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ON A NONCRITICAL SYMMETRIC SQUARE *L*-VALUE OF THE CONGRUENT NUMBER ELLIPTIC CURVES

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Abstract

The congruent number elliptic curves are defined by $E_d : y^2 = x^3 - d^2x$, where $d \in \mathbb{N}$. We give a simple proof of a formula for $L(\mathrm{Sym}^2(E_d), 3)$ in terms of the determinant of the elliptic trilogarithm evaluated at some degree zero divisors supported on the torsion points on $E_d(\overline{\mathbb{Q}})$.

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1. Introduction

Let E be an elliptic curve defined over $\mathbb{C}$. Then there exist $\tau \in \mathbb{C}$ such that $\mathrm{Im}(\tau) > 0$ and isomorphisms

$$\begin{array}{ccccc} E(\mathbb{C}) & \xrightarrow{\sim} & \mathbb{C}/\Lambda & \xrightarrow{\sim} & \mathbb{C}^\times/q^{\mathbb{Z}}, \\ (\wp_\Lambda(u), \wp'_\Lambda(u)) & \longmapsto & u \pmod{\Lambda} & \longmapsto & e^{2\pi i u}, \end{array} \tag{1.1}$$

where $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$, $\wp_\Lambda$ is the Weierstrass $\wp$-function and $q = e^{2\pi i\tau}$. Zagier and Gangl [13, Section 10] defined the two functions $\mathcal{L}^E_{3,j} : E(\mathbb{C}) \to \mathbb{R}$, $j = 1, 2$, by

$$\mathcal{L}^E_{3,1}(P) := \mathcal{L}^E_{3,1}(x) = \sum_{n=-\infty}^{\infty} \mathcal{L}_3(q^n x),$$

$$\mathcal{L}^E_{3,2}(P) := \mathcal{L}^E_{3,2}(x) = \sum_{n=0}^{\infty} J_3(q^n x) + \sum_{n=1}^{\infty} J_3(q^n x^{-1}) + \frac{\log^2|x| \log^2|qx^{-1}|}{4\log|q|},$$

where $\mathcal{L}_3(z) = \mathrm{Re}(\mathrm{Li}_3(z) - \log|z|\,\mathrm{Li}_2(z) + \frac{1}{3}\log^2|z|\,\mathrm{Li}_1(z))$, $\mathrm{Li}_m(z) = \sum_{n=1}^{\infty} z^n/n^m$ is the classical mth polylogarithm function, $J_3(x) = \log^2|x|\log|1-x|$ and $x \in \mathbb{C}^\times$ is the image of the point P on $E(\mathbb{C})$ under the composition of the isomorphisms above. The function $\mathcal{L}^E_{3,1}$ is called the *elliptic trilogarithm*. These two functions serve as higher-dimensional analogues of the *elliptic dilogarithm* D^E and the function J^E defined by

$$D^E(x) = \sum_{n=-\infty}^{\infty} \mathcal{L}_2(q^n x),$$
$$J^E(x) = \sum_{n=0}^{\infty} J(q^n x) - \sum_{n=1}^{\infty} J(q^n x^{-1}) + \frac{1}{3}\log^2|q| B_3\Big(\frac{\log|x|}{\log|q|}\Big),$$

where $\mathcal{L}_2(z) = \mathrm{Im}(\mathrm{Li}_2(z) + \log|z|\log(1-z))$ is known as the *Bloch–Wigner dilogarithm*, $J(z) = \log|z|\log|1-z|$ and $B_3(X) = X^3 - 3X^2/2 + X/2$.

For $a, b \in \mathbb{N}$, the series

$$K_{a,b}(\tau; u) = \sum_{m,n\in\mathbb{Z}}{}' \frac{e^{2\pi i(n\xi - m\eta)}}{(m\tau+n)^a(m\bar{\tau}+n)^b},$$

where $u = \xi\tau + \eta$ and $\xi, \eta \in \mathbb{R}/\mathbb{Z}$, is called the *Eisenstein–Kronecker series*. Here and throughout, $\sum'$ means $(m,n) \neq (0,0)$ in the summation. Bloch [1] defined the regulator function $R^E : E(\mathbb{C}) \to \mathbb{R}$ by

