COMPOSITIO MATHEMATICA

Connected components of affine Deligne–Lusztig varieties for unramified groups

Sian Nie

Compositio Math. **159** (2023), 2051–2088.

doi:10.1112/S0010437X23007339

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY EST. 1865

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

Compositio Math. 159 (2023) 2051–2088
doi:10.1112/S0010437X23007339

Connected components of affine Deligne–Lusztig varieties for unramified groups

Sian Nie

Abstract

For an unramified reductive group, we determine the connected components of affine Deligne–Lusztig varieties in the affine flag variety. Based on work of Hamacher, Kim, and Zhou, this result allows us to verify, in the unramified group case, the He–Rapoport axioms, the almost product structure of Newton strata, and the precise description of isogeny classes predicted by the Langlands–Rapoport conjecture, for the Kisin–Pappas integral models of Shimura varieties of Hodge type with parahoric level structure.

Introduction

0.1 Background

Let F be a non-Archimedean local field with valuation ring $\mathcal{O}_F$ and residue field $\mathbb{F}_q$, where q is a power of some prime p. Let $\breve{F}$ be the completion of a maximal unramified extension of F, and denote by σ the Frobenius automorphism of $\breve{F}/F$.

Let G be a connected reductive group defined over F. Fix an element $b \in G(\breve{F})$, a geometric cocharacter λ of G, and a σ-stable parahoric subgroup $K \subseteq G(\breve{F})$. The attached affine Deligne–Lusztig variety is defined by

$$X(\lambda, b)_K = X^G(\lambda, b)_K = \{g \in G(\breve{F})/K; g^{-1}b\sigma(g) \in K \mathrm{Adm}(\lambda) K\},$$

where $\mathrm{Adm}(\lambda)$ is the admissible set associated to the geometric conjugacy class of λ. If F is of equal characteristic, $X(\lambda, b)_K$ is a locally closed and locally finite-type subvariety of the partial affine flag variety $G(\breve{F})/K$. If F is of mixed characteristic, $X(\lambda, b)_K$ is a perfect subscheme of the Witt vector partial affine flag variety, in the sense of Bhatt and Scholze [BS17] and Zhu [Zhu17].

The variety $X(\lambda, b)_K$, first introduced by Rapoport [Rap05], encodes important arithmetic information of Shimura varieties. Let $(\mathbf{G}, X)$ be a Shimura datum with $G = \mathbf{G}_{\mathbb{Q}_p}$ and λ the inverse of the Hodge cocharacter. Suppose there is a suitable integral model for the corresponding Shimura variety with parahoric level structure. Langlands [Lan76], and latter refined by Langlands and Rapoport [LR87] and Rapoport [Rap05], conjectured a precise description of $\overline{\mathbb{F}}_p$-points of the integral model in terms of the varieties $X(\lambda, b)_K$. In the case of PEL Shimura varieties, $X(\lambda, b)_K$ is also the set of $\overline{\mathbb{F}}_p$-points of a moduli space of p-divisible groups defined by Rapoport and Zink [RZ96].

Received 13 January 2022, accepted in final form 21 March 2023, published online 17 August 2023.
2020 Mathematics Subject Classification 20G25 (primary), 14G35, 11G18 (secondary).
Keywords: affine Deligne–Lusztig varieties, connected components, reduction of Shimura varieties, Langlands–Rapoport conjecture.

0.2 Main result

The main purpose of this paper is to study the set $\pi_0(X(\lambda,b)_K)$ of connected components of $X(\lambda,b)_K$. Note that $X(\lambda,b)_K$ only depends on λ and the σ-conjugacy class $[b]$ of b. Thanks to He [He16], $X(\lambda,b)_K$ is non-empty if and only if $[b]$ belongs to the set $B(G,\lambda)$ of 'neutral acceptable' σ-conjugacy classes of $G(\breve{F})$ with respect to λ.

Let $\pi_1(G)_{\Gamma_0}$ be the set of coinvariants of the fundamental group $\pi_1(G)$ under the Galois group $\Gamma_0 = \mathrm{Gal}(\overline{\breve{F}}/\breve{F})$. Denote by $\eta_G : G(\breve{F}) \to \pi_1(G)_{\Gamma_0}$ the natural group homomorphism. It factors through a map $G(\breve{F})/K \to \pi_1(G)_{\Gamma_0}$ which we still denote by η_G. Let G_{ad} denote the adjoint group of G. Then we have the following Cartesian diagram (see [HZ20, Corollary 4.4]):

$$\begin{array}{ccc} \pi_0(X^G(\lambda,b)_K) & \longrightarrow & \pi_0(X^{G_{\mathrm{ad}}}(\lambda_{\mathrm{ad}},b_{\mathrm{ad}})_{K_{\mathrm{ad}}}) \\ {\scriptstyle \eta_G}\downarrow & & \downarrow{\scriptstyle \eta_{G_{\mathrm{ad}}}} \\ \pi_1(G)_{\Gamma_0} & \longrightarrow & \pi_1(G_{\mathrm{ad}})_{\Gamma_0} \end{array}$$

where b_{ad} and K_{ad} are the natural images of b and K in $G_{\mathrm{ad}}(\breve{F})$, respectively.

Therefore, to compute $\pi_0(X(\lambda,b)_K)$ we may and do assume that G is adjoint and, hence, simple. Note that the map η_G gives a natural obstruction to the connectedness of $X(\lambda,b)_K$. Another more technical obstruction is given by the following Hodge–Newton decomposition theorem.

Theorem 0.1 [GHN19, Theorem 4.17]. *Suppose G is adjoint and simple. If the pair (λ,b) is Hodge–Newton decomposable (with respect to some proper Levi subgroup M) in the sense of [GHN19, § 2.5.5], then $X(\lambda,b)_K$ is a disjoint union of open and closed subsets, which are isomorphic to certain affine Deligne–Lusztig varieties attached to M.*

By Theorem 0.1 and induction on the dimension of G, it suffices to consider the Hodge–Newton indecomposable case. This means that either λ is a central cocharacter or the pair (λ,b) Hodge–Newton irreducible, see [Zho20, Lemma 5.3]. In the former case,

$$X(\lambda,b)_K \cong \mathbb{J}_b/(K \cap \mathbb{J}_b)$$

is a discrete set, where $\mathbb{J}_b$ denotes the σ-centralizer of b. In the latter case, we have the following conjecture.

Conjecture 0.1 (See [Zho20, Conjecture 5.4]). Assume G is adjoint and simple. If (λ,b) is Hodge–Newton irreducible, then there exists a natural bijection

$$\pi_0(X(\lambda,b)_K) \cong \pi_1(G)_{\Gamma_0}^{\sigma},$$

where $\pi_1(G)_{\Gamma_0}^{\sigma}$ is the set of σ-fixed points of $\pi_1(G)_{\Gamma_0}$.

If G is unramified and K is hyperspecial, Conjecture 0.1 is established by Viehmann [Vie08], Chen, Kisin, and Viehmann [CKV15], and the present author [Nie15]. If b is basic, it is proved by He and Zhou [HZ20]. If G is split or $G = \mathrm{Res}_{E/F}\mathrm{GL}_n$ with E/F a finite unramified field extension, it was proved by Chen and the present author in [CN19] and [CN20].

The main result of this paper is the following.

Theorem 0.2. *Conjecture 0.1 is true if G is unramified.*

In particular, Theorem 0.2 completes the computation of connected components of affine Deligne–Lusztig varieties for unramified groups.

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

0.3 Applications

We discuss some applications. Assume $p \neq 2$. Let $(\mathbf{G}, X)$ be a Shimura datum of Hodge type with parahoric level structure such that $p \nmid |\pi_1(\mathbf{G}_{\mathrm{der}})|$, $\mathbf{G}_{\mathbb{Q}_p}$ is tamely ramified, and the corresponding parahoric subgroup K at p is a connected parahoric. Let $\mathscr{S}_K = \mathscr{S}_K(G, X)$ be the Kisin–Pappas integral model constructed in [KP18]. Let $F = \mathbb{Q}_p$, $G = \mathbf{G}_{\mathbb{Q}_p}$, and λ be the inverse of the Hodge cocharacter.

Remark 0.3. In [PR21], Pappas and Rapoport obtained a new construction of integral models for Hodge-type Shimura varieties with parahoric level structure, without the tameness assumption on G. It would be desirable to extend the applications discussed below to their integral models in the unramified group case.

0.3.1 *The Langlands–Rapoport conjecture.* A major motivation to study $\pi_0(X(\lambda, b)_K)$ comes from the Langlands–Rapoport conjecture mentioned in § 0.1. In the hyperspecial level structure case, the conjecture is proved by Kottwitz [Kot92] for PEL Shimura varieties of types A and C, and by Kisin [Kis17] for his integral models [Kis10] of Shimura varieties of abelian type. For the Kisin–Pappas integral models of Hodge type, Zhou [Zho20] proved that each mod p isogeny class has the predicted form when G is residually split. Recently, van Hoften [vH20] proved the Langlands–Rapoport conjecture for a large family of Shimura varieties of abelian type (including the Hodge type) when G is unramified (as well as some other cases), by reducing the problem to the hyperspecial case.

One of the key ingredients in the proofs of Kisin and Zhou is to construct certain lifting maps from the varieties $X(\lambda, b)_K$ to the corresponding isogeny classes of $\mathscr{S}_K(\overline{\mathbb{F}}_p)$ (see also [HK19, Axiom A]), which uses in a crucial way the descriptions of $\pi_0(X(\lambda, b)_K)$ in [CKV15] and [HZ20], respectively. Combining [Zho20, Proposition 6.5] with Theorem 0.2, we deduce that such lifting maps always exist if G is unramified.

Proposition 0.4. *If G is unramified, then the Rapoport–Zink uniformization map admits a unique lift on $\overline{\mathbb{F}}_p$-points*

$$X(\lambda, b)_K \to \mathscr{S}_K(\overline{\mathbb{F}}_p),$$

which respects canonical crystalline Tate tensors on both sides.

If G is unramified and K is hyperspecial, Proposition 0.4 is proved by Kisin [Kis17]. If b is basic or G is residually split, it is proved by Zhou [Zho20]. If G is quasi-split and K is absolutely special, it is proved by Zhou in [vH20, Theorem A.4.3].

Combining the methods in [Zho20] and Proposition 0.4, one can extend [Zho20, Theorem 1.1] to the unramified group case. This was pointed out to us by Zhou.

Corollary 0.5 (van Hoften). *If G is unramified, then the isogeny classes in $\mathscr{S}_K(\overline{\mathbb{F}}_p)$ has the form predicted by the Langlands–Rapoport conjecture. Moreover, each isogeny class contains a point which lifts to a special point in the corresponding Shimura variety.*

0.3.2 *The He–Rapoport axioms.* In [HR17], He and Rapoport formulated five axioms on Shimura varieties with parahoric level structure, which provide a group-theoretic way to study certain characteristic subsets (such as Newton strata, Ekedahl–Oort strata, Kottwitz–Rapoport strata, and so on) in the mod p reductions of Shimura varieties. Based on this axiomatic approach, Zhou [Zho20] proved that all the expected Newton strata are non-empty (see [KMS22] using a different approach). For more applications of these axioms, we refer the reader to [HR17], [HN17], [GHN19], [Zho20], and [SYZ21]. Combining [Zho20, Theorem 8.1] with Proposition 0.4 we have the following result.

Corollary 0.6. *The He–Rapoport axioms hold if G is unramified.*

These axioms are verified by He and Rapoport [HR17] in the Siegel case, and by He and Zhou [HZ20] for certain PEL Shimura varieties (unramified of types A and C and odd ramified unitary groups). In [Zho20], Zhou proved that all the axioms except the surjectivity in Axiom 4(c) of [HR17] hold in the general case, and, moreover, if G is residually split, then all of them hold. For PEL Shimura varieties, Axiom 4(c) is verified by Shen, Yu, and Zhang [SYZ21].

0.3.3 *The almost product structure.* In [Man05], Mantovan established a formula expressing the l-adic cohomology of proper PEL Shimura varieties in terms of the l-adic cohomology with compact supports of the Igusa varieties and of the Rapoport–Zink spaces for any prime $l \neq p$. This formula encodes nicely the local–global compatibility of the Langlands correspondence. A key part of its proof is to show that the products of reduced fibers of Igusa varieties and Rapoport–Zink spaces form nice 'pro-étale covers up to perfection' for the Newton strata, of PEL Shimura varieties with hyperspecial level structure. This is referred as the almost product structure of Newton strata. In [HK19], Hamacher and Kim extended Mantovan's results to the Kisin–Pappas integral models under some mild assumptions. Combining [HK19, Theorem 2] with Proposition 0.4 we have the following result.

Corollary 0.7. *The almost product structure of Newton strata holds if G is unramified.*

When K is hyperspecial, the almost product structure of Newton strata is established by Mantovan [Man05] for PEL Shimura varieties, and by Hamacher [Ham19] for Shimura varieties of Hodge type. The general case is proved by Hamacher–Kim provided the lifting property [HK19, Axiom A] holds. We refer to [CS17], [Ham19], [Ham17], and [Kim19] for the Caraiani–Scholze-type product structure of Newton strata.

0.4 Strategy

We describe the strategy of the proof. Note that the σ-centralizer $\mathbb{J}_b$ acts on $X(\lambda, b)_K$ by left multiplication. First we show that $\mathbb{J}_b$ acts transitively on $\pi_0(X(\lambda, b)_K)$. Then we show that the stabilizer of each connected component is the normal subgroup $\mathbb{J}_b \cap \ker(\eta_G)$ of $\mathbb{J}_b$. Combining these two results we deduce that $\pi_0(X(\lambda, b)_K) \cong \mathbb{J}_b/(\mathbb{J}_b \cap \ker(\eta_G)) \cong \pi_1(G)^\sigma$ as desired.

The stabilizers can be determined by adapting the computations in [Nie18]. The crucial part is to show the transitivity of the $\mathbb{J}_b$-action. Our starting point is the following natural surjection (see Theorem 2.2)

$$\bigsqcup_{\tilde{w} \in \mathcal{S}_{\lambda,b}} \mathbb{J}_{b,\tilde{w}} \twoheadrightarrow \pi_0(X(\lambda, b)_K),$$

where $\mathcal{S}_{\lambda,b}$ is the set of semi-standard elements (see § 1.7) contained in $\mathrm{Adm}(\lambda) \cap [b]$, and $\mathbb{J}_{b,\tilde{w}} = \{g \in G(\breve{F}); g^{-1} b \sigma(g) = \tilde{w}\}$ on which $\mathbb{J}_b$ acts transitively by left multiplication. Thus, it remains to connect all the subsets $\mathbb{J}_{b,\tilde{w}} K/K$ in $X(\lambda, b)_K$. To this end, we consider the following decomposition

$$\mathcal{S}_{\lambda,b} = \bigsqcup_{x \in \mathcal{S}_{\lambda,b}^+} \mathcal{S}_{\lambda,b,x},$$

where $\mathcal{S}_{\lambda,b}^+$ consists of standard elements in $\mathcal{S}_{\lambda,b}$, and $\mathcal{S}_{\lambda,b,x}$ consists of elements in $\mathcal{S}_{\lambda,b}$ that are σ-conjugate to $x \in \mathcal{S}_{\lambda,b}^+$ under the Weyl group of G. Note that $\mathcal{S}_{\lambda,b}^+$ can be naturally identified with a subset of cocharacters dominated by λ, whose structure has been studied extensively in [Nie18]. Thus, we can use the connecting algorithm in [Nie18] as a guideline to connect the pieces $\mathbb{J}_{b,x}K/K$ for $x \in \mathcal{S}_{\lambda,b}^+$ with each other. To finish the proof, it remains to connect (for each $x \in \mathcal{S}_{\lambda,b}^+$) the pieces $\mathbb{J}_{b,\tilde{w}}K/K$ for $\tilde{w} \in \mathcal{S}_{\lambda,b,x}$ with each other. This is an essential difficulty because the structure of $\mathcal{S}_{\lambda,b,x}$ is much more mysterious. To overcome it, we show that each set $\mathcal{S}_{\lambda,b,x}$ contains a unique (distinguished) element $x_{\rm dist}$ which is of minimal length in its Weyl group coset, and then connect $\mathbb{J}_{b,\tilde{w}}K/K$ with $\mathbb{J}_{b,x_{\rm dist}}K/K$ for all $\tilde{w} \in \mathcal{S}_{\lambda,b,x}$. This new connecting algorithm, motivated from the partial conjugation method by He in [He07] and [He10], is the major innovation of the paper.

0.5 Organization

The paper is organized as follows. In § 1 we recall some basic notions and introduce the semi-standard elements. In § 2 we outline the proof of the main result. In § 3 we introduce the set $\mathcal{P}_{\tilde{w}}$ which will play an essential role in our new connecting algorithm. In § 4, we introduce the new connecting algorithm and use it to connect $\mathbb{J}_{b,\tilde{w}}K/K$ for $\tilde{w} \in \mathcal{S}_{\lambda,b,x}$ with each other. In § 5 we connect $\mathbb{J}_{b,x}K/K$ for $x \in \mathcal{S}_{\lambda,b}^+$ with each other. In §§ 6–8 we compute the stabilizer in $\mathbb{J}_b$ of each connected component of $X(\lambda, b)_K$.

1. Preliminaries

In the body of the paper we assume that G is unramified, simple, and adjoint. Without loss of generality, we assume further that $F = \mathbb{F}_q((t))$. Then $\breve{F} = \mathbf{k}((t))$ with valuation ring $\mathcal{O}_{\breve{F}} = \mathbf{k}[[t]]$ and residue field $\mathbf{k} = \overline{\mathbb{F}}_q$.

1.1 Root datum

Let $T \subseteq B$ be a maximal torus and a Borel subgroup defined over $\mathcal{O}_F$. Let $\mathcal{R} = (Y, \Phi^\vee, X, \Phi, \mathbb{S}_0)$ be the root datum associated to the triple (T, B, G), where X and Y are the character group and cocharacter group of T respectively equipped with a perfect pairing $\langle , \rangle : Y \times X \to \mathbb{Z}$; $\Phi = \Phi_G \subseteq X$ (respectively, $\Phi^\vee \subseteq Y$) is the set of roots (respectively, coroots); $\mathbb{S}_0$ is the set of simple roots appearing in B. Let $\Phi^+ = \Phi \cap \mathbb{Z}_{\geqslant 0}\mathbb{S}_0$ be the set of positive roots. Then we have $\Phi = \Phi^+ \sqcup \Phi^-$ with $\Phi^- = -\Phi^+$. For $\alpha \in \Phi$, we denote by s_α the reflection which sends $\mu \in Y$ to $\mu - \langle \mu, \alpha \rangle \alpha^\vee$, where $\alpha^\vee \in \Phi^\vee$ denotes the coroot of α. Via the bijection $\alpha \leftrightarrow s_\alpha$, we also denote by $\mathbb{S}_0$ the set of simple reflections.

