M. V. Basin, "On delay-dependent stability for a class of nonlinear stochastic delay-differential equations", *Math. Control Signals Systems* **18** (2006) 187–197.

[17] A. E. Rodkina and M. V. Basin, "On delay-dependent stability for vector nonlinear stochastic delay-difference equations with Volterra diffusion term", *Syst. Control Lett.* **56** (2006) 423–430.

[18] S. Xu, P. V. Dooren, R. Stefan and J. Lam, "Robust stability and stabilization for singular systems with state delay and parameter uncertainty", *IEEE Trans. Automat. Control* **47** (2002) 1122–1128.

[19] S. Xu, J. Lam and C. Yang, "H_∞ control for uncertain singular systems with state delay", *Internat. J. Robust Nonlinear Control* **13** (2003) 1213–1223.

[20] S. Xu, J. Lam and Y. Zou, "Simplified descriptor system approach to delay-dependent stability and performance analyses for time-delay systems", *IEE Proc. Control Theory Appl.* **152** (2005) 147–151.

[21] S. Xu, J. Lam and Y. Zou, "New results on delay-dependent robust H_∞ control for systems with time-varying delays", *Automatica* **42** (2006) 343–348.

[22] S. Xu, J. Lam and Y. Zou, "An improved characterization of bounded realness for singular delay systems and its applications", *Internat. J. Robust Nonlinear Control* **18** (2008) 263–277.

[23] D. Yue and Q. Han, "H_∞ filter design of uncertain descriptor systems with discrete and distributed delays", *IEEE Trans. Signal Process.* **52** (2003) 3200–3212.

[24] D. Yue and Q. Han, "A delay-dependent stability criterion of neutral systems and its application to a partial element equivalent circuit model", *IEEE Trans. Circuits Systems II* **51** (2004) 685–689.

[25] D. Yue and Q. Han, "Delay-dependent robust H_∞ controller design for uncertain descriptor systems with time-varying discrete and distributed delays", *IEE Proc. Control Theory Appl.* **152** (2005) 628–638.

[26] S. Zhou and J. Lam, "Robust stabilization of delayed singular systems with linear fractional parametric uncertainties", *Circuits Systems Signal Process* **22** (2003) 579–588.

[27] S. Zhu, Z. Cheng and J. Feng, "Delay-dependent robust stability criterion and robust stabilization for uncertain singular time-delay systems", in *Proceedings of American Control Conference*, 2005, 2839–2844.

[28] S. Zhu, Z. Li and Z. Cheng, "Delay-dependent robust resilient H_∞ control for uncertain singular time-delay systems", in *Proceedings of IEEE Conference on Decision and Control and European Control Conference*, 2005, 1373–1378.

[29] S. Zhu, C. Zhang, Z. Cheng and J. Feng, "Delay-dependent robust stability criteria for two classes of uncertain singular time-delay systems", *IEEE Trans. Automat. Control* **52** (2007) 880–885.