$$R^E(e^{2\pi i u}) = \frac{\mathrm{Im}(\tau)^2}{\pi} K_{2,1}(\tau; u).$$

One can extend the functions $D^E, J^E, R^E, \mathcal{L}^E_{3,1}$ and $\mathcal{L}^E_{3,2}$ to the group of divisors on $E(\mathbb{C})$ by linearity. Also, it can be shown that $\mathrm{Re}(R^E) = D^E$ and $\mathrm{Im}(R^E) = J^E$. In [4], Goncharov and Levin proved the following theorem, formerly known as Zagier's conjecture on $L(E,2)$.

THEOREM 1.1. *Let E be a modular elliptic curve over $\mathbb{Q}$. Then there exists a divisor $P = \sum n_j(P_j)$ on $E(\bar{\mathbb{Q}})$ satisfying the following conditions.*

(a) *We have*
$$\sum n_j P_j \otimes P_j \otimes P_j = 0 \quad \textit{in } \mathrm{Sym}^3(E).$$

(b) *For any valuation v of the field $\mathbb{Q}(P)$ generated by the coordinates of the points P_j,*
$$\sum n_j h_v(P_j) \cdot P_j = 0 \quad \textit{on } E,$$
where h_v is the local height associated with the valuation v.

(c) *For every prime p where E has a split multiplicative reduction, P satisfies a certain integrality condition (see* [4, Theorem 1.1]*).*

Moreover, for such a divisor P,

$$L(E,2) \sim_{\mathbb{Q}^\times} \pi \cdot D^E(P),$$

where $A \sim_{\mathbb{Q}^\times} B$ means $A = cB$ for some $c \in \mathbb{Q}^\times$.

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The critical values of the symmetric square L-function attached to a weight k cusp form are those at the odd integers in $\{1, 2, \ldots, k-1\}$ and the even integers in $\{k, k+1, \ldots, 2k-2\}$. In particular, $L(\mathrm{Sym}^2(E), 2)$ and $L(\mathrm{Sym}^2(E), 1)$ are the only critical values of the symmetric square L-function of an elliptic curve. There are several numerical results and conjectures relating noncritical values of L-series of symmetric powers of an elliptic curve over $\mathbb{Q}$ to higher elliptic polylogarithms including those due to Mestre and Schappacher [7], Goncharov [3] and Wildeshaus [11]. Inspired by these examples and their numerical experiments, Zagier and Gangl [13, Section 10] formulated the following conjecture, which is an analogue of Theorem 1.1.

Conjecture 1.2. Let E be an elliptic curve over $\mathbb{Q}$. For any $\xi = \sum n_i(P_i) \in \mathbb{Z}[E(\bar{\mathbb{Q}})]$ and any homomorphism $\phi : E(\bar{\mathbb{Q}}) \to \mathbb{Z}$, let $\iota_\phi(\xi) = \sum n_i \phi(P_i)(P_i)$. Also define

$$C_2(E/\mathbb{Q}) = \Big\langle (f) \diamond (1-f), (P) + (-P), (2P) - 2 \sum_{T \in E[2]} (P+T) \mid f \in \mathbb{Q}(E), P \in E(\mathbb{Q}) \Big\rangle$$

as a subgroup of $\mathbb{Z}[E(\bar{\mathbb{Q}})]^{\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})}$ where, for $(f) = \sum m_i(P_i)$ and $(g) = \sum n_j(Q_j)$, the diamond operator $\diamond$ is defined by $(f) \diamond (g) = \sum m_i n_j (a_i - b_j)$. If $\iota_\phi(\xi) \in C_2(E/\mathbb{Q})$ for all homomorphisms $\phi : E(\bar{\mathbb{Q}}) \to \mathbb{Q}$, then $\vec{\mathcal{L}}_3^E(\xi) := (\mathcal{L}_{3,1}^E(\xi), \mathcal{L}_{3,2}^E(\xi))$ belongs to a two-dimensional lattice whose covolume is related to $L(\mathrm{Sym}^2(E), 3)$.