Let $V = Y \otimes_{\mathbb{Z}} \mathbb{R}$. We say $v \in V$ is dominant if $\langle v, \alpha \rangle \geqslant 0$ for each $\alpha \in \Phi^+$. Let Y^+ and V^+ be the set of dominant vectors in Y and V, respectively. For $v, v' \in V$ we write $v' \leqslant v$ if $v - v' \in \mathbb{R}_{\geqslant 0}(\Phi^+)^\vee$. For $\mu, \lambda \in Y$ we write $\lambda \preceq \mu$ if $\mu - \lambda \in \mathbb{Z}\Phi^\vee$ and $\bar{\lambda} \leqslant \bar{\mu}$. Here for $w \in V$ we denote by $\bar{v}$ the unique dominant W_0-conjugate of v.

Let $\tilde{\Phi} = \tilde{\Phi}_G = \Phi \times \mathbb{Z}$ be the set of (real) affine roots. Let $\tilde{\alpha} = \alpha + k \in \tilde{\Phi}$. Then $\tilde{\alpha}$ is an affine function on V such that $\tilde{\alpha}(v) = -\langle \alpha, v \rangle + k$. Let

$$\mathbf{a} = \{v \in Y_{\mathbb{R}}; 0 < \langle \alpha, v \rangle < 1, \alpha \in \Phi^+\}$$

be the base alcove. Set $\tilde{\Phi}^+ = \tilde{\Phi}_G^+ = \{\tilde{\alpha} \in \tilde{\Phi}; \tilde{\alpha}(\mathbf{a}) > 0\}$ and $\tilde{\Phi}^- = -\tilde{\Phi}^+$. Then $\tilde{\Phi} = \tilde{\Phi}^+ \sqcup \tilde{\Phi}^-$. Note that $\Phi^\pm \subseteq \tilde{\Phi}^\mp$.

1.2 Iwahori–Weyl group

Let $W_0 = W_G = N_T(\breve{F})/T(\breve{F})$ be the Weyl group of G, where N_T is the normalizer of T in G. The Iwahori–Weyl group of G is given by

$$\tilde{W} = \tilde{W}_G = N_T(\breve{F})/T(\mathcal{O}_{\breve{F}}) = Y \rtimes W_0 = \{t^\mu w; \mu \in Y, w \in W_0\}.$$

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

We can view $\tilde{W}$ as a subgroup of affine transformations of V such that the action of $\tilde{w} = t^\mu w$ is given by $v \mapsto \mu + w(v)$ for $v \in V$. The induced action of $\tilde{W}$ on $\tilde{\Phi}$ is given by $\tilde{w}(\tilde{\alpha})(v) = \tilde{\alpha}(\tilde{w}^{-1}(v))$. More precisely, if $\tilde{w} = t^\mu w$ and $\tilde{\alpha} = \alpha + k$, then $\tilde{w}(\tilde{\alpha}) = w(\alpha) + \langle w(\alpha), \mu\rangle + k$.

Let $\tilde{\alpha} = \alpha + k \in \tilde{\Phi}$ and let $s_{\tilde{\alpha}} = t^{k\alpha^\vee} s_\alpha \in \tilde{W}$ be the corresponding affine reflection. Then $\{s_{\tilde{\alpha}}; \tilde{\alpha} \in \tilde{\Phi}\}$ generates the affine Weyl group

$$W^a = W^a_G = \mathbb{Z}\Phi^\vee \rtimes W_0 = \{t^\mu w; \mu \in \mathbb{Z}\Phi^\vee, w \in W_0\}.$$

Moreover, we have $\tilde{W} = W^a \rtimes \Omega$, where $\Omega = \Omega_G = \{\omega \in \tilde{W}; \omega(\mathbf{a}) = \mathbf{a}\}$. Let $\ell : \tilde{W} \to \mathbb{N}$ be the length function given by $\ell(\tilde{w}) = |\tilde{\Phi}^- \cap \tilde{w}(\tilde{\Phi}^+)|$. Let $\mathbb{S}^a = \{s_{\tilde{\alpha}}; \tilde{\alpha} \in \tilde{\Phi}, \ell(s_{\tilde{\alpha}}) = 1\}$ be the set of simple affine reflections. Then $(W^a, \mathbb{S}^a)$ is a Coxeter system, and denote by $\leq = \leq_G$ the associated Bruhat order on $\tilde{W} = W^a \rtimes \Omega$. We frequently use the following fact on Bruhat order.

Lemma 1.1. *Let $\tilde{w}$ and $\tilde{\alpha} \in \tilde{\Phi}^+$. Then $\tilde{w} s_{\tilde{\alpha}} \leq \tilde{w}$ if and only if $\tilde{w}(\tilde{\alpha}) \in \tilde{\Phi}^-$.*

By abuse of notation, we freely identify an element of $\tilde{W}$ with one of its lifts in $N_T(\breve{F})$, according to the context.

1.3 σ-conjugacy classes

Recall that σ is the Frobenius automorphism of $G(\breve{F})$. We also denote by σ the induced automorphism of the root datum $\mathcal{R}$. Then σ acts on V as a linear transformation of finite order fixing $\mathbf{a}$. For $\tilde{w} \in \tilde{W}$ there exists a nonzero integer m such that $(\tilde{w}\sigma)^m = t^\xi$ for some $\xi \in Y$. Define $\nu_{\tilde{w}} = \xi/m \in V$, which does not depend on the choice of m.

Let $b \in G(L)$. We denote by $[b] = [b]_G = \{g^{-1} b \sigma(g); g \in G(L)\}$ the σ-conjugate class of b. By [Kot85], the σ-conjugacy class $[b]$ is determined by two invariants: the Kottwitz point $\kappa_G(b) \in \pi_1(G)_\sigma$ and the Newton point $\nu_G(b) \in (V^+)^\sigma$. Here $\kappa_G : G(\breve{F}) \to \pi_1(G)_\sigma = \pi_1(G)/(\sigma - 1)\pi_1(G)$ is the natural projection. To define $\nu_G(b)$, we note that there exists $\tilde{w} \in \tilde{W}$ such that $\tilde{w} \in [b]$. Then $\nu_G(b) = \bar{\nu}_{\tilde{w}}$, which does not depend on the choice of $\tilde{w}$.

1.4 Affine Deligne–Lusztig varieties

For $\gamma \in \Phi$ let $u_\gamma : \mathbb{G}_a \to G$ be the corresponding root subgroup. Let

$$I = T(\mathcal{O}_{\breve{F}}) \prod_{\alpha \in \Phi^+} u_\alpha(t\mathcal{O}_{\breve{F}}) \prod_{\beta \in \Phi^-} u_\beta(\mathcal{O}_{\breve{F}}) \subseteq G(\breve{F})$$

be the Iwahori subgroup associated to the base alcove $\mathbf{a}$.

For $\tilde{w} \in \tilde{W}$ and $b \in G(\breve{F})$ the associated affine Deligne–Lusztig variety is given by

$$X_{\tilde{w}}(b) = \{g \in G(\breve{F})/I; g^{-1} b \sigma(g) \in I \tilde{w} I\}.$$

We are interested in the following union of affine Deligne–Lusztig varieties

$$X(\lambda, b) = X^G(\lambda, b)_I = \cup_{x \in \mathrm{Adm}(\lambda)} X_x(b),$$

where $\mathrm{Adm}(\lambda)$ is the λ-admissible set defined by

$$\mathrm{Adm}(\lambda) = \{x \in \tilde{W}; x \leq t^{w(\lambda)} \text{ for some } w \in W_0\}.$$

By [He16], $X(\lambda, b) \neq \emptyset$ if and only if $\kappa_G(t^\lambda) = \kappa_G(b)$ and $\nu_G(b) \leqslant \lambda^\diamond$, where $\lambda^\diamond$ is the σ-average of λ. We say the pair (λ, b) is Hodge–Newton irreducible if $\kappa_G(t^\lambda) = \kappa_G(b)$ and $\lambda^\diamond - \nu_G(b) \in \sum_{\alpha \in \mathbb{S}_0} \mathbb{R}_{>0} \alpha^\vee$.

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

1.5 Levi subgroups

Let M be a semi-standard Levi subgroup of G, that is, a Levi subgroup containing T. Then $B \cap M$ is a Borel subgroup of M. By replacing the triple (T, B, G) with $(T, B \cap M, M)$, we can define, as in previous subsections, Φ_M^+, $\tilde{W}_M$, $\mathbb{S}_M^a$, Ω_M, $\tilde{\Phi}_M^+$, I_M, κ_M, $\leq_M$, and so on.

For $v \in V$ we set $\Phi_v = \{\alpha \in \Phi; \alpha(v) = 0\}$ and let $M_v \subseteq G$ be the Levi subgroup generated by T and the root subgroups U_α for $\alpha \in \Phi_v$. We set $\tilde{W}_v = \tilde{W}_{M_v}$, $\tilde{\Phi}_v = \tilde{\Phi}_{M_v}$, and so on. If $v \in V^+$, let $J_v = \{s \in \mathbb{S}_0; s(v) = v\}$.

Let $J \subseteq \mathbb{S}_0$. Then there exists some $v' \in V^+$ such that $J_{v'} = J$. We put $M_J = M_{v'}$, $\Phi_J = \Phi_{M_J}$, $\tilde{W}_J = \tilde{W}_{M_J}$, $W_J^a = W_{M_J}^a$, $\Omega_J = \Omega_{M_J}$, $\leq_J = \leq_{M_J}$, and so on. We say $\mu \in Y$ is J-dominant (respectively, J-minuscule) if $\langle \alpha, \mu \rangle \geqslant 0$ (respectively, $\langle \alpha, \mu \rangle \in \{0, \pm 1\}$) for $\alpha \in \Phi_J^+$.

1.6 The left cyclic shift $\rightharpoonup$

Let $K \subseteq \mathbb{S}^a$. Denote by $W_K \in W^a$ the parabolic subgroup generated by K. Set ${}^K\tilde{W} = \{\tilde{w} \in \tilde{W}; \tilde{w} < s\tilde{w} \text{ for } s \in K\}$ and $\tilde{W}^K = ({}^K\tilde{W})^{-1}$.

Lemma 1.2. *Let $K \subseteq \mathbb{S}^a$ and $\tilde{w} \in {}^K\tilde{W}$. Then:*

(1) *if $\tilde{w} < \tilde{w}s$ with $s \in \mathbb{S}^a$, then $\tilde{w}s \in {}^K\tilde{W}$ or $\tilde{w}s = s'\tilde{w}$ for some $s' \in K$;*
(2) *if $\tilde{w}' \in {}^K\tilde{W}$ lies in the W_K-σ-conjugacy class of $\tilde{w}$, then $\tilde{w}' = \tilde{w}$.*

Proof. If $\tilde{w}s \notin {}^K\tilde{W}$, then there exits $s' \in K$ such that $s'\tilde{w}s < \tilde{w}s$, that is, $s\tilde{w}^{-1}(\tilde{\alpha}') \in \tilde{\Phi}^-$, where $\tilde{\alpha}' \in \tilde{\Phi}^+$ is the simple affine root of s'. Note that $\tilde{w}^{-1}(\tilde{\alpha}') \in \tilde{\Phi}^+$ (since $\tilde{w} \in {}^K\tilde{W}$) and that s is a simple reflection, it follows that $\tilde{w}^{-1}(\tilde{\alpha}')$ is the affine simple root of s. Thus $s'\tilde{w} = \tilde{w}s$ and statement (1) is proved. The statement (2) is proved in [He07, Corollary 2.6]. □

Let $\tilde{w}, \tilde{w}' \in \tilde{W}$ and $s \in \mathbb{S}^a$. Write $\tilde{w} \rightharpoonup_s \tilde{w}'$ (respectively, $\tilde{w} \to_s \tilde{w}'$) if $\tilde{w}' = s\tilde{w}\sigma(s)$ and $s\tilde{w} < \tilde{w}$ (respectively, $\ell(\tilde{w}') \leqslant \ell(\tilde{w})$). Note that $\tilde{w} \rightharpoonup_s \tilde{w}'$ implies that $\tilde{w} \to_s \tilde{w}'$. For $K \subseteq \mathbb{S}^a$ we write $\tilde{w} \rightharpoonup_K \tilde{w}'$ if there is a sequence $\tilde{w} = \tilde{w}_0 \rightharpoonup_{s_0} \tilde{w}_1 \rightharpoonup_{s_1} \cdots \rightharpoonup_{s_n} \tilde{w}_{n+1} = \tilde{w}'$ with $s_i \in K$ for $0 \leqslant i \leqslant n$. We can define $\tilde{w} \to_K \tilde{w}'$ in a similar way.

For $x \in {}^K\tilde{W}$ we define $I(K, x) = \max\{K' \subseteq K; x\sigma(K')x^{-1} = K'\}$. Note that $I(K, x)$ is unique by definition. It also can be an empty set.

Lemma 1.3. *For $K \subseteq \mathbb{S}^a$ and $x \in {}^K\tilde{W}$ we have $I(K, x) \subseteq W_{\nu_x}$.*

Proof. As $I(K, x)$ is a finite set, there exists $n \in \mathbb{Z}_{\geqslant 1}$ such that $(x\sigma)^n = t^{n\nu_x}$ fixes each element of $I(K, x)$, that is, $p(s)(\nu_x) = \nu_x$ for $s \in I(K, x)$. Here $p : \tilde{W} \rtimes \langle \sigma \rangle \to W_0 \rtimes \langle \sigma \rangle$ is the natural projection. Thus, $I(K, x) \subseteq W_{\nu_x}$ as desired. □

Theorem 1.4 [He07, § 3]. *Let $K \subseteq \mathbb{S}^a$ and $\tilde{w} \in \tilde{W}$. If W_K is finite, then there exist $x \in {}^K\tilde{W}$ and $u \in W_{I(x,K)}$ such that $\tilde{w} \to_K ux$. Moreover, the element $x \in {}^K\tilde{W}$ is uniquely determined by the W_K-σ-conjugacy class of $\tilde{w}$, which may be empty.*

1.7 Semi-standard elements

We say $\tilde{w} \in \tilde{W}$ is semi-standard if $\tilde{w}\sigma(\tilde{\Phi}_{\nu_{\tilde{w}}}^+) = \tilde{\Phi}_{\nu_{\tilde{w}}}^+$, or equivalently, ${}^{\tilde{w}\sigma}I_{M_{\nu_{\tilde{w}}}} := \tilde{w}\sigma(I_{M_{\nu_{\tilde{w}}}})\tilde{w}^{-1} = I_{M_{\nu_{\tilde{w}}}}$. We say $\tilde{w}$ is standard if it is semi-standard and $\nu_{\tilde{w}}$ is dominant. Let $\mathcal{S}$ and $\mathcal{S}^+$ denote the set of semi-standard elements and standard elements respectively.

Lemma 1.5. *Let $\tilde{w} \in \mathcal{S}$. Then:*

(1) *$z\tilde{w}\sigma(z)^{-1} \in \mathcal{S}$ if $z \in \tilde{W}$ such that $z(\tilde{\Phi}_{\nu_{\tilde{w}}}^+) \subseteq \tilde{\Phi}^+$;*
(2) *there is a unique $\tilde{w}' \in \mathcal{S}^+$ in the W_0-σ-conjugacy class of $\tilde{w}$, and moreover, there is a unique element $z' \in W_0^{J_{\bar{\nu}_{\tilde{w}}}}$ such that $\tilde{w} = z'\tilde{w}'\sigma(z')^{-1}$;*

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

(3) $\tilde{w}' \in \mathcal{S}$ *if* $\tilde{w} \to_{\mathbb{S}^a} \tilde{w}'$*;*
(4) $\mathbb{J}_{\tilde{w}}$ *is generated by* $I \cap \mathbb{J}_{\tilde{w}}$ *and* $\tilde{W} \cap \mathbb{J}_{\tilde{w}}$*;*
(5) $\tilde{w} \in \Omega_{J_{\nu_{\tilde{w}}}}$ *if* $\tilde{w} \in \mathcal{S}^+$*;*
(6) $\tilde{w} \leq u\tilde{w}$ *for any* $u \in W_{\nu_{\tilde{w}}}$*.*

Proof. Note that $\nu_{z\tilde{w}\sigma(z)^{-1}} = p(z)(\nu_{\tilde{w}})$ and, hence, $z(\tilde{\Phi}_{\nu_{\tilde{w}}}) = \tilde{\Phi}_{\nu_{z\tilde{w}\sigma(z)^{-1}}}$, where $p : \tilde{W} \rtimes \langle\sigma\rangle \to W_0 \rtimes \langle\sigma\rangle$ is the natural projection. Thus, $z(\tilde{\Phi}^{\pm}_{\nu_{\tilde{w}}}) = \tilde{\Phi}^{\pm}_{\nu_{z\tilde{w}\sigma(z)^{-1}}}$, and part (1) follows by definition.

Let $z' \in W_0^{J_{\bar{\nu}_{\tilde{w}}}}$ such that $z'(\bar{\nu}_{\tilde{w}}) = \nu_{\tilde{w}}$. Let $\tilde{w}' = z'^{-1}\tilde{w}\sigma(z')$. Note that $z'(\tilde{\Phi}^+_{\bar{\nu}_{\tilde{w}}}) = \tilde{\Phi}^+_{\nu_{\tilde{w}}}$. So $\tilde{w}' \in \mathcal{S}^+$ by (1). Assume there is another $\tilde{w}'' \in \mathcal{S}^+$ such that $\tilde{w} = w\tilde{w}''\sigma(w)^{-1}$ for some $w \in W_0$. Write $w = z''u$ with $z'' \in W_0^{J_{\bar{\nu}_{\tilde{w}}}}$ and $u \in W_{J_{\bar{\nu}_{\tilde{w}}}}$. Then $\nu_{\tilde{w}'} = \nu_{\tilde{w}''} = \bar{\nu}_{\tilde{w}}$ and $z'^{-1}z'' \in W_{J_{\bar{\nu}_{\tilde{w}}}}$. Thus, $z' = z'' \in W_0^{J_{\bar{\nu}_{\tilde{w}}}}$, and $\tilde{w}'', \tilde{w}' \in \Omega_{J_{\bar{\nu}_{\tilde{w}}}}$ are σ-conjugate by $W_{J_{\bar{\nu}_{\tilde{w}}}}$. Thus, $\tilde{w}' = \tilde{w}''$ as desired.

To prove part (3) we can assume $\tilde{w} \to_s \tilde{w}'$ for some $s \in \mathbb{S}^a$ and $\tilde{w} \neq \tilde{w}'$. Thus, either $s\tilde{w} < \tilde{w}$ or $\tilde{w}\sigma(s) < \tilde{w}$. In view of part (1) it suffices to show $s(\tilde{\Phi}^+_{\nu_{\tilde{w}}}) \subseteq \tilde{\Phi}^+$. Otherwise, the simple affine root of s lies in $\tilde{\Phi}^+_{\nu_{\tilde{w}}}$. Hence, $s\tilde{w}, \tilde{w}\sigma(s) > \tilde{w}$ (since $\tilde{w}\sigma(\tilde{\Phi}^+_{\nu_{\tilde{w}}}) = \tilde{\Phi}^+_{\nu_{\tilde{w}}}$), contradicting our assumption.