Zagier and Gangl verified numerically that if E is the conductor 37 elliptic curve defined by $y^2 - y = x^3 - x$, $E \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, and

$$\eta_4 = 3(4P) - 13(3P) + 18(2P) - 3(P) - 5(O),$$
$$\eta_6 = 2(6P) - 45(3P) + 60(2P) + 93(P) - 110(O),$$

where $P = [0, 0]$, then

$$\mathrm{Reg}_3(E) := \begin{vmatrix} \mathcal{L}_{3,1}^E(\eta_4) & \mathcal{L}_{3,2}^E(\eta_4) \\ \mathcal{L}_{3,1}^E(\eta_6) & \mathcal{L}_{3,2}^E(\eta_6) \end{vmatrix} \overset{?}{=} -\frac{37^3}{4} \operatorname{Im}(\tau)^2 L(\mathrm{Sym}^2(E), 3). \tag{1.2}$$

Here $A \overset{?}{=} B$ means A and B are equal to at least 15 decimal places. (Note that the negative sign in the above identity is missing in [13].) Recall from [2] that $L(\mathrm{Sym}^2(E), s)$ satisfies the functional equation

$$\Lambda(\mathrm{Sym}^2(E), s) = \Lambda(\mathrm{Sym}^2(E), 3-s),$$

where $\Lambda(\mathrm{Sym}^2(E), s) = C^{s/2}\pi^{-s/2}\Gamma(s/2)(2\pi)^{-s}\Gamma(s)L(\mathrm{Sym}^2(E), s)$ and C is the conductor of the Galois representation associated with the symmetric square of the Tate module of E. Therefore, (1.2) can be rephrased as

$$\mathrm{Reg}_3(E) \overset{?}{=} 2\pi^4 \operatorname{Im}(\tau)^2 L''(\mathrm{Sym}^2(E), 0).$$

This conjecture is consistent with a special case of [3, Conjecture 6.8] that, for any elliptic curve E over $\mathbb{Q}$, there exist degree zero divisors ξ_1 and ξ_2 on $E(\bar{\mathbb{Q}})$ such that

$$\begin{vmatrix} \mathrm{Re}\,(K_{1,3}(\tau; \xi_1)) & K_{2,2}(\tau; \xi_1) \\ \mathrm{Re}\,(K_{1,3}(\tau; \xi_2)) & K_{2,2}(\tau; \xi_2) \end{vmatrix} \overset{?}{\sim}_{\mathbb{Q}^\times} \frac{\pi^6}{\operatorname{Im}(\tau)^4} L''(\mathrm{Sym}^2(E), 0).$$

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The relationship between the determinant above and the one in (1.2) was established in [9, Section 4] and can be stated as follows.

PROPOSITION 1.3. *Let E be an elliptic curve over $\mathbb{C}$ and suppose that $E \cong \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$. If ξ_1 and ξ_2 are divisors of degree zero on E, then*

$$\begin{vmatrix} \mathcal{L}^E_{3,1}(\xi_1) & \mathcal{L}^E_{3,2}(\xi_1) \\ \mathcal{L}^E_{3,1}(\xi_2) & \mathcal{L}^E_{3,2}(\xi_2) \end{vmatrix} = -\frac{2\,\mathrm{Im}(\tau)^6}{\pi^2} \begin{vmatrix} \mathrm{Re}\,(K_{1,3}(\tau;\xi_1)) & K_{2,2}(\tau;\xi_1) \\ \mathrm{Re}\,(K_{1,3}(\tau;\xi_2)) & K_{2,2}(\tau;\xi_2) \end{vmatrix},$$

where $K_{a,b}(\tau;\xi) = \sum_{P\in E} n_P K_{a,b}(\tau;u_P)$ if $\xi = \sum_{P\in E} n_P(P)$ and u_P is the image of P in $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$.

A square-free positive integer n is called a *congruent number* if it is the area of a right triangle all of whose sides are rational numbers. The congruent number problem, one of the oldest unsolved problems in number theory, asks if there is an algorithm for determining whether any given number is a congruent number in a finite number of steps. The main result in this paper concerns a symmetric square L-value of the congruent number elliptic curves, which are defined by

$$E_d : y^2 = x^3 - d^2x \quad \text{for all } d \in \mathbb{N}. \tag{1.3}$$

These curves play a crucial role in the study of the congruent number problem. In fact, assuming the Birch and Swinnerton-Dyer conjecture, it can be proved that a square-free positive integer d is a congruent number if and only if $L(E_d, 1) = 0$ (see, for example, [6]). Some useful facts about E_d include $E_d \cong \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\sqrt{-1})$, that E_d has complex multiplication by $\mathbb{Z}[\sqrt{-1}]$ and that E_d is a quadratic twist of E_1. We will give a rigorous proof of a formula for $L(\mathrm{Sym}^2(E_d), 3)$, which provides evidence supporting Conjecture 1.2.

THEOREM 1.4. *For any positive integer d, let $E := E_d$ be the elliptic curve defined by (1.3) and let P, Q and O be points on $E(\bar{\mathbb{Q}})$ corresponding to $\frac{1}{2}\sqrt{-1}$, $\frac{1}{4}$ and 1, respectively, via the isomorphism $E \cong \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\sqrt{-1})$. If $\xi_1 = (Q) + (P+Q) - 2(O)$ and $\xi_2 = (2Q) - (P)$, then*

$$\begin{vmatrix} \mathcal{L}^E_{3,1}(\xi_1) & \mathcal{L}^E_{3,2}(\xi_1) \\ \mathcal{L}^E_{3,1}(\xi_2) & \mathcal{L}^E_{3,2}(\xi_2) \end{vmatrix} = -\frac{43}{2}L(\mathrm{Sym}^2(E), 3) = -\frac{43\pi^4}{128}L''(\mathrm{Sym}^2(E), 0). \tag{1.4}$$

REMARK 1.5. (i) The points P and Q in Theorem 1.4 can be written explicitly as

$$P = [d, 0], \quad Q = [-d(1+\sqrt{2}), \sqrt{-(6+4\sqrt{2})d^3}].$$

This can be checked using a computer algebra system such as `PARI/GP` or `SAGE`.

(ii) Since the symmetric square L-function is invariant under a quadratic twist (by [2, Section 1.1]), it suffices to prove Theorem 1.4 for a particular value of d. As the reader will see in Sections 3 and 4, we choose $d = 2$.

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2. Some identities involving $\mathcal{L}_{3,1}^E$ and $\mathcal{L}_{3,2}^E$

Before proving the main result, we shall state some useful facts about the functions $\mathcal{L}_{3,1}^E$ and $\mathcal{L}_{3,2}^E$. The reader is referred to [9] and [12] for further details.

PROPOSITION 2.1 [9, Corollary 2.3]. *Suppose that $E \cong \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ with $\tau \in \mathcal{H}$ and let $q = e^{2\pi i\tau}$ and $x = e^{2\pi i u}$, where $u = \xi\tau + \eta$ and $\xi, \eta \in \mathbb{R}/\mathbb{Z}$. Then*

$$\mathcal{L}_{3,1}^E(x) = \frac{4\,\mathrm{Im}(\tau)^5}{3\pi}\,\mathrm{Re}\Big(\sum_{m,n\in\mathbb{Z}}{}' e^{2\pi i(n\xi - m\eta)}\frac{m^2}{|m\tau+n|^6}\Big), \tag{2.1}$$

$$\mathcal{L}_{3,2}^E(x) = \frac{\mathrm{Im}(\tau)^3}{\pi}\Big[\sum_{m,n\in\mathbb{Z}}{}' \frac{e^{2\pi i(n\xi - m\eta)}}{|m\tau+n|^4} + 2\,\mathrm{Re}\Big(\sum_{m,n\in\mathbb{Z}}{}' e^{2\pi i(n\xi - m\eta)}\frac{(m\tau+n)^2}{|m\tau+n|^6}\Big)\Big] + \frac{\log^3|q|}{120}. \tag{2.2}$$

The following result is an immediate consequence of (2.1).