Note that $\mathbb{J}_{\tilde{w}} \subseteq M_{\nu_{\tilde{w}}}$. Then part (4) follows from that ${}^{\tilde{w}\sigma}I_{M_{\nu_{\tilde{w}}}} = I_{M_{\nu_{\tilde{w}}}}$, ${}^{\tilde{w}\sigma}\tilde{W}_{M_{\nu_{\tilde{w}}}} = \tilde{W}_{M_{\nu_{\tilde{w}}}}$, and the Bruhat decomposition $M_{\nu_{\tilde{w}}}(\breve{F}) = I_{M_{\nu_{\tilde{w}}}}\tilde{W}_{M_{\nu_{\tilde{w}}}}I_{M_{\nu_{\tilde{w}}}}$.

Assume $\tilde{w} \in \mathcal{S}^+$, that is, $\nu_{\tilde{w}} = \sigma(\nu_{\tilde{w}})$ is dominant. As $p(\tilde{w}\sigma)(\nu_{\tilde{w}}) = \nu_{\tilde{w}}$, we have $p(\tilde{w})(\nu_{\tilde{w}}) = \nu_{\tilde{w}}$ and, hence, $\tilde{w} \in \tilde{W}_{\nu_{\tilde{w}}} = \tilde{W}_{J_{\nu_{\tilde{w}}}}$. Moreover, $\tilde{w}(\tilde{\Phi}^+_{J_{\nu_{\tilde{w}}}}) = \tilde{w}\sigma(\tilde{\Phi}^+_{J_{\nu_{\tilde{w}}}}) = \tilde{\Phi}^+_{J_{\nu_{\tilde{w}}}}$ by definition. This means that $\tilde{w} \in \Omega_{J_{\nu_{\tilde{w}}}}$ and part (5) is proved.

By part (2) there exits $z \in {}^{J_{\bar{\nu}_{\tilde{w}}}}W_0$ and $\tilde{w}' \in \mathcal{S}^+$ such that $\tilde{w}' = z^{-1}\tilde{w}\sigma(z)$. Let $u \in W_{\nu_{\tilde{w}}}$. Then $z^{-1}uz \in W^a_{J_{\bar{\nu}_{\tilde{w}}}}$ since $z(\nu_{\tilde{w}}) = \nu_{\bar{\nu}_{\tilde{w}}}$. By part (5) we have $\tilde{w}'' \in \Omega_{J_{\nu_{\tilde{w}}}}$ and, hence, $\tilde{w}'' \leq (z^{-1}uz)$. It follows from [CN19, Lemma 1.3] that

$$\tilde{w} = z\tilde{w}'\sigma(z)^{-1} \leq z(z^{-1}uz)\tilde{w}'\sigma(z)^{-1} = u\tilde{w}$$

and part (b) follows. □

2. Outline of the proof

We fix $\lambda \in Y^+$ and $b \in G(\breve{F})$ such that $X(\lambda, b) \neq \emptyset$. Let $J = J_{\nu_G(b)} \subseteq \mathbb{S}_0$. We may and do assume that $b \in M_J(\breve{F})$ and $\nu_{M_J}(b) = \nu_G(b)$. As b is basic in $M_J(\breve{F})$ we assume further that $b \in \Omega_J$. Set $\mathcal{S}_{\lambda,b} = \mathrm{Adm}(\lambda) \cap \mathcal{S} \cap [b]$.

For $x \in \Omega_J \cong \pi_1(M_J) \cong Y/\mathbb{Z}\Phi_J^\vee$ we set $\mu_x \in Y$ such that $x = t^{\mu_x}p(x)$, where $p : \tilde{W} \rtimes \langle\sigma\rangle \to W_0 \rtimes \langle\sigma\rangle$ is the natural projection. Define

$$\mathcal{S}^+_{\lambda,b} = \{x \in \Omega_J; \kappa_{M_J}(x) = \kappa_{M_J}(b), \mu_x \preceq \lambda\},$$

$$\mathcal{S}_{\lambda,b,x} = \{zx\sigma(z)^{-1} \in \mathrm{Adm}(\lambda); x \in \mathcal{S}^+_{\lambda,b}, z \in W_0^J\}.$$

Lemma 2.1. *We have* $\mathcal{S}^+_{\lambda,b} = \mathcal{S}^+ \cap \mathcal{S}_{\lambda,b} \subseteq \Omega_J$ *and* $\mathcal{S}_{\lambda,b} = \bigsqcup_{x \in \mathcal{S}^+_{\lambda,b}} \mathcal{S}_{\lambda,b,x}$*. In particular,* $x \in S_{\lambda,b,x}$ *for* $x \in S^+_{\lambda,b}$*.*

Proof. Let $x \in \mathcal{S}^+_{\lambda,b} \subseteq \Omega_J$. Then $\kappa_{M_J}(x) = \kappa_{M_J}(b) \in \pi_1(M_J)/(1-\sigma)\pi_1(M_J)$ by definition. In view of the natural identification $\Omega_J \cong \pi_1(M_J)$, the previous equality means that $x \equiv b \mod (1-\sigma)\Omega_J$, or equivalently, x and b are Ω_J-σ-conjugate. In particular, $\nu_x = \nu_{M_J}(b) = \nu_G(b)$ is dominant and $x \in \mathcal{S}^+$. Moreover, as $x \in \Omega_J$, $x \leq_J t^{\mu_x} \in \mathrm{Adm}(\lambda)$ (since $\mu_x \preceq \lambda$). Thus, $x \in \mathcal{S}^+$ and $\mathcal{S}^+_{\lambda,b} \subseteq \mathcal{S}^+ \cap \mathcal{S}_{\lambda,b}$. Let $x' \in \mathcal{S}^+ \cap \mathcal{S}_{\lambda,b}$. Then $\nu_{x'}$ is dominant and, hence, $\nu_{x'} = \nu_G(b)$ (since $x' \in [b]$). In particular, we have $x' \in \Omega_J$ (by Lemma 1.5(5)) and $\kappa_{M_J}(x') = \kappa_{M_J}(b)$

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

(by [CKV15, Remark 2.5.8]). Since $x' \in \mathrm{Adm}(\lambda)$, it follows that $\mu_{x'} \preceq \lambda$. Therefore, $x' \in \mathcal{S}_{\lambda,b}^+$ and, hence, $\mathcal{S}_{\lambda,b}^+ = \mathcal{S}^+ \cap \mathcal{S}_{\lambda,b}$.

Let $\tilde{w} \in \mathcal{S}_{\lambda,b}$. Then $\bar{\nu}_{\tilde{w}} = \nu_G(b)$. By Lemma 1.5 there exist $x \in \mathcal{S}^+$ and $z \in W_0^J$ such that $\tilde{w} = zx\sigma(z)^{-1}$. As $\tilde{w} \in [b]$, we have $\nu_x = \bar{\nu}_{\tilde{w}} = \nu_G(b)$ and $\kappa_G(x) = \kappa_G(\tilde{w}) = \kappa_G(b)$. By the proof of [GHN15, Proposition 3.5.1] we have $\kappa_{M_J}(x) = \kappa_{M_J}(b)$. Moreover, as $\tilde{w} \in \mathrm{Adm}(\lambda)$ and $W_0\tilde{w}W_0 = W_0xW_0 = W_0t^{\mu_x}W_0$, we have $\mu_x \preceq \lambda$. Thus, $x \in \mathcal{S}_{\lambda,b}^+$ and the second statement follows. □

For $b' \in G(\breve{F})$ we set $\mathbb{J}_{b,b'} = \mathbb{J}_{b,b'}^G = \{g \in G(\breve{F}); g^{-1}b\sigma(g) = b'\}$ and put $\mathbb{J}_b = \mathbb{J}_{b,b'}$ if $b = b'$. Then $\mathbb{J}_b$ acts on $\mathbb{J}_{b,b'}$ and $X(\lambda, b)$ by left multiplication.

THEOREM 2.2 [HZ20]. *Each connected component of $X(\lambda, b)$ intersects $\mathbb{J}_{b,\tilde{w}}I/I$ for some $\tilde{w} \in \mathcal{S}_{\lambda,b}$.*

Proof. By [HZ20, Theorem 4.1], each connected component of $X(\lambda, b)$ intersects $X_{\tilde{w}}(b)$ for some σ-straight element $\tilde{w} \in \mathrm{Adm}(\lambda)$ which is σ-conjugate to b. Then the statement follows from [He14, Proposition 4.5] and the proof of [Nie15, Theorem 1.3] that $X_{\tilde{w}}(b) = \mathbb{J}_{b,\tilde{w}}I/I$ and $\tilde{w} \in \mathcal{S}_{\lambda,b}$, respectively. □

For $g, g' \in G(\breve{F})$ we write $gI \sim_{\lambda,b} g'I$ if they are in the same connected component of $X(\lambda, b)$. For $\tilde{w}, \tilde{w}' \in \mathcal{S}_{\lambda,b}$, we write $\mathbb{J}_{b,\tilde{w}} \sim_{\lambda,b} \mathbb{J}_{b,\tilde{w}'}$ if their natural images in $\pi_0(X(\lambda, b))$ coincide.

In the following four propositions, we retain the assumptions in Theorem 0.2. The proofs are given in the remaining sections.

PROPOSITION 2.3. *For $x \in \mathcal{S}_{\lambda,b}^+$ and $\tilde{w}, \tilde{w}' \in \mathcal{S}_{\lambda,b,x}$ we have $\mathbb{J}_{b,\tilde{w}} \sim_{\lambda,b} \mathbb{J}_{b,\tilde{w}'}$.*

PROPOSITION 2.4. *For $x, x' \in \mathcal{S}_{\lambda,b}^+$ we have $\mathbb{J}_{b,x} \sim_{\lambda,b} \mathbb{J}_{b,x'}$.*

PROPOSITION 2.5. *The natural action of $\ker(\eta_{M_J}) \cap \mathbb{J}_b$ on $\pi_0(X(\lambda, b))$ is trivial.*

PROPOSITION 2.6. *The natural action of*

$$(\ker(\eta_G) \cap \mathbb{J}_b)/(\ker(\eta_{M_J}) \cap \mathbb{J}_b) \cong (\mathbb{Z}\Phi^\vee/\mathbb{Z}\Phi_J^\vee)^\sigma$$

on $\pi_0(X(\lambda, b))$ is trivial.

Proof of Theorem 0.2. By [He16, Theorem 1.1], it suffices to consider the Iwahori case $K = I$. By Proposition 2.2, the natural projection

$$\cup_{\tilde{w} \in \mathcal{S}_{\lambda,b}} \mathbb{J}_{b,\tilde{w}} \to \pi_0(X(\lambda, b))$$

is surjective. Note that $\mathcal{S}_{\lambda,b} = \cup_{x \in \mathcal{S}_{\lambda,x}^+} \mathcal{S}_{\lambda,b,x}$. It follows from Propositions 2.3 and 2.4 that the natural projection

$$\mathbb{J}_{b,\tilde{w}} \to \pi_0(X(\lambda, b))$$

is surjective for any $\tilde{w} \in \mathcal{S}_{\lambda,b}$. Since $\mathbb{J}_b$ acts on $\mathbb{J}_{b,\tilde{w}}$ transitively, $\mathbb{J}_b$ also acts on $\pi_0(X(\lambda, b))$ transitively. Thus, by Propositions 2.5 and 2.6,

$$\pi_0(X(\lambda, b)) \cong \mathbb{J}_b/(\ker(\eta_G) \cap \mathbb{J}_b).$$

As $b \in \Omega_J$ and $\mathbb{J}_b = \mathbb{J}_b^{M_J}$, it follows by Lemma 1.5(4) that $\mathbb{J}_b$ is generated by $I_{M_J} \cap \mathbb{J}_b$, $W_{M_J}^a \cap \mathbb{J}_b$ and $\Omega_J \cap \mathbb{J}_b = \Omega_J^\sigma$. Hence, $\mathbb{J}_b = (\ker(\eta_{M_J}) \cap \mathbb{J}_b) \rtimes \Omega_J^\sigma$. Since $\ker(\eta_{M_J}) \subseteq \ker(\eta_G)$, $\ker(\eta_G) \cap \mathbb{J}_b = (\ker(\eta_{M_J}) \cap \mathbb{J}_b) \rtimes (\ker(\eta_G) \cap \Omega_J^\sigma)$. Thus, we have

$$\mathbb{J}_b/(\ker(\eta_G) \cap \mathbb{J}_b) \cong \Omega_J^\sigma/(\ker(\eta_G) \cap \Omega_J^\sigma) \cong \pi_1(G)^\sigma,$$

where the last isomorphism follows from [CKV15, Lemma 2.5.11] that the natural map $Y^\sigma \to \pi_1(M_J)^\sigma \cong \Omega_J^\sigma \to \pi_1(G)^\sigma$ is surjective. The proof is finished. □

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

3. The set $\mathcal{P}_{\tilde{w}}$

In the rest of the paper, we assume that G is adjoint, simple, and its root system Φ has d irreducible factors.

3.1 The set $\mathcal{P}_{\tilde{w}}$

For $\tilde{w} \in \mathrm{Adm}(\lambda)$ we denote by $\mathcal{P}_{\tilde{w}}$ the set of roots $\alpha \in \Phi^+ \setminus \Phi_{\nu_{\tilde{w}}}$ such that $\tilde{w}\sigma(s_\alpha) \in \mathrm{Adm}(\lambda)$ and $(\tilde{w}\sigma)^{-m_{\alpha,\tilde{w}}}(\alpha) \in \tilde{\Phi}^+$. Here

$$\begin{aligned} m_{\alpha,\tilde{w}} &= \min\{i \in \mathbb{Z}_{\geqslant 1}; (\tilde{w}\sigma)^{-i}(\alpha) \in \tilde{\Phi} \setminus \Phi\}, \\ &= \min\{i \in \mathbb{Z}_{\geqslant 1}; \langle \alpha, p(\tilde{w}\sigma)^{i-1}(\mu_{\tilde{w}})\rangle \neq 0\}, \end{aligned}$$

where $\mu_{\tilde{w}} \in Y$ such that $\tilde{w} \in t^{\mu_{\tilde{w}}} W_0$. Note that $m_{\alpha,\tilde{w}}$ is well defined since $\langle \alpha, \nu_{\tilde{w}}\rangle \neq 0$, and $\alpha \in \mathcal{P}_{\tilde{w}}$ if and only if $\langle \alpha, p(\tilde{w}\sigma)^{m_{\alpha,\tilde{w}}-1}(\mu_{\tilde{w}})\rangle \leqslant -1$.

The sets $\mathcal{P}_{\tilde{w}}$ will be used to construct affine lines of $X(\lambda, b)$ in the next section. The main result of this section is as follows.

Proposition 3.1. *Assume (λ, b) is Hodge–Newton irreducible. Then $\mathcal{P}_{\tilde{w}} \neq \emptyset$ for $\tilde{w} \in \mathcal{S}_{\lambda,b} \setminus {}^{\mathbb{S}_0}\tilde{W}$.*

The proposition is proved in § 3.4. The proof is based on induction on left cyclic shifts studied in § 3.3. In a single induction step, we will come up against an extreme (and harder) case, which involves distinct elements introduced in § 3.2.

3.2 Distinct elements

Let R be a σ-orbit of $\mathbb{S}_0$. We say $\tilde{w} \in \mathrm{Adm}(\lambda)$ is left R-distinct (respectively, right R-distinct) if $s\tilde{w} \notin \mathrm{Adm}(\lambda)$ (respectively, $\tilde{w}s \notin \mathrm{Adm}(\lambda)$) for $s \in R$. Let w_R denote the longest root of W_R. As $\sigma(R) = R$ we have $\sigma(w_R) = w_R$.

Lemma 3.2. *Let R be a σ-orbit of $\mathbb{S}_0$. If $\tilde{w} \in \mathrm{Adm}(\lambda)$ is left R-distinct, then $\tilde{w} \in {}^R\tilde{W}$. Moreover: (1) $w_R\tilde{w}w_R \in \mathrm{Adm}(\lambda)$ is right R-distinct; and (2) $\mathcal{P}_{w_R\tilde{w}w_R} \neq \emptyset$ if $\mathcal{P}_{\tilde{w}} \neq \emptyset$.*

Proof. Let $s \in R$. If $s\tilde{w} < \tilde{w}$, then $s\tilde{w} \in \mathrm{Adm}(\lambda)$ since $\tilde{w} \in \mathrm{Adm}(\lambda)$, which is a contradiction. Thus, $s\tilde{w} > \tilde{w}$ and, hence, $\tilde{w} \in {}^R\tilde{W}$.

To show part (1) we can assume $d = 1$. Then one checks that R is either commutative or is of type A_2. Thus, part (1) follows from Lemma A.4.

Now we show part (2). Let $\alpha \in \mathcal{P}_{\tilde{w}}$ and set $\alpha^i = (\tilde{w}\sigma)^i(\alpha) \in \tilde{\Phi}$ for $i \in \mathbb{Z}$. Let

$$n_\alpha = \min\{i \in \mathbb{Z}_{\geqslant 0}; \alpha^{-i} \notin \Phi_R^+\} \leqslant m_{\alpha,\tilde{w}}.$$

It suffices to show that $w_R(\alpha^{-n_\alpha}) \in \mathcal{P}_{w_R\tilde{w}w_R}$. First we check that

(a) $$\alpha^{-n_\alpha} \in \Phi^+ \text{ and } w_R\tilde{w}w_R\sigma(s_{w_R(\alpha^{-n_\alpha})}) = w_R\tilde{w}\sigma(s_{\alpha^{-n_\alpha}})w_R \in \mathrm{Adm}(\lambda).$$

If $n_\alpha = 0$, then $\alpha^{-n_\alpha} = \alpha \in \Phi^+ \setminus \Phi_R$, and part (a) follows from Corollary A.6. Otherwise, $\alpha^{-n_\alpha+1} \in \Phi_R^+ \subseteq \tilde{\Phi}^-$. Noting that $\tilde{w} \in {}^R\tilde{W}$ (since $\tilde{w} \in \mathrm{Adm}(\lambda)$ is left R-distinct), by Lemma 1.1 we have $\alpha^{-n_\alpha} = (\tilde{w}\sigma)^{-1}(\alpha^{-n_\alpha+1}) \in \tilde{\Phi}^-$. Hence, $\alpha^{-n_\alpha} \in \Phi^+$ since $n_\alpha \leqslant m_{\alpha,\tilde{w}}$ and $\alpha^{-m_{\alpha,\tilde{w}}} \in \tilde{\Phi}^+$. Moreover,

$$w_R\tilde{w}w_R\sigma(w_R(\alpha^{-n_\alpha})) = w_R(\alpha^{-n_\alpha+1}) \in \Phi_R^- \subseteq \tilde{\Phi}^+.$$

By Lemma 1.1, $w_R\tilde{w}w_R\sigma(s_{w_R(\alpha^{-n_\alpha})}) \leq w_R\tilde{w}w_R \in \mathrm{Adm}(\lambda)$, and part (a) follows.

Note the following three facts: $(w_R\tilde{w}w_R)^i(w_R(\alpha^{-n_\alpha})) = w_R(\alpha^{-n_\alpha+i})$ for $i \in \mathbb{Z}$; $w_R \in W_0$ preserves $\tilde{\Phi} \setminus \Phi$; and $n_\alpha < n_{\alpha,\tilde{w}}$ (since $\alpha^{-n_\alpha} \in \Phi$). Then it follows, by definition, that

$n_{w_R(\alpha^{-n_\alpha}),w_R\tilde{w}w_R} = n_{\alpha,\tilde{w}} - n_\alpha$ and

$$(w_R\tilde{w}w_R)^{-n_{w_R(\alpha^{-n_\alpha}),w_R\tilde{w}w_R}}(w_R(\alpha^{-n_\alpha})) = w_R(\alpha^{-n_{\alpha,\tilde{w}}}) \in \tilde{\Phi}^+ \setminus \Phi,$$

where the inclusion follows from $\alpha^{-n_{\alpha,\tilde{w}}} \in \tilde{\Phi}^+ \setminus \Phi$. Therefore, we have $w_R(\alpha^{-n_\alpha}) \in \mathcal{P}_{w_R\tilde{w}w_R}$ as desired. □

Lemma 3.3. *Let R be a σ-orbit of $\mathbb{S}_0$. Let $\tilde{w} \in \mathrm{Adm}(\lambda) \cap \mathcal{S}$. If $\tilde{w} \notin {}^R\tilde{W}$ and $\tilde{w}$ is not right R-distinct. Then $\mathcal{P}_{\tilde{w}} \neq \emptyset$.*

Proof. By assumption, there exist $s' \in R$ and $0 \leqslant i \leqslant |R|-1$ such that $\sigma^{-i}(s')\tilde{w} < \tilde{w}$ and $\tilde{w}\sigma(s') \in \mathrm{Adm}(\lambda)$. Thus, we can define

$$k = \min\{0 \leqslant i \leqslant |R|-1; \sigma^{-i}(s')\tilde{w} < \tilde{w}, \tilde{w}\sigma(s') \in \mathrm{Adm}(\lambda) \text{ for some } s' \in R\}.$$

Choose $s \in R$ such that $\sigma^{-k}(s)\tilde{w} < \tilde{w}$ and $\tilde{w}\sigma(s) \in \mathrm{Adm}(\lambda)$. Let $\alpha \in \Phi^+$ be the simple root of s. Set $\gamma^i = (\tilde{w}\sigma)^i(\gamma) \in \tilde{\Phi}$ for $\gamma \in \Phi$ and $i \in \mathbb{Z}$. We claim that

(a) $$\alpha^{-i} = \sigma^{-i}(\alpha) \text{ for } 0 \leqslant i \leqslant k, \text{ and hence } m_{\alpha,\tilde{w}} \geqslant k+1.$$

Let $0 \leqslant i \leqslant k-1$. By the choice of k we have $\tilde{w} < \sigma^{-i}(s)\tilde{w}$ and $\tilde{w}\sigma^{-i}(s) \notin \mathrm{Adm}(\lambda)$, which means that $\tilde{w}\sigma^{-i}(s) = \sigma^{-i}(s)\tilde{w}$ by Lemma A.2, that is, $\sigma^{-i}(\alpha) = \tilde{w}\sigma^{-i}(\alpha)$ (since $\tilde{w} < \sigma^{-i}(s)\tilde{w}$). Thus,

$$\alpha^{-i-1} = (\sigma^{-1}\tilde{w}^{-1})^{i+1}(\alpha) = (\sigma^{-1}\tilde{w}^{-1})^i\sigma^{-1}(\alpha) = \cdots = (\sigma^{-1}\tilde{w}^{-1})\sigma^{-i}(\alpha) = \sigma^{-i-1}(\alpha).$$

Thus, part (a) is proved.

By part (a) we have $\alpha^{-k} \in \Phi^+$. Thus, $\alpha^{-k-1} = (\tilde{w}\sigma)^{-1}(\alpha^{-k}) \in \tilde{\Phi}^+$ since $\sigma^{-k}(s)\tilde{w} < \tilde{w}$. As $\tilde{w} \in \mathcal{S}$, it follows that $\alpha^{-k} \notin \Phi_{\nu_{\tilde{w}}}$ and, hence, $\alpha^i \notin \Phi_{\nu_{\tilde{w}}}$ for $i \in \mathbb{Z}$. If $\alpha \notin \mathcal{P}_{\tilde{w}}$, then $\alpha^{-m_{\alpha,\tilde{w}}} \in \tilde{\Phi}^- \setminus \Phi$ by definition. Thus, $\alpha^{-k-1} \in \tilde{\Phi}^+ \cap \Phi = \Phi^-$ since $k+1 \leqslant m_{\alpha,\tilde{w}}$ by part (a). Let $\beta = -\alpha^{-k-1} \in \Phi^+ \setminus \Phi_{\nu_{\tilde{w}}}$. Then $\beta^{-m_{\beta,\tilde{w}}} = -\alpha^{-m_{\alpha,\tilde{w}}} \in \tilde{\Phi}^+ \setminus \Phi$, and $\tilde{w}\sigma(s_\beta) < \tilde{w} \in \mathrm{Adm}(\lambda)$ since $\tilde{w}\sigma(\beta) = -\alpha^{-k} \in \Phi^-$. Thus, $\beta \in \mathcal{P}_{\tilde{w}}$ as desired. □

3.3 Reduction by cyclic shifts

To show Proposition 3.1, we introduce a reduction method via the left cyclic shift. We adopt the notation from § 1.6.

Proposition 3.4. *Let $K \subseteq \mathbb{S}_0$ and $\tilde{w} \in \mathcal{S}_{\lambda,b}$. Then there exists a unique semi-standard element $\tilde{w}' \in {}^K\tilde{W}$ which is σ-conjugate to $\tilde{w}$ by W_K. If, moreover, $K = \mathbb{S}_0$ and (λ, b) is Hodge–Newton irreducible, then $\tilde{w}'$ is not left R-distinct for any σ-orbit R of $\mathbb{S}_0$.*

Proof. By Theorem 1.4, there exist unique $\tilde{w}' \in {}^K\tilde{W}$ and some $u \in I(K, \tilde{w}')$ such that $\tilde{w} \to_K u\tilde{w}'$. Thus, $\Phi_{I(K,\tilde{w}')} \subseteq \Phi_{\nu_{\tilde{w}'}}$ (by Lemma 1.3) and $\ell(u\tilde{w}') = \ell(u) + \ell(\tilde{w}')$. As $\tilde{w} \in \mathcal{S}$, by Lemma 1.5(3) and (6) we have $u\tilde{w}' \in \mathcal{S}$ and $u\tilde{w}' \leq u^{-1}u\tilde{w}' = \tilde{w}'$. Thus, $u = 1$, and the first statement follows. The second statement is proved in [CN19, Lemma 6.11]. □

Lemma 3.5. *Let $K \subseteq \mathbb{S}_0$ and $\tilde{w} \in \mathcal{S}$. Then is no infinite sequence*

$$\tilde{w} = \tilde{w}_0 \rightharpoonup_{s_1} \tilde{w}_1 \rightharpoonup_{s_2} \cdots,$$

where $s_i \in K$ for $i \in \mathbb{Z}_{\geqslant 0}$. In particular, $\tilde{w} \rightharpoonup_K \tilde{w}'$ for some $\tilde{w}' \in \mathcal{S} \cap {}^K\tilde{W}$.

Proof. We argue by induction on $|K|$. If $K = \emptyset$, the statement is trivial. Assume $|K| \geqslant 1$. Suppose there exists such an infinite sequence. As $\tilde{w} \in \mathcal{S}$, $\tilde{w}_i \in \mathcal{S}$ for $i \in \mathbb{Z}_{\geqslant 0}$ by Lemma 1.5(3). Noting that $\ell(\tilde{w}_0) \geqslant \ell(\tilde{w}_1) \geqslant \cdots$, we can assume that: (a) $\ell(\tilde{w}_0) = \ell(\tilde{w}_1) = \cdots$. Write $\tilde{w}_i = u_i y_i$ with $u_i \in W_K$ and $y_i \in {}^K\tilde{W}$. Then $s_{i+1}u_i < u_i$ and, hence, $y_i\sigma(s_{i+1}) > y_i$. By Lemma 1.2(1), for each $i \in \mathbb{Z}_{\geqslant 0}$ we have $y_i \leq y_{i+1}$, and more precisely, either $y_{i+1} = y_i\sigma(s_{i+1}) > y_i$ or $y_{i+1} = y_i$ and

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

$y_i\sigma(s_{i+1})y_i^{-1} \in K$. Thus, we can assume further that $y_0 = y_1 = \cdots$, that is: (b) there exists $y \in {}^K\tilde{W}$ such that $\tilde{w}_i \in W_K y$ and $y\sigma(s_i)y^{-1} \in K$ for $i \in \mathbb{Z}_{\geqslant 1}$. Let $K' = \{s_i; i \in \mathbb{Z}_{\geqslant 1}\} \subseteq K$. If $K' = K$, then by part (b) we have $K = I(K, y) \subseteq W_{\nu_{\tilde{w}}}$, see § 1.5. Thus, $\tilde{w}_i = y \in {}^K\tilde{W}$ since $\tilde{w}_i \in \mathcal{S}$, which is impossible. Otherwise, $K' \subsetneq K$ and it contradicts the induction hypothesis for the proper subset K'. □

Proposition 3.6 [CN19, Proposition 6.16]. *Let $\tilde{w} \in \mathcal{S}$ and let $\tilde{w}' \in {}^{\mathbb{S}_0}\tilde{W}$ be the unique element in the W_0-σ-conjugacy class of $\tilde{w}$. Then there is a sequence*

$$\tilde{w} = \tilde{w}_0 \rightharpoonup_{R_1} \tilde{w}_1 \rightharpoonup_{R_2} \cdots \rightharpoonup_{R_n} \tilde{w}_n = \tilde{w}',$$

where $R_i \subseteq K$ is a σ-orbit and $\tilde{w}_i \in \mathcal{S} \cap {}^{R_i}\tilde{W}$ for $1 \leqslant i \leqslant n$.

Proof. Assume otherwise. Then by Lemma 3.5 there is an infinite sequence

$$\tilde{w} = \tilde{w}_0 \rightharpoonup_{R_0} \tilde{w}_1 \rightharpoonup_{R_1} \cdots,$$

where $\tilde{w}_{i+1} \in {}^{R_i}\tilde{W}$ and R_i is some σ-orbit of $\mathbb{S}_0$ for $i \in \mathbb{Z}_{\geqslant 0}$. This contradicts Lemma 3.5. Thus, the statement follows. □

3.4 Proof of Proposition 3.1

By Proposition 3.6, there exists a sequence

$$\tilde{w} = \tilde{w}_0 \rightharpoonup_{R_1} \tilde{w}_1 \rightharpoonup_{R_2} \cdots \rightharpoonup_{R_n} \tilde{w}_n = \tilde{w}',$$

where $\tilde{w}_0, \ldots, \tilde{w}_n \in \mathcal{S}$ are distinct elements, $R_0, \ldots, R_n$ are σ-orbits of $\mathbb{S}_0$, $\tilde{w}' \in {}^{\mathbb{S}_0}\tilde{W}$, and $\tilde{w}_i \in {}^{R_i}\tilde{W}$ for $1 \leqslant i \leqslant n$.

We argue by induction on n. If $n = 0$, then $\tilde{w} \in {}^{\mathbb{S}_0}\tilde{W}$ and there is nothing to prove. Assume $n \geqslant 1$. If $\tilde{w} \in {}^{R_1}\tilde{W}$, then $\tilde{w} = \tilde{w}_1$ by Lemma 1.2(2), contradicting our assumption that $\tilde{w} = \tilde{w}_0 \neq \tilde{w}_1$. Thus, $\tilde{w} \notin {}^{R_1}\tilde{W}$. If $\tilde{w} = \tilde{w}_0$ is not right R_1-distinct, then $\mathcal{P}_{\tilde{w}} \neq \emptyset$ by Lemma 3.3. Otherwise, by Lemma A.4, $w_{R_1}\tilde{w}w_{R_1} \in \mathrm{Adm}(\lambda)$ is left R_1-distinct, where w_{R_1} is the longest element of W_{R_1}. In view of Lemmas 3.2 and 1.2(2), we have $w_{R_1}\tilde{w}w_{R_1} = \tilde{w}_1 \in {}^{R_1}\tilde{W}$. Thus, $\tilde{w}_1 \in \mathcal{S}_{\lambda,b}$. Moreover, $\tilde{w}_1 \notin {}^{\mathbb{S}_0}\tilde{W}$ by Proposition 3.4. Thus, $\mathcal{P}_{\tilde{w}_1} \neq \emptyset$ by the induction hypothesis, which implies that $\mathcal{P}_{\tilde{w}} \neq \emptyset$ by Lemma 3.2.

4. Proof of Proposition 2.3

In this section, we prove Proposition 2.3 for all $x \in \mathcal{S}^+_{\lambda,b}$. We introduce a new algorithm in § 4.3 to construct affine lines connecting the sets $\mathbb{J}_{b,\tilde{w}}$ for $\tilde{w} \in \mathcal{S}_{\lambda,b,x}$ with each other. This algorithm is based on an induction on the vectors $\nu^\flat_{\tilde{w}}$ introduced in § 4.1. The construction of affine lines is given in § 4.2, which relies on the sets $\mathcal{P}_{\tilde{w}}$ studied in previous section.

Assume that (λ, b) is Hodge–Newton irreducible. Recall that d is the number of connected components of $\mathbb{S}_0$.

4.1 The vector $\nu^\flat_{\tilde{w}}$

Let $\eta \in Y^+$. Let $A = \max\{|\langle\alpha, \eta\rangle|; \alpha \in \Phi\}$. Fix $M \in \mathbb{Z}_{\geqslant 2}$ such that $M|\langle\alpha, \eta\rangle| > 2A$ for any $\alpha \in \Phi$ with $\langle\alpha, \eta\rangle \neq 0$. Motivated from the $\underline{\mathbf{a}}$-function in [He10], for $\tilde{w} \in t^\mu W_0 \subseteq W_0 t^\eta W_0$ we define

$$\nu^\flat_{\tilde{w}} = \sum_{i=0}^{N-1} \frac{p(\tilde{w}\sigma)^i(\mu)}{M^i} \in V,$$

where N is the order of the natural projection image $p(\tilde{w}\sigma) \in W_0\sigma$.

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

Lemma 4.1. *Let $\alpha \in \Phi$ and $0 \leqslant n \leqslant N-1$ such that $\langle \alpha, p(\tilde{w}\sigma)^n(\mu)\rangle \neq 0$ and $\langle \alpha, p(\tilde{w}\sigma)^i(\mu)\rangle = 0$ for $0 \leqslant i \leqslant n-1$. Then $\langle \alpha, \nu_{\tilde{w}}^\flat\rangle\langle \alpha, p(\tilde{w}\sigma)^n(\mu)\rangle > 0$. In particular, if $\alpha \in \Phi \setminus \Phi_{\nu_{\tilde{w}}}$, then $\langle \alpha, \nu_{\tilde{w}}^\flat\rangle < 0$ if and only if $(\tilde{w}\sigma)^{-m_{\alpha,\tilde{w}}}(\alpha) \in \tilde{\Phi}^+ \setminus \Phi$.*

Proof. Note that μ, η are conjugate by W_0. By the choice of $M \geqslant 2$ we have

$$\left|\frac{\langle \alpha, p(\tilde{w}\sigma)^n(\mu)\rangle}{M^n}\right| > \frac{2A}{M^{n+1}} > \frac{A}{M^{n+1}}\sum_{i=n+1}^{N-1}\frac{1}{M^{i-n-1}} \geqslant \sum_{i=n+1}^{N-1}\left|\frac{\langle \alpha, p(\tilde{w}\sigma)^i(\mu)\rangle}{M^i}\right|.$$

Thus, the statement follows. □

Corollary 4.2. *We have the following properties:*

(1) *$\langle \alpha, \nu_{\tilde{w}}^\flat\rangle = 0$ if and only if $\langle \alpha, p(\tilde{w}\sigma)^i(\mu)\rangle = 0$ for $i \in \mathbb{Z}$;*
(2) *$\nu_{\tilde{w}}^\flat$ is dominant for $\Phi_{\nu_{\tilde{w}}}^+$ if $\tilde{w} \in \mathcal{S}$;*
(3) *$\nu^\flat_{z\tilde{w}\sigma(z)^{-1}} = z(\nu_{\tilde{w}}^\flat)$ for $z \in W_0$;*
(4) *$\tilde{w}\sigma(\tilde{\Phi}^\pm_{\nu^\flat_{\tilde{w}}}) = \tilde{\Phi}^\pm_{\nu^\flat_{\tilde{w}}}$ if $\tilde{w} \in \mathcal{S}$;*
(5) *if $\alpha \in \mathcal{P}_{\tilde{w}}$, then the roots $(\tilde{w}\sigma)^i(\alpha) \in \Phi$ for $1 - m_{\alpha,\tilde{w}} \leqslant i \leqslant 0$ are linearly independent, and moreover, $\langle (\tilde{w}\sigma)^i(\alpha), \nu_{\tilde{w}}^\flat\rangle < 0$ for $1 - m_{\alpha,\tilde{w}} \leqslant i \leqslant 0$.*

Proof. Statement (1) follows from Lemma 4.1 and the definition of $\nu_{\tilde{w}}^\flat$.

Suppose there exists $\alpha \in \Phi^+_{\nu_{\tilde{w}}} \subseteq \tilde{\Phi}^-$ such that $\langle \alpha, \nu_{\tilde{w}}^\flat\rangle < 0$. By Lemma 4.1, there exists $n \in \mathbb{Z}_{\geqslant 0}$ such that $\langle \alpha, p(\tilde{w}\sigma)^n(\mu)\rangle < 0$ and $\langle \alpha, p(\tilde{w}\sigma)^i(\mu)\rangle = 0$ for $0 \leqslant i \leqslant n-1$. In particular, we have $(\tilde{w}\sigma)^{-i}(\alpha) = p(\tilde{w}\sigma)^{-i}(\alpha)$ for $1 \leqslant i \leqslant n$ and $(\tilde{w}\sigma)^{-n-1}(\alpha) \in \tilde{\Phi}^+ \setminus \Phi$, contradicting that $\tilde{w} \in \mathcal{S}$. Thus, statement (2) follows.

Statement (3) follows by definition.

By statement (1) we have $\tilde{\Phi}_{\nu^\flat_{\tilde{w}}} = \tilde{w}\sigma(\tilde{\Phi}_{\nu^\flat_{\tilde{w}}}) \subseteq \tilde{\Phi}_{\nu_{\tilde{w}}}$. As $\tilde{w} \in \mathcal{S}$, we have $\tilde{w}\sigma(\tilde{\Phi}^\pm_{\nu_{\tilde{w}}}) = \tilde{\Phi}^\pm_{\nu_{\tilde{w}}}$. Thus, statement (4) follows from that $\tilde{\Phi}^\pm_{\nu^\flat_{\tilde{w}}} = \tilde{\Phi}_{\nu^\flat_{\tilde{w}}} \cap \tilde{\Phi}^\pm_{\nu_{\tilde{w}}}$.