PROPOSITION 2.2. *Let E be an elliptic curve isomorphic to $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$. If P and Q are the points on E corresponding to $\tau/2$ and $1/4$, respectively, via the isomorphism (1.1), then*

$$\mathcal{L}_{3,1}^E((Q) + (P+Q)) = \tfrac{1}{8}\mathcal{L}_{3,1}^E(2Q). \tag{2.3}$$

PROOF. Using (2.1) and the fact that $e^{m\pi i/2} = i^m$ for any $m \in \mathbb{Z}$,

$$\begin{aligned}
\mathcal{L}_{3,1}^E(8(Q) + 8(P+Q)) &= \frac{32\,\mathrm{Im}(\tau)^5}{3\pi}\,\mathrm{Re}\Big(\sum_{m,n\in\mathbb{Z}}{}'(i^m(1+(-1)^n))\frac{m^2}{|m\tau+n|^6}\Big)\\
&= \frac{128\,\mathrm{Im}(\tau)^5}{3\pi}\sum_{m,n\in\mathbb{Z}}{}'((-1)^m(1+(-1)^n))\frac{m^2}{|2m\tau+n|^6}\\
&= \frac{4\,\mathrm{Im}(\tau)^5}{3\pi}\sum_{m,n\in\mathbb{Z}}{}'\frac{(-1)^m m^2}{|m\tau+n|^6}\\
&= \mathcal{L}_{3,1}^E(2Q).
\end{aligned}$$

The last equality follows from the fact that $2Q$ is a point corresponding to $1/2$. □

3. Grössencharakters and modular forms

It is well known that the L-function of an elliptic curve over $\mathbb{Q}$ with complex multiplication (CM) coincides with that of a Hecke character (a Grössencharakter) of an imaginary quadratic field. In this section, we will explicitly construct the Hecke character corresponding to the CM elliptic curve $E := E_2$. Then we invoke a result of Coates and Schmidt [2] to obtain an expression of $L(\mathrm{Sym}^2(E), s)$ in terms of a product of L-functions attached to a weight three modular form and a Dirichlet character. More precisely, we will prove the following identity.

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THEOREM 3.1. *Let E be the elliptic curve of conductor* 64 *defined by $E : y^2 = x^3 - 4x$. Then, for any $s \in \mathbb{C}$,*

$$L(\mathrm{Sym}^2(E), s) = L(g, s)L(\chi_{-4}, s-1),$$

where $g(\tau) = q - 6q^5 + 9q^9 + \cdots$ is a weight three cusp form of level 16 *and $\chi_{-4} = \left(\frac{-4}{\cdot}\right)$ is the Dirichlet character associated with $\mathbb{Q}(\sqrt{-1})$.*

PROOF. Let $K = \mathbb{Q}(\sqrt{-1})$. Then the ring of integers of K is $\mathcal{O}_K = \mathbb{Z}[\sqrt{-1}]$. Let $\Lambda = (4) \subset \mathcal{O}_K$ and let $P(\Lambda)$ be the set of (integral) ideals of $\mathcal{O}_K$ that are relatively prime to Λ. It is easily seen that each element of $P(\Lambda)$ can be represented uniquely by $(m + ni)$, where $m > 0$ is an odd integer and n is an even integer.

Define a map $\phi : P(\Lambda) \to \mathbb{C}^\times$ by

$$\phi((m+ni)) = \chi_{-4}(m)(m+ni) = \begin{cases} m+ni & \text{if } m \equiv 1 \pmod 4, \\ -(m+ni) & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

Then $\phi((\alpha)) = \alpha$, for any $\alpha \in \mathcal{O}_K$ such that $\alpha \equiv 1 \pmod \Lambda$. It follows that we can extend ϕ multiplicatively to a Hecke character of conductor Λ. By [8, Theorem 1.31],

$$\mathfrak{f}(\tau) = \sum_{\mathfrak{a} \in P(\Lambda)} \phi(\mathfrak{a}) q^{N(\mathfrak{a})}$$

is a weight two newform of level 64. Computing the first few terms of $\mathfrak{a}$, we obtain

$$\mathfrak{f}(\tau) = \sum_{\substack{m \in \mathbb{N} \\ n \in \mathbb{Z}}} \chi_{-4}(m) m q^{m^2+4n^2} = q + 2q^5 - 3q^9 - 6q^{13} + \cdots,$$

which is the weight two newform corresponding to E via the modularity theorem.