Let $\alpha \in \mathcal{P}_{\tilde{w}}$. Set $m = m_{\alpha,\tilde{w}}$ and $\alpha^i = (\tilde{w}\sigma)^i(\alpha)$ for $i \in \mathbb{Z}$. By definition, $\langle \alpha^{1-m}, \mu\rangle < 0$, $\alpha^{-i} = p(\tilde{w}\sigma)^{-i}(\alpha)$, and $\langle \alpha^{1-i}, \mu\rangle = \langle \alpha, p(\tilde{w}\sigma)^{i-1}(\mu)\rangle = 0$ for $1 \leqslant i \leqslant m-1$. Thus, it follows from Lemma 4.1 that $\langle \alpha^i, \nu_{\tilde{w}}^\flat\rangle < 0$ for $1-m \leqslant i \leqslant 0$. Suppose $\sum_{i=0}^{1-m} c_i\alpha^i = 0$, where the coefficients $c_i \in \mathbb{R}$ are not all zero. Let $i_0 = \min\{1-m \leqslant i \leqslant 0; c_i \neq 0\}$. Then

$$0 = \left\langle p(\tilde{w}\sigma)^{1-m-i_0}\left(\sum_{i=0}^{1-m} c_i\alpha^i\right), \mu\right\rangle = \sum_{i=0}^{i_0} c_i\langle \alpha^{1-m-i_0+i}, \mu\rangle = c_{i_0}\langle \alpha^{1-m}, \mu\rangle \neq 0,$$

which is a contradiction. Thus, statement (5) follows. □

Lemma 4.3. *Let $\tilde{w} \in \mathcal{S}$. Then $\tilde{w} \in {}^{\mathbb{S}_0}\tilde{W}$ if $\nu_{\tilde{w}}^\flat$ is dominant.*

Proof. Assume $\nu_{\tilde{w}}^\flat$ is dominant. Let $\mu \in Y$ such that $\tilde{w} \in t^\mu W_0$. Then μ is dominant by Lemma 4.1. We show $\tilde{w} < s_\alpha\tilde{w}$ for $\alpha \in \Phi^+$. If $\langle \alpha, \nu_{\tilde{w}}^\flat\rangle > 0$, then either $\langle \alpha, \mu\rangle > 0$, or $\langle \alpha, \mu\rangle = 0$ and $\langle p(\tilde{w}\sigma)^{-1}(\alpha), \nu_{\tilde{w}}^\flat\rangle > 0$ (hence, $p(\tilde{w}\sigma)^{-1}(\alpha) \in \Phi^+$) by Corollary 4.2, which means $\tilde{w} < s_\alpha\tilde{w}$ as desired. Suppose $\langle \alpha, \nu_{\tilde{w}}^\flat\rangle = 0$, that is, $\alpha \in \Phi^+_{\nu^\flat_{\tilde{w}}} \subseteq \tilde{\Phi}^-$. Thus, $(\tilde{w}\sigma)^{-1}(\alpha) \in \tilde{\Phi}^-_{\nu^\flat_{\tilde{w}}}$ by Corollary 4.2(4), which also means $\tilde{w} < s_\alpha\tilde{w}$ as desired. □

4.2 Construction of affine lines

Let $\tilde{\alpha} = \alpha + k \in \tilde{\Phi}$. We denote by $U_{\tilde{\alpha}} \subseteq LG$ be the corresponding affine root subgroup of the loop subgroup LG associated to G. More precisely, $U_{\tilde{\alpha}}(z) = u_\alpha(zt^k)$ for $z \in \mathbf{k}$, where $u_\alpha : \mathbb{G}_a \to G$ is the root subgroup corresponding to α. For simplicity we write $U_{\tilde{\alpha}} = U_{\tilde{\alpha}}(\mathbf{k}) \subseteq G(\breve{F})$ if no confusion is caused.

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

For $g \in G(\breve{F})$, $\tilde{\gamma} \in \tilde{\Phi}$, $\tilde{w} \in \tilde{W}$, and $m \in \mathbb{Z}_{\geqslant 0}$, we define

$$g_{g,\tilde{\gamma},\tilde{w},m} : \mathbb{A}^1 \to G(\breve{F})/I, \; z \mapsto g^{(\tilde{w}\sigma)^{1-m}} U_{\tilde{\gamma}}(z) \cdots {}^{(\tilde{w}\sigma)^{-1}} U_{\tilde{\gamma}}(z) U_{\tilde{\gamma}}(z) I.$$

It extends to a unique morphism from $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$ to $G(\breve{F})/I$ which we still denoted by $g_{g,\tilde{\gamma},\tilde{w},m}$. Here ${}^{g\sigma^i} U_{\tilde{\gamma}}(z) = g\sigma^i(U_{\tilde{\gamma}}(z))g^{-1}$ for $g \in G(\breve{F})$ and $i \in \mathbb{Z}$.

Hypothesis 4.1. Recall that $\mathbb{F}_q$ is the residue field of F. Assume that $q^d > 2$ (respectively, $q^d > 3$) if some/any connected component of $\mathbb{S}_0$ is non-simply-laced except of type G_2 (respectively, is of type G_2).

Note that if Hypothesis 4.1 is not true, then $d = 1$ and $\mathbb{S}_0$ is non-simply-laced, which implies that G is residually split, and hence split (since G is unramified).

Lemma 4.4. *Suppose Hypothesis 4.1 holds. Let $\tilde{w} \in \tilde{W}$, $\gamma \in \Phi$, and $m \in \mathbb{Z}_{\geqslant 0}$ such that the roots $\gamma^i := (\tilde{w}\sigma)^i(\gamma) \in \Phi$ for $1 - m \leqslant i \leqslant 0$ are linearly independent. Let $g = g_{1,\gamma,\tilde{w},m}$. Then there exist a sequence of integers $1 - m \leqslant i_r < \cdots < i_0 \leqslant 0$ (which may be empty) such that*

$$g(\infty) = s_{\gamma^{i_r}} \cdots s_{\gamma^{i_0}} I, \text{ and } s_{\gamma^{i_0}} \cdots s_{\gamma^{i_{k-1}}}(\gamma^{i_k}) \in \Phi^+ \text{ for } 0 \leqslant k \leqslant r.$$

Moreover, if there exists $v \in V$ such that $\langle \gamma^i, v\rangle < 0$ for $1 - m \leqslant i \leqslant 0$, then $v \leqslant (s_{\gamma^{i_r}} \cdots s_{\gamma^{i_0}})^{-1}(v)$, where the equality holds if and only if the sequence $i_r, \ldots, i_0$ is empty, or equivalently, $\gamma^i \in \Phi^-$ for $1 - m \leqslant i \leqslant 0$.

Proof. By assumption, we have

(a) $$^{(\tilde{w}\sigma)^i} U_\gamma(z) = U_{\gamma^i}(c_i z^{q^i}) \text{ with } c_i \in \mathcal{O}_{\breve{F}}^\times \text{ for } 1 - m \leqslant i \leqslant 0.$$

For $\alpha \neq -\alpha' \in \Phi$ there exist constants $c_{\alpha,\alpha',i,j} \in \mathcal{O}_{\breve{F}}$ for $i, j \in \mathbb{Z}_{\geqslant 1}$ such that

(b) $$U_\alpha(x)U_{\alpha'}(y)U_\alpha(-x) = U_{\alpha'}(y) \prod_{i,j \in \mathbb{Z}_{\geqslant 1}} U_{i\alpha + j\alpha'}(c_{\alpha,\alpha',i,j} x^i y^j).$$

Now we argue by induction on m. If $m = 0$, the statement is trivial. Assume $m \geqslant 1$. If $\gamma \in \Phi^-$, then $U_\gamma(z) \in I$ and, hence, $g(\infty) = g_{1,\gamma^{-1},\tilde{w},m-1}(\infty)$, from which the statement follows by induction hypothesis. Otherwise, we have

$$g(z) = {}^{(\tilde{w}\sigma)^{1-m}} U_\gamma(z) \cdots {}^{(\tilde{w}\sigma)^{-1}} U_\gamma(z) U_{-\gamma}(z^{-1}) s_\gamma I \text{ for } z \neq 0.$$

As the roots γ^i for $1 - m \leqslant i \leqslant 0$ are linearly independent, it follows by parts (a) and (b) that

$$^{(\tilde{w}\sigma)^{1-m}} U_\gamma(z) \cdots {}^{(\tilde{w}\sigma)^{-1}} U_\gamma(z) U_{-\gamma}(z^{-1}) = \prod_{(\beta, a_\bullet)} U_\beta(c_{a_\bullet} z^{n_{a_\bullet}}),$$

where $a_\bullet = (a_i)_{0 \leqslant i \leqslant m-1} \in (\mathbb{Z}_{\geqslant 0})^m$ with $a_0 \geqslant 1$, $\beta = -a_0\gamma + \sum_{i=1}^{m-1} a_i \gamma^{-i} \in \Phi$, $c_{a_\bullet} \in \mathcal{O}_{\breve{F}}$ and $n_{a_\bullet} = -a_0 + \sum_{i=1}^{m-1} a_i q^{-i}$. Note that $a_i = 0$ unless $i \in d\mathbb{Z}$ since $\beta \in \Phi$. Moreover, $a_{jd}/a_0 \leqslant 1$ (respectively, $a_{jd}/a_0 \leqslant 2$; respectively, $a_{jd}/a_0 \leqslant 3$) for $j \geqslant 1$ if some/any connected component of $\mathbb{S}_0$ is simply-laced (respectively, is non-simply-laced except of type G_2; respectively, is of type G_2). Thus, by Hypothesis 4.1 we have $a_{jd}/a_0 \leqslant q^d - 1$ for $j \geqslant 1$, which implies that $n_{a_\bullet} < 0$ and, hence,

$$\lim_{z \to \infty} {}^{(\tilde{w}\sigma)^{1-m}} U_\gamma(z) \cdots {}^{(\tilde{w}\sigma)^{-1}} U_\gamma(z) U_{-\gamma}(z^{-1}) = 1.$$

Then we have

$$g(\infty) = s_\gamma g_{1, s_\gamma(\gamma^{-1}), s_\gamma \tilde{w}\sigma(s_\gamma), m-1}(\infty).$$

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

By the induction hypothesis, there exist a sequence $2-m \leqslant j_r < \cdots < j_1 \leqslant 0$ of integers such that

$$g_{1,s_\gamma(\gamma^{-1}),s_\gamma\tilde{w}\sigma(s_\gamma),m-1}(\infty) \\ = s_{(s_\gamma\tilde{w}\sigma s_\gamma)^{j_r}(s_\gamma(\gamma^{-1}))}\cdots s_{(s_\gamma\tilde{w}\sigma s_\gamma)^{j_1}(s_\gamma(\gamma^{-1}))}I = s_\gamma s_{\gamma^{j_r-1}}\cdots s_{\gamma^{j_1-1}}s_\gamma I,$$

and for $1 \leqslant k \leqslant r$,

$$s_{(s_\gamma\tilde{w}\sigma s_\gamma)^{j_1}(s_\gamma(\gamma^{-1}))}\cdots s_{(s_\gamma\tilde{w}\sigma s_\gamma)^{j_{k-1}}(s_\gamma(\gamma^{-1}))}((s_\gamma\tilde{w}\sigma s_\gamma)^{j_k}(s_\gamma(\gamma^{-1}))) \\ = s_\gamma s_{\gamma^{j_1-1}}\cdots s_{\gamma^{j_{k-1}-1}}(\gamma^{j_k-1}) \in \Phi^+.$$

Take $i_0 = 0$ and $i_k = j_k - 1$ for $1 \leqslant k \leqslant r$. Then one checks directly that the first statement is true.

Set $\beta_k = s_{\gamma^{i_0}}\cdots s_{\gamma^{i_{k-1}}}(\gamma^{i_k}) \in \Phi^+$ and $v_k = s_{\gamma^{i_0}}\cdots s_{\gamma^{i_k}}(v)$ for $0 \leqslant k \leqslant r$. As $\langle\gamma^{i_k}, v\rangle < 0$ we have

$$v_k = s_{\beta_k}(v_{k-1}) = v_{k-1} - \langle\beta_k, v_{k-1}\rangle\beta_k^\vee = v_{k-1} - \langle\gamma^{i_k}, v\rangle\beta_k^\vee > v_{k-1}.$$

Thus, the 'Moreover' part follows. □

Remark 4.5. In view of Corollary 4.2(5), we apply the above lemma (by taking $(\gamma, m, v) = (\alpha, m_{\tilde{w},\alpha}, \nu_{\tilde{w}}^\flat)$ for $\tilde{w} \in \mathcal{S}_{\lambda,b}$ and $\alpha \in \mathcal{P}_{\tilde{w}}$) to construct affine lines in $X(\lambda, b)$.

4.3 A connecting algorithm

Let $J = J_{\nu_G(b)}$. Let $x \in \mathcal{S}_{\lambda,b}^+$. Let $J_{x,0} = \sigma(J_{x,0}) \subseteq J$ be the union of connected components K of J such that $\sigma^i(\mu_x)$ is central on K for all $i \in \mathbb{Z}$. Let $J_{x,1} = J \setminus J_{x,0}$. Let $H_x \subseteq M_J(\breve{F})$ be the subgroup generated by I_{M_J}, $W_{J_{x,0}}$, and $W_{J_{x,1}}^a$, see § 1.5. By definition, $J_{x,1}$ commutes with $J_{x,0}$, and $x \in \tilde{W}_{J_{x,1}}$.

Remark 4.6. The reason for distinguishing $J_{x,0}$ and $J_{x,1}$ is that we will employ different methods to study the actions of two normal subgroups of $\ker(\eta_{M_J}) \cap \mathbb{J}_b$ on $X_0(X(\lambda, b))$ coming from $J_{x,1}$ and $J_{x,0}$ in § 6. Moreover, this distinction also plays a delicate role in handling the case of Lemma 6.5, see § 6.3.

Note that $\tilde{W} = \bigsqcup_{z\in W_0^J} z\tilde{W}_J = \bigsqcup_{z\in W_0^J}\bigsqcup_{\omega\in\Omega_J} z\omega^{-1}W_J^a$.

LEMMA 4.7. *Let $x \in \mathcal{S}_{\lambda,b}^+$, $\tilde{w} \in \mathcal{S}_{\lambda,b,x}$, $z \in W_0^J$ with $\tilde{w} = zx\sigma(z)^{-1}$. Let $y \in \tilde{W}$ (respectively, $y \in W_0$) such that $y\tilde{w}\sigma(y)^{-1} \in \mathrm{Adm}(\lambda)$. Let $z' \in W_0^J$, $\omega \in \Omega_J$ such that $yz \in z'\omega^{-1}W_J^a$. Let $x' = \omega^{-1}x\sigma(\omega)$ and $\tilde{w}' = z'x'\sigma(z')^{-1}$. Then:*

(1) $x' \in \mathcal{S}_{\lambda,b}^+$ *and* $\tilde{w}' \in \mathcal{S}_{\lambda,b}$;
(2) $y\tilde{w}\sigma(y)^{-1}$ *and* $\tilde{w}'$ *are σ-conjugate under $W_{\nu_{\tilde{w}'}}^a$ (respectively, $W_{\nu_{\tilde{w}'}}$);*
(3) *there exists $h \in \ker(\eta_{M_J}) \cap \mathbb{J}_x$ (respectively, $h \in H_x \cap \mathbb{J}_x$) such that $gy^{-1}I \sim_{\lambda,b} gzh\omega z'^{-1}I$ for $g \in \mathbb{J}_{b,\tilde{w}}$.*

Proof. Write $yz = z'\omega^{-1}u$ with $u \in W_J^a$. Let $\delta = ux\sigma(u)^{-1}x^{-1} \in W_J^a$. Then $x \leq_J \delta x$ (see § 1.5) since $x \in \Omega_J$. As $z'\omega^{-1}(\tilde{\Phi}_J^+) \subseteq \tilde{\Phi}^+$, it follows from [CN19, Lemma 1.3] that

$$\tilde{w}' = (z'\omega^{-1})x\sigma(z'\omega^{-1})^{-1} \leq (z'\omega^{-1})\delta x\sigma(z'\omega^{-1})^{-1} = y\tilde{w}\sigma(y)^{-1} \in \mathrm{Adm}(\lambda).$$

Thus, $\tilde{w}' \in \mathrm{Adm}(\lambda) \cap \mathcal{S}$ by Lemma 1.5. Note that $\tilde{w}' \in W_0 t^{\mu_{x'}} W_0$ and $x' \in \Omega_J$. Then $\mu_{x'} \preceq \lambda$ and $x' \leq_J t^{\mu_{x'}} \in \mathrm{Adm}(\lambda)$, which means $x' \in \mathrm{Adm}(\lambda)$ and $x' \in \mathcal{S}_{\lambda,b}^+$. Thus, part (1) follows.

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

Note that $J = J_{\nu_x} = J_{\nu_{x'}}$. By definition $y\tilde{w}\sigma(y)^{-1}, \tilde{w}'$ are σ-conjugate by

$$z'\omega^{-1}u\omega z'^{-1} \in z'W_J^a z'^{-1} = z'W_{\nu_{x'}}^a z'^{-1} = W_{\nu_{\tilde{w}'}}^a.$$

Moreover, if $y \in W_0$, then $\omega = 1$, $u \in W_J$ and hence $z'\omega^{-1}u\omega z'^{-1} \in W_{\nu_{\tilde{w}'}}$. Thus, part (2) follows.

Now we consider the following closed affine Deligne–Lusztig variety

$$X_{\leq_J \delta x}^{M_J}(x) = \{m \in M_J(\breve{F})/I_{M_J}; m^{-1}x\sigma(m) \in \cup_{\delta' \leq_J \delta} I_{M_J}\delta' x I_{M_J}\}.$$

Note that $u^{-1}I_{M_J} \in X_{\leq_J \delta x}^{M_J}(x)$. As $x \in \Omega_J$, by [HZ20, Theorem 4.1] (respectively, [CN20, Lemma 6.13]), there exists $h \in \ker(\eta_{M_J}) \cap \mathbb{J}_x$ (respectively, $h \in H_x \cap \mathbb{J}_x$ if $y \in W_0$) such that $u^{-1}I_{M_J}, hI_{M_J}$ are connected in $X_{\leq_J \delta x}^{M_J}(x)$. For $g \in \mathbb{J}_{b,\tilde{w}}$ there is an embedding

$$X_{\leq_J \delta x}^{M_J}(x) \hookrightarrow X(\lambda, b), \quad mI_{M_J} \mapsto gzm\omega z'^{-1}I,$$

from which we have $gy^{-1}I = gzu^{-1}\omega z'^{-1}I \sim_{\lambda,b} gzh\omega z'^{-1}I$ as desired. □

Lemma 4.8. *Assume Hypothesis 4.1 holds. Let $x \in \mathcal{S}_{\lambda,b}^+$ and $\tilde{w} \in \mathcal{S}_{\lambda,b,x}$. If $\tilde{w} \notin {}^{\mathbb{S}_0}\tilde{W}$, then there exist $h \in H_x \cap \mathbb{J}_x$ and $\tilde{w}' \in \mathcal{S}_{\lambda,b,x}$ such that $\nu_{\tilde{w}}^\flat < \nu_{\tilde{w}'}^\flat$ and $gI \sim_{\lambda,b} gzhz'^{-1}I$ for $g \in \mathbb{J}_{b,\tilde{w}}$. Here $z, z' \in W_0^J$ such that $\tilde{w} = zx\sigma(z)^{-1}$ and $\tilde{w}' = z'x\sigma(z')^{-1}$.*

Proof. By Proposition 3.1, there exists $\alpha \in \mathcal{P}_{\tilde{w}}$. Set $m = m_{\alpha,\tilde{w}}$ and $\alpha^i = (\tilde{w}\sigma)^i(\alpha)$ for $i \in \mathbb{Z}$. Let $g = g_{g,\alpha,\tilde{w},m}$ for $g \in \mathbb{J}_{b,\tilde{w}}$. Let $A \subseteq I$ be the kernel of the natural reduction map $G(\mathcal{O}_{\breve{F}}) \overset{t\mapsto 0}{\to} G(\mathbf{k})$. Since $\alpha^{-m} \in \tilde{\Phi}^+ \setminus \Phi$ and $\alpha^i \in \Phi$ for $1 - m \leqslant i \leqslant 0$, we have ${}^{(\tilde{w}\sigma)^{-m}}U_\alpha = U_{\alpha^{-m}} \subseteq A$ and ${}^{(\tilde{w}\sigma)^i}U_\alpha = U_{\alpha^i} \subseteq G(\mathcal{O}_{\breve{F}})$ for $1 - m \leqslant i \leqslant 0$. Thus, for $z \in \mathbf{k}$ we have

$$F(z) := {}^{({}^{(\tilde{w}\sigma)^{1-m}}U_\alpha(z)\cdots U_\alpha(z))^{-1}(\tilde{w}\sigma)^{-m}}U_\alpha(-z)) \in A \subseteq I.$$

Now one computes that

$$\begin{aligned} g(z)^{-1}b\sigma g(z) &= U_\alpha(-z)\cdots {}^{(\tilde{w}\sigma)^{1-m}}U_\alpha(-z)\tilde{w}\sigma^{(\tilde{w}\sigma)^{1-m}}U_\alpha(z)\cdots U_\alpha(z) \\ &= U_\alpha(-z)\cdots {}^{(\tilde{w}\sigma)^{2-m}}U_\alpha(-z)\tilde{w}\sigma^{(\tilde{w}\sigma)^{-m}}U_\alpha(-z)^{(\tilde{w}\sigma)^{1-m}}U_\alpha(z)\cdots U_\alpha(z) \\ &= U_\alpha(-z)\cdots {}^{(\tilde{w}\sigma)^{2-m}}U_\alpha(-z)\tilde{w}\sigma^{(\tilde{w}\sigma)^{1-m}}U_\alpha(z)\cdots U_\alpha(z)F(z) \\ &\subseteq U_\alpha(-z)\cdots {}^{(\tilde{w}\sigma)^{2-m}}U_\alpha(-z)\tilde{w}\sigma^{(\tilde{w}\sigma)^{1-m}}U_\alpha(z)\cdots U_\alpha(z)I \\ &= \tilde{w}\sigma U_\alpha(z)I \subseteq I\{\tilde{w}\sigma, \tilde{w}\sigma s_\alpha\}I \subseteq I\mathrm{Adm}(\lambda)\sigma I. \end{aligned}$$

By Hypothesis 4.1 and Corollary 4.2(5), the conditions in Lemma 4.4 are satisfied (for $(\gamma, m, v) = (\alpha, m_{\tilde{w},\alpha}, \nu_{\tilde{w}}^\flat)$). Thus, by Lemma 4.4 we have $gI = g(0) \sim_{\lambda,b} g(\infty) = gy^{-1}I$ for some $y \in W_0$ such that $y(\nu_{\tilde{w}}^\flat) > \nu_{\tilde{w}}^\flat$. Then $y\tilde{w}\sigma(y)^{-1} \in \mathrm{Adm}(\lambda)$ and $\nu_{\tilde{w}}^\flat < y(\nu_{\tilde{w}}^\flat) = \nu_{y\tilde{w}\sigma(y)^{-1}}^\flat$. Let $h \in H_x$, $\tilde{w}' \in \mathcal{S}_{\lambda,b,x}$, and $z' \in W_0^J$ be as in Lemma 4.7 such that $gI \sim_{\lambda,b} gy^{-1}I \sim_{\lambda,b} gzhz'^{-1}I$. Then $y\tilde{w}\sigma(y)^{-1}$ and $\tilde{w}'$ are σ-conjugate by $W_{\nu_{\tilde{w}'}}$, and, hence, $\nu_{\tilde{w}'}^\flat$ and $\nu_{y\tilde{w}\sigma(y)^{-1}}^\flat$ are conjugate by $W_{\nu_{\tilde{w}'}}$. By Corollary 4.2(2), $\nu_{\tilde{w}'}^\flat$ is dominant for $\Phi_{\nu_{\tilde{w}'}}^+$, which means $\nu_{\tilde{w}}^\flat < \nu_{y\tilde{w}\sigma(y)^{-1}}^\flat \leqslant \nu_{\tilde{w}'}^\flat$ as desired. □

Corollary 4.9. *Let $x \in \mathcal{S}_{\lambda,b}^+$ and $\tilde{w}, \tilde{w}' \in \mathcal{S}_{\lambda,b,x}$ with $\tilde{w}'$ the unique element in ${}^{\mathbb{S}_0}\tilde{W}$. Then there exists $h \in H_x \cap \mathbb{J}_x$ such that $gI \sim_{\lambda,b} gzhz'^{-1}I$ for $g \in \mathbb{J}_{b,\tilde{w}}$, where $z, z' \in W_0^J$ such that $\tilde{w} = zx\sigma(z)^{-1}$ and $\tilde{w}' = z'x\sigma(z')^{-1}$.*

Proof. Note that the statement follows from Theorem 0.2, which is proved in [CN19] when G is split. Thus, we can assume G is not split and Hypothesis 4.1 holds. If $\tilde{w} \notin {}^{\mathbb{S}_0}\tilde{W}$, by Lemma 4.8, there exist $h \in H_x \cap \mathbb{J}_x$ and $\tilde{w}' \in \mathcal{S}_{\lambda,b,x}$ such that $\nu_{\tilde{w}}^\flat < \nu_{\tilde{w}'}^\flat$ and $gI \sim_{\lambda,b} gzhz'^{-1}I$

for $g \in \mathbb{J}_{b,\tilde{w}}$, where $z' \in W_0^J$ such that $\tilde{w}' = z'x\sigma(z')^{-1}$. Thus, the statement follows by repeating this process. □

Proposition 2.3 is a consequence of the following result.

Proposition 4.10. *Let $x \in \mathcal{S}_{\lambda,b}^+$ and $\tilde{w} \in \mathcal{S}_{\lambda,b,x}$. Then there exists $h \in H_x \cap \mathbb{J}_x$ such that $gI \sim_{\lambda,b} ghz^{-1}I$ or, equivalently, $gh^{-1}I \sim_{\lambda,b} gz^{-1}I$ for $g \in \mathbb{J}_{b,x}$, where $z \in W_0^J$ such that $\tilde{w} = zx\sigma(z)^{-1}$. In particular, $\mathbb{J}_{b,\tilde{w}} \sim_{\lambda,b} \mathbb{J}_{b,x}$.*

Proof. By Proposition 3.6 and Lemma 1.5(2), there exists $z' \in W_0$ such that $z'x\sigma(z')^{-1} \in \mathcal{S} \cap {}^{\mathbb{S}_0}\tilde{W}$. By Corollary 4.9, there exist $h_1, h_2 \in H_x \cap \mathbb{J}_x$ such that $gI \sim_{\lambda,b} gh_1 z'^{-1}I$ and $gz^{-1}I \sim_{\lambda,b} gh_2 z'^{-1}I$ for $g \in \mathbb{J}_{b,x}$. Then we have

$$ghz^{-1}I = jgz^{-1}I \sim_{\lambda,b} jgh_2z'^{-1}I = gh_1z'^{-1}I \sim_{\lambda,b} gI,$$

where $h = h_1h_2^{-1} \in H_x \cap \mathbb{J}_x$ and $j = gh_1h_2^{-1}g^{-1} \in \mathbb{J}_b$. □

The following result will be used to compute the stabilizers of connected components of $X(\lambda, b)$ in the remaining sections.

Proposition 4.11. *Let $x \in \mathcal{S}_{\lambda,b}^+$ and $y \in \tilde{W}$ (respectively, $y \in W_0$) such that $yx\sigma(y)^{-1} \in \mathrm{Adm}(\lambda)$. Then there exists $h \in \ker(\eta_{M_J}) \cap \mathbb{J}_x$ (respectively, $h \in H_x \cap \mathbb{J}_x$) such that $gy^{-1}I \sim_{\lambda,b} gh\omega I$ (respectively, $gy^{-1}I \sim_{\lambda,b} ghI$) for $g \in \mathbb{J}_{b,x}$, where $\omega \in \Omega_J$ such that $y \in W_0^J\omega^{-1}W_J^a$.*

Proof. It follows from Lemma 4.7 and Proposition 4.10. □

5. Proof of Proposition 2.4

In this section we show Proposition 2.4, which is based on an algorithm introduced in [CKV15], see § 5.3. To this end, we need a detailed study on a single reduction step. This is carried out in §§ 5.1 and 5.2.

Recall that d is the number of connected components of $\mathbb{S}_0$.

5.1 The set $\mathrm{Adm}(\lambda)$

We collect more properties on $\mathrm{Adm}(\lambda)$. For $K \subseteq \mathbb{S}_0$ we denote by $\mathrm{pr}_K : \mathbb{R}\Phi^\vee \to (\mathbb{R}\Phi_K^\vee)^\perp$ the orthogonal projection with respect to the usual Killing form $(\,,)$ on $\mathbb{R}\Phi^\vee$ such that $\langle \alpha, \beta^\vee \rangle = 2(\alpha, \beta)/(\beta, \beta)$ for $\alpha, \beta \in \Phi$.

Lemma 5.1. *Let $x \in \mathcal{S}_{\lambda,b}^+$ and let $\mathcal{O}$ be a σ-orbit of J-anti-dominant roots in $\Phi^+ \setminus \Phi_J$ with $J = J_{\nu_G(b)}$. Then we have: (1) $\sum_{\alpha \in \mathcal{O}} \langle \alpha, \mathrm{pr}_J(\mu_x) \rangle > 0$; and (2) $\langle w_J(\beta), \mu_x \rangle \geqslant 1$ for some $\beta \in \mathcal{O}$. Here w_J is the longest element of W_J.*

Proof. Let $\gamma \in \mathcal{O}$. By definition, $\langle \gamma, \nu_G(b) \rangle = \langle \gamma, \mathrm{pr}_J(\mu_x)^\diamond \rangle > 0$, where $\mathrm{pr}_J(\mu_x)^\diamond$ is the σ-average of $\mathrm{pr}_J(\mu_x)$. Thus, part (1) follows since

$$\sum_{\alpha \in \mathcal{O}} \langle \alpha, \mathrm{pr}_J(\mu_x) \rangle = \sum_{\alpha \in \mathcal{O}} \langle \alpha, \mathrm{pr}_J(\mu_x)^\diamond \rangle = |\mathcal{O}| \langle \gamma, \nu_G(b) \rangle > 0.$$

By part (1), there exists $\beta \in \mathcal{O}$ such that $\langle \beta, \mathrm{pr}_J(\mu_x) \rangle > 0$. As $w_J(\beta)$ is J-dominant and $\mu_x - \mathrm{pr}_J(\mu_x) \in \mathbb{R}_{\geqslant 0}(\Phi_J^+)^\vee$, we have

$$\langle w_J(\beta), \mu_x \rangle \geqslant \langle w_J(\beta), \mathrm{pr}_J(\mu_x) \rangle = \langle \beta, \mathrm{pr}_J(\mu_x) \rangle > 0.$$

Thus, part (2) follows. □

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

Lemma 5.2 [CN20, Lemma 1.6]. *Let $K \subseteq \mathbb{S}_0$ and $\tilde{w} = (t^\mu W_K) \cap \Omega_K$ with $\mu \in Y$. Let $\alpha \in \Phi^+$ be K-anti-dominant. Then: (1) $s_\alpha \tilde{w} \in \mathrm{Adm}(\lambda)$ if $\mu + \alpha^\vee \preceq \lambda$; (2) $\tilde{w} s_\alpha \in \mathrm{Adm}(\lambda)$ if $\mu - p(\tilde{w})(\alpha)^\vee \preceq \lambda$; and (3) $z\tilde{w}z^{-1} \in \mathrm{Adm}(\lambda)$ for $z \in \tilde{W}^K$.*

Lemma 5.3. *Let $K \subseteq \mathbb{S}_0$, $\tilde{w} = (t^\mu W_K) \cap \Omega_K$ with $\mu \in Y$. Let $r \in \mathbb{Z}_{\geqslant 0}$ and $\gamma \in \Phi^+ \setminus \Phi_K$ with $\gamma^\vee$ K-dominant and K-minuscule such that*

$$\mu, \mu - \gamma^\vee, \mu + p(\tilde{w})\sigma^r(\gamma^\vee), \mu - \gamma^\vee + p(\tilde{w})\sigma^r(\gamma^\vee) \preceq \lambda.$$

Let $\tilde{\gamma} = \gamma + 1 \in \tilde{\Phi}^+$. Then we have:

(1) *$\mu - \gamma^\vee, \mu + p(\tilde{w})(\sigma^r(\gamma^\vee)), \mu - \gamma^\vee + p(\tilde{w})\sigma^r(\gamma^\vee)$ are K-minuscule;*
(2) *$\tilde{w}, s_{\tilde{\gamma}}\tilde{w}, \tilde{w}s_{\sigma^r(\tilde{\gamma})}, s_{\tilde{\gamma}}\tilde{w}s_{\tilde{\gamma}} \in \mathrm{Adm}(\lambda)$;*
(3) *$s_{\tilde{\gamma}}\tilde{w}s_{\sigma^r(\tilde{\gamma})} \in \mathrm{Adm}(\lambda)$ if $\gamma \neq \sigma^r(\gamma)$ and $\langle p(\tilde{w})\sigma^r(\gamma), \mu\rangle, \langle \gamma, \mu\rangle \geqslant -1$;*

Proof. Note that parts (1) and (2) were proved in [CKV15, Lemma 4.4.6] and [CN20, Lemma 1.5], respectively. To show part (3) we claim that

(a) there exists $\eta \in W_K(\mu)$ such that $\eta - \gamma^\vee + \sigma^r(\gamma^\vee)$ is K-minuscule.

Indeed, let η be a W_K-conjugate of μ such that $\eta - \gamma^\vee + \sigma^r(\gamma^\vee)$ is minimal under the partial order $\preceq$. If $\eta - \gamma^\vee + \sigma^r(\gamma^\vee)$ is not K-minuscule, then there exists $\alpha \in \Phi_K$ such that $\langle \alpha, \eta - \gamma^\vee + \sigma^r(\gamma^\vee)\rangle \geqslant 2$. As η is K-minuscule, and $\gamma^\vee, \sigma^r(\gamma^\vee)$ are K-dominant and K-minuscule, we deduce that $\langle \alpha, \eta\rangle = 1$. Let $\eta' = s_\alpha(\eta) = \eta - \alpha^\vee$. Then we have

$$\eta' - \gamma^\vee + \sigma^r(\gamma^\vee) = \eta - \gamma^\vee + \sigma^r(\gamma^\vee) - \alpha^\vee \prec \eta - \gamma^\vee + \sigma^r(\gamma^\vee),$$

which contradicts the choice of η. Thus, part (a) is proved.

Let $w = p(\tilde{w}) \in W_K$. By parts (1) and (a), $\eta - \gamma^\vee + \sigma^r(\gamma^\vee)$, $\mu - \gamma^\vee + w\sigma^r(\gamma^\vee)$ are conjugate by W_K. In particular, $\eta - \gamma^\vee + \sigma^r(\gamma^\vee) \preceq \lambda$. Then part (3) follows from that

$$s_{\tilde{\gamma}}\tilde{w}s_{\sigma^r(\tilde{\gamma})} \leq s_{\tilde{\gamma}}t^\eta s_{\sigma^r(\tilde{\gamma})} = s_\gamma t^{\eta - \gamma^\vee + \sigma^r(\gamma^\vee)} s_{\sigma^r(\gamma)} \leq t^{\eta - \gamma^\vee + \sigma^r(\gamma^\vee)} \in \mathrm{Adm}(\lambda),$$

where the first $\leq$ follows from [CN19, Lemma 1.3], and the second $\leq$ follows from that

$$\langle \gamma, \sigma^r(\gamma^\vee)\rangle \leqslant 0 \text{ since } \gamma \neq \sigma^r(\gamma);$$
$$\langle \gamma, \eta - \gamma^\vee + \sigma^r(\gamma^\vee)\rangle \leqslant \langle \gamma, \mu\rangle - 2 \leqslant -1;$$
$$\langle \sigma^r(\gamma), \eta - \gamma^\vee + \sigma^r(\gamma^\vee)\rangle \geqslant \langle w\sigma^r(\gamma), \mu\rangle + 2 \geqslant 1.$$

The proof is finished. □

5.2 Strongly K-minuscule coroots

For $K \subseteq \mathbb{S}_0$ we say $\gamma^\vee \in \Phi^{\vee,+} \setminus \Phi_K^\vee$ is strongly K-minuscule if $\gamma^\vee$ is K-minuscule, and moreover, γ is a long root if: (1) some/any connected component of $\mathbb{S}_0$ is of type G_2; and (2) K is the set of short simple roots.

For any $K \subseteq \mathbb{S}_0$ there is a natural isomorphism $\Omega_K \cong \pi_1(M_K)$. We identify these two sets according to the context.

Lemma 5.4. *Let K, $\tilde{w}$, γ, $\tilde{\gamma}$, and r be as in Lemma 5.3. Assume further that $\gamma^\vee$ is strongly K-minuscule. Then $U_{-\tilde{\gamma}}\tilde{w}U_{-\sigma^r(\tilde{\gamma})} \subseteq I\mathrm{Adm}(\lambda)I$ unless*

(∗) $$\langle \gamma, \mu\rangle = -\langle p(\tilde{w})\sigma^r(\gamma), \mu\rangle = 1 \text{ and } \langle \gamma, p(\tilde{w})\sigma^r(\gamma^\vee)\rangle = -1.$$

Moreover, if (∗) holds, then

$$\tilde{w} \neq \tilde{w}',\ U_{-\sigma^r(\tilde{\gamma})}\tilde{w}'U_{-\tilde{\gamma}} \subseteq I\mathrm{Adm}(\lambda)I, \text{ and } \mu \pm (\gamma + p(\tilde{w})\sigma^r(\gamma))^\vee \preceq \lambda,$$

where $\tilde{w}' = \mu - \gamma^\vee + \sigma^r(\gamma^\vee) \in \pi_1(M_K) \cong \Omega_K$.

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

Proof. Let $w = p(\tilde{w}) \in W_K$. First we claim that

(a) $\Psi := \Phi \cap (\mathbb{Z}\gamma + \mathbb{Z}w\sigma^r(\gamma))$ is of type A_2, or $A_1 \times A_1$, or A_1.