Let ϕ^2 be the primitive Hecke character attached to the square of ϕ. Then ϕ^2 is a Hecke character of conductor $\Lambda' = (2)$ and satisfies

$$\phi^2((\alpha)) = \alpha^2,$$

for any ideal (α) in $\mathcal{O}_K$ satisfying $\alpha \equiv 1 \pmod{\Lambda'}$. Moreover, it is known that $L(\phi^2, s) = L(g, s)$ (see, for example, [10, Lemma 2.3]). Finally, by a result due to Coates and Schmidt [2, Proposition 5.1],

$$L(\mathrm{Sym}^2(E), s) = L(\phi^2, s)L(\chi_{-4}, s-1) = L(g, s)L(\chi_{-4}, s-1). \qquad \square$$

It has been shown that $L(\chi_{-4}, s)$ and $L(g, s)$ have simple lattice sum expressions, which will be particularly useful in the proof of our main result.

PROPOSITION 3.2 ([10, Lemma 2.3], [5, Section IV]). *Let $s, t \in \mathbb{C}$, where* $\mathrm{Re}(s) > 2$ *and* $\mathrm{Re}(t) > 1$. *Then*

$$L(g, s) = \frac{1}{2} \sum_{m,n \in \mathbb{Z}}{}' \frac{m^2 - 4n^2}{(m^2+4n^2)^s},$$

$$L(\chi_{-4}, t) = \frac{1}{4\zeta(t)} \sum_{m,n \in \mathbb{Z}}{}' \frac{1}{(m^2+n^2)^t} = \frac{1}{2(1 - 2^{-t} + 2^{1-2t})\zeta(t)} \sum_{m,n \in \mathbb{Z}}{}' \frac{1}{(m^2+4n^2)^t},$$

where $\zeta(t)$ is the Riemann zeta function.

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COROLLARY 3.3. *We have*

$$L(\chi_{-4},2) = \frac{3}{2\pi^2}\sideset{}{'}\sum_{m,n\in\mathbb{Z}} \frac{1}{(m^2+n^2)^2}, \tag{3.1}$$

$$= \frac{24}{7\pi^2}\sideset{}{'}\sum_{\substack{m \text{ even}\\ n\in\mathbb{Z}}} \frac{1}{(m^2+n^2)^2}, \tag{3.2}$$

$$L(g,3) = \frac{1}{2}\sideset{}{'}\sum_{\substack{m\in\mathbb{Z}\\ n \text{ even}}} \frac{m^2-n^2}{(m^2+n^2)^3}, \tag{3.3}$$

$$= \frac{1}{2}\sum_{\substack{m \text{ odd}\\ n \text{ even}}} \frac{m^2-n^2}{(m^2+n^2)^3}. \tag{3.4}$$

4. Proof of the main result

We first give a series of identities relating values of $\mathcal{L}^E_{3,1}$ and $\mathcal{L}^E_{3,2}$ to modular and Dirichlet L-values.

LEMMA 4.1. *With the same assumptions as in Theorem 1.4,*

$$\mathcal{L}^E_{3,1}((Q)+(P+Q)) = -\frac{1}{3\pi}L(g,3) - \frac{\pi}{144}L(\chi_{-4},2), \tag{4.1}$$

$$\mathcal{L}^E_{3,1}(O) = \frac{4\pi}{9}L(\chi_{-4},2), \tag{4.2}$$

$$\mathcal{L}^E_{3,1}((2Q)-(P)) = -\frac{16}{3\pi}L(g,3), \tag{4.3}$$

$$\mathcal{L}^E_{3,2}((2Q)-(P)) = \frac{16}{\pi}L(g,3), \tag{4.4}$$

$$\mathcal{L}^E_{3,2}((Q)+(P+Q)-2(O)) = \frac{1}{\pi}L(g,3) - \frac{43\pi}{32}L(\chi_{-4},2). \tag{4.5}$$

PROOF. First, note that, by symmetry,

$$\sideset{}{'}\sum_{m,n\in\mathbb{Z}} \frac{n^2-m^2}{(m^2+n^2)^3} = 0.$$

Therefore, by (2.1) and (3.1),

$$\begin{aligned}
\mathcal{L}^E_{3,1}(O) &= \frac{4}{3\pi}\operatorname{Re}\Bigg(\sideset{}{'}\sum_{m,n\in\mathbb{Z}} e^{-2\pi i m}\frac{m^2}{|n+mi|^6}\Bigg) \\
&= \frac{4}{3\pi}\sideset{}{'}\sum_{m,n\in\mathbb{Z}} \frac{m^2}{(m^2+n^2)^3} \\
&= \frac{2}{3\pi}\sideset{}{'}\sum_{m,n\in\mathbb{Z}}\Bigg(\frac{1}{(m^2+n^2)^2} - \frac{n^2-m^2}{(m^2+n^2)^3}\Bigg)
\end{aligned}$$

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$$= \frac{2}{3\pi} \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+n^2)^2} = \frac{4\pi}{9} L(\chi_{-4}, 2),$$

which yields (4.2).

Next, using (2.2) and (3.4),

$$\begin{aligned}
\mathcal{L}^E_{3,2}((2Q)-(P)) &= \frac{1}{\pi}\Big(\sum_{m,n\in\mathbb{Z}}{}' \frac{(-1)^m-(-1)^n}{(m^2+n^2)^2} + 2\sum_{m,n\in\mathbb{Z}}{}' ((-1)^m-(-1)^n)\frac{n^2-m^2}{(m^2+n^2)^3}\Big) \\
&= \frac{1}{\pi}\Big(-2 \sum_{\substack{m \text{ odd} \\ n \text{ even}}}{}' \frac{1}{(m^2+n^2)^2} - 4 \sum_{\substack{m \text{ odd} \\ n \text{ even}}}{}' \frac{n^2-m^2}{(m^2+n^2)^3} \\
&\quad + 2 \sum_{\substack{m \text{ even} \\ n \text{ odd}}}{}' \frac{1}{(m^2+n^2)^2} + 4 \sum_{\substack{m \text{ even} \\ n \text{ odd}}}{}' \frac{n^2-m^2}{(m^2+n^2)^3}\Big) \\
&= \frac{8}{\pi} \sum_{\substack{m \text{ even} \\ n \text{ odd}}}{}' \frac{n^2-m^2}{(m^2+n^2)^3} = \frac{16}{\pi} L(g,3),
\end{aligned}$$

which is (4.4).

On the other hand, it is easily seen by symmetry that

$$\sum_{m,n\in\mathbb{Z}}{}' \frac{(-1)^m-(-1)^n}{(m^2+n^2)^2} = 0,$$

so that

$$\begin{aligned}
\mathcal{L}^E_{3,2}((2Q)-(P)) &= \frac{2}{\pi} \sum_{m,n\in\mathbb{Z}}{}' ((-1)^m-(-1)^n)\frac{n^2-m^2}{(m^2+n^2)^3} \\
&= -\frac{4}{\pi} \sum_{m,n\in\mathbb{Z}}{}' ((-1)^m-(-1)^n)\frac{m^2}{(m^2+n^2)^3} = -3\mathcal{L}^E_{3,1}((2Q)-(P)).
\end{aligned}$$

Together with (4.4), this gives (4.3).

To establish (4.1), we first employ (3.1) and (3.2) to deduce that

$$\frac{1}{2} \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+n^2)^2} - \sum_{\substack{m \text{ even} \\ n \text{ even}}}{}' \frac{1}{(m^2+n^2)^2} = \frac{7}{16} \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+n^2)^2} = \sum_{\substack{m \text{ even} \\ n\in\mathbb{Z}}}{}' \frac{1}{(m^2+n^2)^2}.$$

Therefore,

$$\begin{aligned}
\sum_{\substack{m \text{ even} \\ n \text{ even}}}{}' \frac{1}{(m^2+n^2)^2} + \sum_{\substack{m \text{ even} \\ n\in\mathbb{Z}}}{}' \frac{1}{(m^2+n^2)^2} &= \frac{1}{2} \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+n^2)^2} \\
&= \frac{1}{2} \sum_{m,n\in\mathbb{Z}}{}' \frac{m^2+n^2}{(m^2+n^2)^3} \\
&= \sum_{m,n\in\mathbb{Z}}{}' \frac{m^2}{(m^2+n^2)^3}. \qquad (4.6)
\end{aligned}$$