Otherwise, then Ψ is of type B_2 or G_2. In particular, $\gamma = \sigma^r(\gamma)$ (since $\sigma^d = \mathrm{id}$ with d the number of connected components of $\mathbb{S}_0$), $\gamma \neq w\sigma^r(\gamma) = w(\gamma)$ are short roots, and, hence, $K \neq \emptyset$. If Ψ is of B_2, then $\gamma \pm w\sigma^r(\gamma) \in \Phi$ and $\langle \gamma, w\sigma^r(\gamma^\vee)\rangle = 0$ since $\gamma, w\sigma^r(\gamma)$ are of the same length. Thus, $\gamma - w\sigma^r(\gamma) \in \Phi_K$ and $\langle \gamma - w\sigma^r(\gamma), \gamma^\vee\rangle = 2$, contradicting that $\gamma^\vee$ is K-minuscule. Thus, Ψ is of type G_2. As $\gamma \neq w\sigma^r(\gamma)$ are short roots and $\gamma^\vee$ is strongly K-minuscule, we deduce that K consists of long simple roots. This contradicts that $\gamma^\vee$ is K-minuscule. Thus, part (a) is proved.

Now we claim that

(b) $U_{-\tilde{\gamma}}\tilde{w}U_{-\sigma^r(\tilde{\gamma})} \subseteq I\mathrm{Adm}(\lambda)I$ if one of the following holds:

(b1) either $\langle \gamma, \mu\rangle \geqslant 2$ or $\langle \gamma, \mu\rangle = 1$ and $\langle \gamma, w\sigma(\gamma^\vee)\rangle \geqslant 0$;

(b2) either $\langle w\sigma^r(\gamma), \mu\rangle \leqslant -2$ or $\langle w\sigma^r(\gamma), \mu\rangle = -1$ and $\langle \gamma, w\sigma^r(\gamma^\vee)\rangle \geqslant 0$.

By symmetry we may assume (b1) occurs. By part (a) we have

$$U_{-\tilde{w}^{-1}(\tilde{\gamma})}, [U_{-\tilde{w}^{-1}(\tilde{\gamma})}, U_{-\sigma^r(\tilde{\gamma})}] \subseteq I,$$

where $[g, g'] = gg'g^{-1}{g'}^{-1}$ denotes the commutator of $g, g' \in G(\breve{F})$. Thus,

$$U_{-\tilde{\gamma}}\tilde{w}U_{-\sigma^r(\tilde{\gamma})} \subseteq \tilde{w}U_{-\sigma^r(\tilde{\gamma})}I \subseteq I\{\tilde{w}, \tilde{w}s_{\sigma^r(\tilde{\gamma})}\} \subseteq I\mathrm{Adm}(\lambda)I,$$

where the last inclusion follows from Lemma 5.3(2). Thus, part (b) is proved.

Suppose $U_{-\tilde{\gamma}}\tilde{w}U_{-\sigma^r(\tilde{\gamma})} \nsubseteq I\mathrm{Adm}(\lambda)I$. Then $-\langle w\sigma^r(\gamma), \mu\rangle, \langle \gamma, \mu\rangle \leqslant 1$ by (b), which implies that $\tilde{w}^{-1}(\gamma) \neq \sigma^r(\gamma)$. Assume $\langle \gamma, \mu\rangle \leqslant 0$. We claim that

$$U_{\tilde{w}^{-1}(\tilde{\gamma})}, [U_{\tilde{w}^{-1}(\tilde{\gamma})}, U_{-\sigma^r(\tilde{\gamma})}] \subseteq I.$$

The first inclusion follows from that $\tilde{w}^{-1}(\tilde{\gamma}) = w^{-1}(\gamma) + 1 - \langle \gamma, \mu\rangle \in \tilde{\Phi}^+$. Note that $[U_{\tilde{w}^{-1}(\tilde{\gamma})}, U_{-\sigma^r(\tilde{\gamma})}] = U_{\tilde{w}^{-1}(\tilde{\gamma}) - \sigma^r(\tilde{\gamma})}$ by part (a). Thus, we can assume that $U_{\tilde{w}^{-1}(\tilde{\gamma}) - \sigma^r(\tilde{\gamma})}$ is nontrivial, that is,

$$\tilde{w}^{-1}(\tilde{\gamma}) - \sigma^r(\tilde{\gamma}) = w^{-1}(\gamma) - \sigma^r(\gamma) - \langle \gamma, \mu\rangle \in \tilde{\Phi}.$$

As γ is K-dominant and $w \in W_K$, $w^{-1}(\gamma) - \gamma \in \mathbb{Z}_{\geqslant 0}\Phi_K^-$. Thus, the σ-average of $w^{-1}(\gamma) - \sigma^r(\gamma)$, which equals the σ-average of $w^{-1}(\gamma) - \gamma$, lies in $\mathbb{R}_{\geqslant 0}\Phi^-$. This means that $w^{-1}(\gamma) - \sigma^r(\gamma) \in \Phi^-$ and, hence, $\tilde{w}^{-1}(\tilde{\gamma}) - \sigma^r(\tilde{\gamma}) \in \tilde{\Phi}^+$ (since $\langle \gamma, \mu\rangle \leqslant 0$). Thus, the second inclusion follows, and the claim is proved.

Thus, by Lemma 5.3 we compute that

$$U_{-\tilde{\gamma}}\tilde{w}U_{-\sigma^r(\tilde{\gamma})} \subseteq Is_{\tilde{\gamma}}\tilde{w}U_{-\sigma^r(\tilde{\gamma})}I \subseteq I\{s_{\tilde{\gamma}}\tilde{w}, s_{\tilde{\gamma}}\tilde{w}s_{\sigma^r(\tilde{\gamma})}\}I \subseteq I\mathrm{Adm}(\lambda)I,$$

which contradicts our assumption. Thus, $\langle \gamma, \mu\rangle = 1$, and $\langle w\sigma^r(\gamma), \mu\rangle = -1$ by symmetry. Moreover, we have $\langle \gamma, w\sigma^r(\gamma^\vee)\rangle = -1$ by parts (b) and (a).

Write $\tilde{w}' = t^{\mu'}w' \in \Omega_K$ with $\mu' \in Y$ and $w' \in W_K$. Applying Lemma 5.3(1) to $\tilde{w}$ and $\tilde{w}'$ we deduce that

$$\mu', \mu - \gamma^\vee + w\sigma^r(\gamma^\vee), \mu' - \sigma^r(\gamma^\vee), \mu - \gamma^\vee, \mu' + w'(\gamma^\vee), \mu + w\sigma^r(\gamma^\vee)$$

are K-minuscule, and, hence, are conjugate by W_K. Since

$$\langle \gamma, \mu\rangle = -\langle w\sigma^r(\gamma), \mu\rangle = -\langle \gamma, w\sigma^r(\gamma^\vee)\rangle = 1,$$

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

it follows that $\mu - \gamma^\vee + w\sigma^r(\gamma^\vee)$ and $\mu \pm (\gamma^\vee + w\sigma^r(\gamma^\vee))$ are conjugate by W_0. Hence, $\mu \pm (\gamma^\vee + w\sigma^r(\gamma^\vee)), \mu' \preceq \lambda$. As $w_K(\gamma)$ (with w_K the longest element of W_K) is K-anti-dominant, we have

$$\langle w'(\gamma), \mu' \rangle = \langle w_K(\gamma), \mu' \rangle \leqslant \langle \gamma, \mu - \gamma^\vee + w\sigma^r(\gamma^\vee) \rangle = -2.$$

Thus, $\sigma^r(\gamma) \neq \gamma$ (which means $\tilde{w} \neq \tilde{w}'$) and $U_{-\sigma^r(\tilde{\gamma})}\tilde{w}'U_{-\tilde{\gamma}} \subseteq I\mathrm{Adm}(\lambda)I$ by (b2). □

5.3 The second connecting algorithm

Let $J = J_{\nu_G(b)}$. Let $x, x' \in \mathcal{S}^+_{\lambda,b} \subseteq \pi_1(M_J)$. Write $x \overset{(\gamma,r)}{\to} x'$ for some $\gamma \in \Phi \setminus \Phi_J$ and $r \in \mathbb{Z}_{\geqslant 1}$ if $x' - x = \sigma^r(\gamma^\vee) - \gamma^\vee$ and $\mu_{x-\gamma^\vee}, \mu_{x+\sigma^r(\gamma^\vee)} \preceq \lambda$, see § 2. Moreover, write $x \overset{(\gamma,r)}{\rightarrowtail} x'$ if $x \overset{(\gamma,r)}{\to} x'$, and for each $1 \leqslant i \leqslant r-1$ we have

$$\text{neither} \quad x \overset{(\gamma,i)}{\to} x - \gamma^\vee + \sigma^i(\gamma^\vee) \overset{(\sigma^i(\gamma),r-i)}{\to} x',$$
$$\text{nor} \quad x \overset{(\sigma^i(\gamma),r-i)}{\to} x - \sigma^i(\gamma^\vee) + \sigma^r(\gamma^\vee) \overset{(\gamma,i)}{\to} x'.$$

Note that $x \overset{(\gamma,r)}{\to} x'$ is equivalent to $x' \overset{(-\gamma,r)}{\to} x$.

Lemma 5.5 [CKV15, Remark 4.5.2]. *Let $x \neq x' \in \mathcal{S}^+_{\lambda,b}$ such that $x \overset{(\gamma,r)}{\rightarrowtail} x'$ for some $\gamma \in \Phi \setminus \Phi_J$ and $r \in \mathbb{Z}_{\geqslant 1}$. Then $x\sigma^i(\delta) = \sigma^i(\delta)$ for any W_0-conjugate δ of γ and $1 \leqslant i \leqslant r-1$ with $i, i-r \notin d\mathbb{Z}$.*

For $\gamma \in \Phi$ we denote by $\mathcal{O}_\gamma$ the σ-orbit of γ.

Proposition 5.6 [Nie18, Lemma 6.7]. *Let $x \neq x' \in \mathcal{S}^+_{\lambda,b}$. Then there exist distinct elements $x = x_0, x_1, \dots, x_m = x' \in \mathcal{S}^+_{\lambda,b}$ such that for each $1 \leqslant i \leqslant m$ we have:*

(1) $x_{i-1} \overset{(\gamma_i,r_i)}{\rightarrowtail} x_i$ *with $\gamma_i \in \Phi \setminus \Phi_J$ such that $\gamma_i^\vee$ J-dominant and J-minuscule;*
(2) $1 \leqslant r_i \leqslant d-1$ *if $|\mathcal{O}_{\gamma_i}| = d$; $1 \leqslant r_i \leqslant d$ if $|\mathcal{O}_{\gamma_i}| = 2d$; $1 \leqslant r_i \leqslant 2d-1$ if $|\mathcal{O}_{\gamma_i}| \leqslant 3d$.*

Proof of Proposition 2.4. The case when σ has order $3d$ is handled in § 8.2. We consider the case when σ has order $\leqslant 2d$. Without loss of generality, we can assume that $|\mathcal{O}_\gamma| = 2d$. By Proposition 5.6 and symmetry, we may assume $x \overset{(\gamma,r)}{\to} x'$ for some $1 \leqslant r \leqslant d$ and $\gamma \in \Phi^+ \setminus \Phi_J$ with $\gamma^\vee$ J-dominant and J-minuscule. Then $\gamma^\vee$ is also strongly J-minuscule since $|\mathcal{O}_\gamma| = 2d$. Moreover, we can assume that

$$\text{(a)} \qquad U_{-\tilde{\gamma}}xU_{-\sigma^r(\tilde{\gamma})} \subseteq I\mathrm{Adm}(\lambda)I.$$

Indeed, if $1 \leqslant r \leqslant d-1$, part (a) follows from Lemma 5.3(2). If $r = d$, by Lemma 5.4 we can switch the pairs (x, γ) and $(x', \sigma^d(\gamma))$ if necessary so that part (a) still holds.

Now we can assume further that $x \overset{(\gamma,r)}{\rightarrowtail} x'$. Let $\tilde{\gamma} = \gamma + 1 \in \tilde{\Phi}^+$, and let $g = g_{g,-\sigma^{r-1}(\tilde{\gamma}),x,r}$ for $g \in \mathbb{J}_{b,x}$ (see § 4). By Lemma 5.5, $(x\sigma)^i(\gamma) = \sigma^i(\gamma)$ for $1 \leqslant i \leqslant r-1$. Then by part (a) we have $g^{-1}b\sigma(g) \subseteq U_{-\tilde{\gamma}}xU_{-\sigma^r(\tilde{\gamma})} \subseteq I\mathrm{Adm}(\lambda)I$, which means that $gI = g(0) \sim_{\lambda,b} g(\infty) = gsI$, where $s = s_{\tilde{\gamma}} \cdots s_{\sigma^{r-1}(\tilde{\gamma})}$. By [CN20, Lemma 1.3] we can write $s = \omega z^{-1}$, where $z \in W_0^J$ and $\omega = \gamma^\vee + \cdots + \sigma^{r-1}(\gamma^\vee) \in \Omega_J \cong \pi_1(M_J)$. By Proposition 4.11, there is $h \in \mathbb{J}_{b,x}$ such that $gI \sim_{\lambda,b}\sim_{\lambda,b} gh\omega I$. So $\mathbb{J}_{b,x} \sim_{\lambda,b} \mathbb{J}_{b,x'}$ as desired. □

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

6. Proof of Proposition 2.5

In this section we prove Proposition 2.5, that is, $\ker(\eta_{M_J}) \cap \mathbb{J}_b$ acts trivially on $\pi_1(X(\lambda, b))$. To this end, we divide $\ker(\eta_{M_J}) \cap \mathbb{J}_b$ into two part: the J_1-part and the J_0-part, see § 6.1. The triviality of the action of J_1-part follows from a main result in [HZ20], see Lemma 6.3. For the J_0-part, we first use Lemma 6.4 to reduce it to the situation of Lemma 6.5. Finally in § 6.3 we addressing this remaining case in an *ad hoc* way.

Assume that (λ, b) is Hodge–Newton irreducible. Let $J = J_{\nu_G(b)}$.

6.1 The stabilizer

Define $J_1 = \cup_{x \in \mathcal{S}^+_{\lambda,b}} J_{x,1}$ (see § 4.3) and $J_0 = J \setminus J_1$. Note that $J_i = \sigma(J_i)$ is a union of connected components of J for $i \in \{0, 1\}$. By definition, $x \in \Omega_{J_1}$ and μ_x is central on J_0 for all $x \in \mathcal{S}^+_{\lambda,b}$. Let $\mathcal{K}_i$ (with $i \in \{0, 1\}$) be the set of subsets $K \subseteq J_i$ such that $K = \cup_{i \in \mathbb{Z}} \sigma^i(K')$ with K' a connected component of J. Set $\mathcal{K} = \mathcal{K}_1 \sqcup \mathcal{K}_0$.

Theorem 6.1 [HZ20, Theorem 6.3]. *Let $x \in \mathcal{S}^+_{\lambda,b}$. Then $\ker(\eta_{M_{J_{x,1}}}) \cap \mathbb{J}_x$ fixes each connected components of $X^{M_{J_{x,1}}}(\mu_x, x)$.*

Let J^a denote the set of simple reflections of $W^a_J = W^a_{M_J}$. For $x \in \Omega_J$ the group $W^a_J \cap \mathbb{J}_x$ is a Coxeter group whose simple reflections are parameterized by the $(\mathrm{Ad}(x) \circ \sigma)$-orbits of J^a. For $w \in W^a_J \cap \mathbb{J}_x$ denote by $\mathrm{supp}^x(w)$ the set of simple reflections of $W^a_J \cap \mathbb{J}_x$ which appear in some/any reduced expression of w. Moreover, for $h \in \ker(\eta_{M_J}) \cap \mathbb{J}_x$ we set $\mathrm{supp}^x(h) = \mathrm{supp}^x(u)$, where $u \in W^a_J \cap \mathbb{J}_x$ such that $h \in I_{M_J} u I_{M_J}$.

Lemma 6.2. *Let C be a connected component of $X(\lambda, b)$. Let $x \in \mathcal{S}^+_{\lambda,b}$. Then there exists $g \in \mathbb{J}_{b,x}$ such that $gI \in C$. Moreover, the stabilizer $\mathrm{Stab}_{\mathbb{J}_b}(C)$ of C in $\mathbb{J}_b$ equals gQg^{-1} with $Q \subseteq \mathbb{J}_x$ a subgroup containing $I \cap \mathbb{J}_x = I_{M_J} \cap \mathbb{J}_x$. In particular, $\mathrm{supp}^x(h) \subseteq Q$ for $h \in \ker(\eta_{M_J}) \cap Q$.*

Proof. The existence of g follows from Proposition 2.4. As $g(I_{M_J} \cap \mathbb{J}_x)g^{-1}$ fixes gI, it also fixes C. Let $Q = g^{-1}\mathrm{Stab}_{\mathbb{J}_b}(C)g$. Then $I_{M_J} \cap \mathbb{J}_x \subseteq Q$ as desired. Note that the conjugation by x preserves the standard Bruhat decomposition $M_J(\breve{F}) = I_{M_J} \tilde{W}_J I_{M_J}$ of $M_J(\breve{F})$. Thus, $I_{M_J} \cap \mathbb{J}_x$ is a standard Iwahori subgroup of $\mathbb{J}_x$. Hence, there exists a unique subset $E = x\sigma(E)x^{-1} \subseteq J^a$ such that $Q = (I_{M_J} \cap \mathbb{J}_x)(W_E \cap \mathbb{J}_x)(I_{M_J} \cap \mathbb{J}_x)$, from which the 'In particular' part follows. □

Let K be a union of connected components of J. We denote by $L_K \subseteq M_J$ the normal subgroup generated by U_α for $\alpha \in \Phi_K$.

Lemma 6.3. *For $x \in \mathcal{S}^+_{\lambda,b}$ the group $L_{J_{x,1}}(\breve{F}) \cap \mathbb{J}_b$ fixes each connected component of $X(\lambda, b)$.*

Proof. Let C, x, g be as in Lemma 6.2. Moreover, gI lies in the image of the embedding

$$X^{M_{J_{x,1}}}(\mu_x, x) \hookrightarrow X(\lambda, b), \ hI_{M_{J_{x,1}}} \mapsto ghI.$$

By Theorem 6.1, $L_{J_{x,1}}(\breve{F}) \cap \mathbb{J}_x \subseteq \ker(\eta_{J_{x,1}}) \cap \mathbb{J}_x$ fixes the connected component of $X^{M_{J_{x,1}}}(\mu_x, x)$ containing $I_{M_{J_{x,1}}}$. Thus, $g(L_{J_{x,1}}(\breve{F}) \cap \mathbb{J}_x)g^{-1} = L_{J_{x,1}}(\breve{F}) \cap \mathbb{J}_b$ fixes C. □

Lemma 6.4. *Let $K \in \mathcal{K}_0$. If $\mu_x + \alpha^\vee \preceq \lambda$ for some $x \in \mathcal{S}^+_{\lambda,b}$ and some $\alpha \in K$, then $L_K(\breve{F}) \cap \mathbb{J}_b$ fixes each connected component of $X(\lambda, b)$.*

Proof. Let C, x, g, Q be as in Lemma 6.2. Then $I_{M_J} \cap \mathbb{J}_x \subseteq Q$. Note that $L_K(\breve{F}) \cap \mathbb{J}_x$ is contained in the subgroup generated by $I_{M_J} \cap \mathbb{J}_x$ and $W^a_K \cap \mathbb{J}_x$, it suffices to show $W^a_K \cap \mathbb{J}_x \subseteq Q$.