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Using (4.6),

$$\begin{aligned}
&\sideset{}{'}\sum_{\substack{m \text{ even} \\ n \text{ even}}} \frac{1}{(m^2+n^2)^2} + \sideset{}{'}\sum_{\substack{m \text{ even} \\ n\in\mathbb{Z}}} \frac{n^2}{(m^2+n^2)^3} \\
&\quad = \sideset{}{'}\sum_{\substack{m \text{ even} \\ n \text{ even}}} \frac{1}{(m^2+n^2)^2} + \sideset{}{'}\sum_{\substack{m \text{ even} \\ n\in\mathbb{Z}}} \Big(\frac{1}{(m^2+n^2)^2} - \frac{m^2}{(m^2+n^2)^3}\Big) \\
&\quad = \sideset{}{'}\sum_{m,n\in\mathbb{Z}} \frac{m^2}{(m^2+n^2)^3} - \sideset{}{'}\sum_{\substack{m \text{ even} \\ n\in\mathbb{Z}}} \frac{m^2}{(m^2+n^2)^3} \\
&\quad = \sideset{}{'}\sum_{\substack{m \text{ odd} \\ n\in\mathbb{Z}}} \frac{m^2}{(m^2+n^2)^3}.
\end{aligned} \tag{4.7}$$

By (2.3), (2.1) and (4.7),

$$\begin{aligned}
\mathcal{L}^E_{3,1}((Q)+(P+Q)) &= \frac{1}{6\pi} \sideset{}{'}\sum_{m,n\in\mathbb{Z}} \frac{(-1)^m m^2}{(m^2+n^2)^3} \\
&= \frac{1}{6\pi} \sideset{}{'}\sum_{\substack{m \text{ even} \\ n\in\mathbb{Z}}} \frac{m^2}{(m^2+n^2)^3} - \frac{1}{6\pi} \sideset{}{'}\sum_{\substack{m \text{ odd} \\ n\in\mathbb{Z}}} \frac{m^2}{(m^2+n^2)^3} \\
&= \frac{1}{6\pi} \sideset{}{'}\sum_{\substack{m \text{ even} \\ n\in\mathbb{Z}}} \frac{m^2}{(m^2+n^2)^3} - \frac{1}{6\pi} \sideset{}{'}\sum_{\substack{m \text{ even} \\ n\in\mathbb{Z}}} \frac{n^2}{(m^2+n^2)^3} \\
&\qquad - \frac{1}{96\pi} \sideset{}{'}\sum_{m,n\in\mathbb{Z}} \frac{1}{(m^2+n^2)^2} \\
&= -\frac{1}{3\pi} L(g,3) - \frac{\pi}{144} L(\chi_{-4},2),
\end{aligned}$$

where the last equality follows from (3.3) and (3.1).

Finally, (4.5) follows from (2.2), (3.1), (3.3) and some tedious manipulations. □

Theorem 1.4 now easily follows from Lemma 4.1 and Theorem 3.1.

Proof of Theorem 1.4. Let $\alpha = L(g,3)$ and $\beta = L(\chi_{-4},2)$. By (4.1)–(4.5) and Theorem 3.1,

$$\begin{aligned}
\begin{vmatrix} \mathcal{L}^E_{3,1}(\xi_1) & \mathcal{L}^E_{3,2}(\xi_1) \\ \mathcal{L}^E_{3,1}(\xi_2) & \mathcal{L}^E_{3,2}(\xi_2) \end{vmatrix} &= \mathcal{L}^E_{3,1}(\xi_1)\mathcal{L}^E_{3,2}(\xi_2) - \mathcal{L}^E_{3,1}(\xi_2)\mathcal{L}^E_{3,2}(\xi_1) \\
&= \frac{16}{\pi}\alpha\Big(-\frac{1}{3\pi}\alpha - \frac{43\pi}{48}\beta\Big) + \frac{16}{3\pi}\alpha\Big(\frac{1}{\pi}\alpha - \frac{43\pi}{32}\beta\Big) \\
&= -\frac{43}{2}\alpha\beta \\
&= -\frac{43}{2} L(\mathrm{Sym}^2(E),3).
\end{aligned}$$

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The second equality in (1.4) follows from the functional equation for the symmetric square L-function.

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