As μ_x is central on Φ_K (since $K \in \mathcal{K}_0$), by replacing α with a suitable W_K-conjugate we can assume α is K-dominant and, hence, $\sigma^d(\alpha) = \alpha$. Moreover, the action $\mathrm{Ad}(x) \circ \sigma$ restricts to σ

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

on $W^a_{J_0} \supseteq W^a_K$. Let $g = g_{g,\alpha,x,d}$ and $g' = g_{g,-\alpha-1,x,d}$ be as in § 4. By Lemmas 5.2 and 5.3,

$$g^{-1} b\sigma(g) \subseteq U_\alpha x \subseteq I\mathrm{Adm}(\lambda)I \quad \text{and} \quad g'^{-1} b\sigma(g') \subseteq x U_{-\alpha-1} \subseteq I\mathrm{Adm}(\lambda)I,$$

which means

$$gsI = g(\infty) \sim_{\lambda,b} g(0) = g'(0) \sim_{\lambda,b} g'(\infty) = gs'I,$$

where $s = s_\alpha \cdots s_{\sigma^{d-1}(\alpha)}, s' = s_{\alpha+1} \cdots s_{\sigma^{d-1}(\alpha)+1} \in \mathbb{J}_x$. Thus, $s, s' \in Q$.

As α is a highest root of Φ^+_K, $\mathrm{supp}^x(s) \cup \mathrm{supp}^x(s')$ consists of all simple reflections of $W^a_K \cap \mathbb{J}_x$. Hence, $W^a_K \cap \mathbb{J}_x \subseteq Q$ by Lemma 6.2. □

6.2 A technical lemma

Let $\mathcal{O}$ be a σ-orbit of $\Phi^+ \setminus \Phi_J$ whose roots are J-anti-dominant and J-minuscule. We define $\Psi_{J,\mathcal{O}} = \Phi \cap \mathbb{Z}(J \cup \mathcal{O})$. If Φ is simply laced, then $J \cup \mathcal{O}$ is a set of simple roots of $\Psi_{J,\mathcal{O}}$.

The following lemma is proved in Appendix B.

Lemma 6.5. *Let $K \in \mathcal{K}_0$. Suppose $\mu_{x''} + \delta^\vee \not\preceq \lambda$ for any $x'' \in \mathcal{S}^+_{\lambda,b}$ and any $\delta \in K$. Then there exist $x \in \mathcal{S}^+_{\lambda,b}$ and $\beta \in \Phi^+ \setminus \Phi_J$ with $\beta^\vee$ J-anti-dominant and J-minuscule such that:*

(1) *$\mu_x + \beta^\vee \preceq \lambda$, and $\beta^\vee$ is non-central on K;*
(2) *$x\sigma^i(\beta) = \sigma^i(\beta)$ for $i \in \mathbb{Z} \setminus n\mathbb{Z}$;*
(3) *$\langle p(x)\sigma^n(\beta), \mu_x \rangle \geqslant 1$;*
(4) *if σ^n does not act trivially on $\Psi_\beta \cap J_0$, then $\Psi = \Phi$, Ψ_β is of type E_6, $\Psi_\beta \cap J_0 = \{\alpha_1, \alpha_6\}$, $\Psi_\beta \cap J_1 = \{\alpha_2, \alpha_4\}$, $\beta = \alpha_3$, $\mu_x|_{\Psi_\beta} = \omega^\vee_4 - \omega^\vee_3$, and $\mu_x|_{\Psi \setminus \Psi_\beta} = 0$.*

Here, $\Psi = \Psi_{J,\mathcal{O}_\beta}$ with $\mathcal{O}_\beta$ the σ-orbit of β; Ψ_β is the irreducible factor of Ψ containing β; $n \in \{d, 2d, 3d\}$ is the minimal integer such that $\sigma^n(\beta) \in \Psi_\beta$; and in part (4) the simple roots α_i (with $\omega^\vee_i$ the corresponding fundamental coweights) for the root system of type E_6 are labeled as in [Hum72].

Lemma 6.6. *Retain the situation of Lemma 6.5. Let $\alpha = \sigma(\alpha) \in \Phi^+_K$ such that $\langle \alpha, \beta^\vee \rangle = -1$. Then $U_\beta x U_{\sigma^n(\beta)}, U_\alpha s_\beta x s_{\sigma^n(\beta)} U_\alpha \subseteq I\mathrm{Adm}(\lambda)I$.*

Proof. As $K \in \mathcal{K}_0$, $\alpha \in \Phi_K$, and $x \in \Omega_{J_1}$, we have $x(\alpha) = \alpha$ and

$$\mu_x + \beta^\vee + \alpha^\vee = \mu_x + s_\alpha(\beta^\vee) = s_\alpha(\mu_x + \beta^\vee) \preceq \lambda.$$

Moreover, since $\beta^\vee$, $\alpha^\vee$ and $s_\alpha(\beta^\vee) = \alpha^\vee + \beta^\vee$ are J_1-anti-dominant, we have $s_\alpha, s_\beta, s_{s_\alpha(\beta)} \in W_0^{J_1}$. By Lemma 5.2, $s_\beta x, s_{s_\alpha(\beta)} x \in \mathrm{Adm}(\lambda)$. By Lemma 6.5(3) we have $x\sigma^n(\beta) \in \tilde{\Phi}^+ \setminus \Phi$. Moreover, as $x(\alpha) = \alpha = \sigma^n(\alpha)$ we have

$$s_\beta x s_{\sigma^n(\beta)}(\alpha) = s_\beta x \sigma^n s_\beta(\alpha) = s_\beta x \sigma^n(\alpha + \beta) = s_\beta(\alpha) + s_\beta x \sigma^n(\beta) \in \tilde{\Phi}^+ \setminus \Phi,$$

where the inclusion follows from that $x\sigma^n(\beta) \in \tilde{\Phi}^+ \setminus \Phi$. Therefore,

$$U_\beta x U_{\sigma^n(\beta)} \subseteq I U_\beta x, \ U_\alpha s_\beta x s_{\sigma^n(\beta)} U_\alpha \subseteq I U_\alpha s_\beta x s_{\sigma^n(\beta)}.$$

Then it remains to show $s_\alpha s_\beta x s_{\sigma^n(\beta)}, s_\beta x s_{\sigma^n(\beta)} \in \mathrm{Adm}(\lambda)$. As $x\sigma^n(\beta) \in \tilde{\Phi}^+ \setminus \Phi$, we have $wx\sigma^n(\beta) \in w(\tilde{\Phi}^+ \setminus \Phi) = \tilde{\Phi}^+ \setminus \Phi$ and, hence, $wxs_{\sigma^n(\beta)} < wx$ for any $w \in W_0$. In particular, $s_\beta x s_{\sigma^n(\beta)} \leq s_\beta x \in \mathrm{Adm}(\lambda)$ and

$$s_\alpha s_\beta x s_{\sigma^n(\beta)} < s_\alpha s_\beta x < s_\alpha s_\beta x s_\alpha = s_\alpha s_\beta s_\alpha x = s_{s_\alpha(\beta)} x \in \mathrm{Adm}(\lambda),$$

where the second inequality follows from that $s_\alpha s_\beta x(\alpha) = s_\alpha s_\beta(\alpha) \in \Phi^+$. The proof is finished. □

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

6.3 The action of $\ker(\eta_{M_J}) \cap \mathbb{J}_b$

We are ready to show that $\ker(\eta_{M_J}) \cap \mathbb{J}_b$ acts on $\pi_0(X(\lambda, b))$ trivially.

Lemma 6.7. *If $L_K(\breve{F}) \cap \mathbb{J}_b$ fixes each connected component of $X(\lambda, b)$ for $K \in \mathcal{K}$, then so does $\ker(\eta_{M_J}) \cap \mathbb{J}_b$.*

Proof. Let C, x, g, Q be as in Lemma 6.2. Then $I_{M_J} \cap \mathbb{J}_x \subseteq Q$. By assumption, $g^{-1}(L_K(\breve{F}) \cap \mathbb{J}_b)g = L_K(\breve{F}) \cap \mathbb{J}_x \subseteq Q$ for $K \in \mathcal{K}$. Note that $\ker(\eta_{M_J}) \cap \mathbb{J}_x$ is generated by $I_{M_J} \cap \mathbb{J}_x$ and $L_K(\breve{F}) \cap \mathbb{J}_x$ for $K \in \mathcal{K}$. Thus, $\ker(\eta_{M_J}) \cap \mathbb{J}_x \subseteq Q$, which means that $g(\ker(\eta_{M_J}) \cap \mathbb{J}_x)g^{-1} = \ker(\eta_{M_J}) \cap \mathbb{J}_b$ fixes C as desired. □

Proof of Proposition 2.5. Let C be a connected component of $X(\lambda, b)$. By Lemma 6.7 it suffices to show $L_K(\breve{F}) \cap \mathbb{J}_b$ fixes C for all $K \in \mathcal{K}$. If $K \in \mathcal{K}_1$, by definition $K \subseteq J_{x',1}$ for some $x' \in \mathcal{S}_{\lambda,b}^+$, and the statement follows from Lemma 6.3. Now we assume $K \in \mathcal{K}_0$. If $\nu_{x'} + \alpha^\vee \preceq \lambda$ for some $x' \in \mathcal{S}_{\lambda,x}^+$ and some $\alpha \in \Phi_K$, it follows from Lemma 6.4. Thus, it remains to handle the situation of Lemma 6.5. Let $x, \beta, \Psi, \Psi_\beta, n$ be as in Lemma 6.5. Let $g \in \mathbb{J}_{b,x}$ with $gI \in C$ and $Q \subseteq \mathbb{J}_x$ be as in Lemma 6.2. As in the proof of Lemma 6.4, it remains to show $W_K^a \cap \mathbb{J}_x \subseteq Q$.

Case (1): σ^n acts trivially on $\Psi_\beta \cap J_0$. As $\beta^\vee$ is noncentral on K, there exists a highest root $\alpha \in \Phi_K^+$ such that $\langle \alpha, \beta^\vee \rangle = -1$. As in proof of Lemma 6.4 it suffices to show $s, s' \in Q$, where $s = s_\alpha \cdots s_{\sigma^{n-1}(\alpha)}$, $s' = s_{\alpha+1} \cdots s_{\sigma^{n-1}(\alpha)+1} \in \mathbb{J}_x$.

Let $r = s_\beta \cdots \sigma^{n-1}(s_\beta)$. We claim that

(a) $$gI \sim_{\lambda,b} grI \sim_{\lambda,b} grsI \sim_{\lambda,b} gsI, \text{ and, hence, } s \in Q.$$

To show the first relation $\sim_{\lambda,b}$ in part (a) we define $\mathrm{g} = \mathrm{g}_{g,\sigma^{n-1}(\beta),x,n}$. By Lemmas 6.5(2) and 6.6 we have

$$\mathrm{g}^{-1} b\sigma(\mathrm{g}) \subseteq U_\beta x U_{\sigma^n(\beta)} \subseteq I \mathrm{Adm}(\lambda) I,$$

which means $gI = \mathrm{g}(0) \sim_{\lambda,b} \mathrm{g}(\infty) = grI$ as desired. The last relation $\sim_{\lambda,b}$ in part (a) follows the same way by replacing g, β with gs, $s_\alpha(\beta)$, respectively.

To show the second relation $\sim_{\lambda,b}$ in (a) we define $\mathrm{g}' = \mathrm{g}_{gr,\sigma^{n-1}(\alpha),x,n}$. Note that $r^{-1}\tilde{w}_x\sigma(r) = s_\beta x s_{\sigma^n(\beta)}$. Then by Lemma 6.6 we have

$$\mathrm{g}'^{-1} b\sigma(\mathrm{g}') \subseteq U_\alpha s_\beta x s_{\sigma^n(\beta)} U_\alpha \subseteq I \mathrm{Adm}(\lambda) I,$$

which means $grI = \mathrm{g}'(0) \sim_{\lambda,b} \mathrm{g}'(\infty) = grsI$. Thus, part (a) is proved.

Let $x' = x + \beta^\vee - \sigma^n(\beta)^\vee \in \pi_1(M_J)$. We claim that

(b) $$\text{if } \beta \neq \sigma^n(\beta) \text{ then } \langle w(\beta), \sigma^n(\beta)^\vee \rangle = 0 \text{ for any } w \in W_{J_1}.$$

Indeed, as $\beta \neq \sigma^n(\beta)$, Φ is simply laced and, hence, $J \cup \mathcal{O}_\beta$ is the set of simple roots of Ψ^+. Thus, $\beta, \sigma^n(\beta)$ are neighbors of $\Psi_\beta \cap K$ (in the Dynkin diagram $J \cup \mathcal{O}_\beta$) on which σ^n acts trivially. This means they are in distinct connected components of $(J \cup \mathcal{O}_\beta) \setminus K \supseteq J_1$. Thus, part (b) follows.

By Lemma 6.5(1) and (3) and part (b) we have $x' \in \mathcal{S}_{\lambda,b}^+$. Moreover, $\mu_{x'}$ and $\mu_x + \beta^\vee - p(x)(\beta^\vee)$ are conjugate by W_{J_1} as they are conjugate by W_J and $\mu_{x'}$ is central on J_0. Let $\gamma_1 = w_{J_1}(\beta)$ and $\gamma_2 = w_{J_1}(s_\alpha(\beta))$ which are J_1-dominant (since $\alpha \in \Phi_{J_0}$ is central on J_1). By Lemma 5.3(1) and that σ^n acts trivially on $\Psi_\beta \cap J_0$,

$$\mu_x, \mu_x - \sigma^n(\gamma_i^\vee), \mu_x + p(x)(\gamma_i^\vee), \mu_x - \sigma^n(\gamma_i^\vee) + p(x)(\gamma_i^\vee) \preceq \lambda$$

are conjugate to

$$\mu_{x'} - \gamma_i^\vee + p(x')\sigma^n(\gamma_i^\vee), \mu_{x'} - \gamma_i^\vee, \mu_{x'} + p(x')\sigma^n(\gamma_i^\vee), \mu_{x'} \preceq \lambda$$

under W_{J_1}, respectively.

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

Let $\tau = \beta^\vee + \cdots + \sigma^{n-1}(\beta)^\vee \in \pi_1(M_{J_1}) \cong \Omega_{J_1}$. Then $x = \tau^{-1}x'\sigma(\tau)$ and, hence, $g\tau^{-1} \in \mathbb{J}_{b,x'}$. Define $g_i = g_{g\tau^{-1},-\sigma^{n-1}(\gamma_i)-1,x',n}$. As $J_0 \neq \emptyset$, $\gamma_i^\vee$ is strongly J_1-minuscule. By Lemmas 6.5(2) and 5.4 and part (b) we have

$$g_i{}^{-1}b\sigma(g_i) \subseteq U_{-\gamma_i-1}x'U_{-\sigma^n(\gamma_i)-1} \subseteq I\mathrm{Adm}(\lambda)I,$$

which means $g\tau^{-1}I = g_i(0) \sim_{\lambda,b} g_i(\infty) = g\tau^{-1}s_iI$ with $s_i = s_{\gamma_i+1}\cdots s_{\sigma^{n-1}(\gamma_i)+1}$. As $\gamma_i^\vee$ is J_1-minuscule and J_1-dominant, we have $s_i = \tau_i y_i^{-1}$, where $\tau_i \in \Omega_{J_1}$ and $y_i \in W_0$. Note that $g\tau^{-1}\tau_i \in \mathbb{J}_{b,x}$, $\tau = \tau_1$, and $\tau^{-1}\tau_2 = s's \in \mathbb{J}_x$. By Proposition 4.11, there exist $h_i \in H_x \cap \mathbb{J}_x$ for $i \in \{1,2\}$ such that

$$g\tau^{-1}s_iI = g\tau^{-1}\tau_iy_i^{-1}I \sim_{\lambda,b} g\tau^{-1}\tau_ih_iI.$$

In particular, $gh_1I \sim_{\lambda,b} g\tau^{-1}I \sim_{\lambda,b} g\tau^{-1}\tau_2h_2I$, that is, $gI \sim_{\lambda,b} h_1^{-1}\tau\tau_2^{-1}h_2I$ and, hence,

$$h_1^{-1}\tau^{-1}\tau_2h_2 = h_1^{-1}s'sh_2 \in Q.$$

Note that s, h_1, h_2 belong to the subgroup $H_x \cap \mathbb{J}_x$ generated by $I_{M_J} \cap \mathbb{J}_x$ and $(W_{J_1}^a W_{J_0}) \cap \mathbb{J}_x$, while s' is a simple reflection of $W_J^a \cap \mathbb{J}_x$ not contained in $(W_{J_1}^a W_{J_0}) \cap \mathbb{J}_x$. Thus, $s' \in \mathrm{supp}^x(h_1^{-1}s'sh_2)$. By Lemma 6.2 we have $s' \in Q$ as desired.

Case (2): σ^n acts non-trivially on $\Psi_\beta \cap J_0$. By Lemma 6.5(4), $\Psi = \Phi$ and $\mu_x|_{\Psi\setminus\Psi_\beta} = 0$. Thus, we can assume that $n = d = 1$, σ is of order 2, and Φ is of type E_6. Then $p(x) = s_{\alpha_4}s_{\alpha_2}$, and it suffices to show $s, s' \in Q$, where $s = s_{\alpha_1}s_{\alpha_6}$ and $s' = s_{\alpha_1+1}s_{\alpha_6+1}$ are all the simple reflections of $W_J^a \cap \mathbb{J}_x$.

Let $\theta_0 = \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6$, $\theta_1 = \alpha_2 + \alpha_4 + \alpha_5$, $\eta_i = (p(x\sigma)^{-1}(\theta_i)$ and $\vartheta_i = \eta_i + \theta_i$. Define $g_i = g_{g,-\theta_i-1,x,2}$ for $g \in \mathbb{J}_{b,x}$. As $\mu + \alpha_3^\vee, \mu + \alpha_3^\vee + \alpha_1^\vee \preceq \lambda$, we have $\tilde{w}_x s_{\sigma(\theta_i)+1} \in \mathrm{Adm}(\lambda)$ by Lemma 5.3. Then

$$g_i^{-1}b\sigma(g_i) \subseteq IU_{-\vartheta_i-1}xU_{-\sigma(\theta_i)-1} \subseteq IxU_{-\sigma(\theta_i)-1}I \subseteq I\mathrm{Adm}(\lambda)I,$$

which means

$$gs_{\vartheta_0+1}s_{\eta_0}I = g_0(\infty) \sim_{\lambda,b} g_0(0) = gI = g_1(0) \sim_{\lambda,b} g_1(\infty) = gs_{\vartheta_1+1}s_{\eta_1}I.$$

As $\vartheta_0^\vee$ is J-dominant and J-minuscule, $s_{\vartheta_0+1}s_{\eta_0} = \omega y_0^{-1}$, where $\omega = \vartheta_0^\vee \in \Omega_J \cap \mathbb{J}_x$ and $y_0 \in W_0$. Then $s_{\vartheta_1+1}s_{\eta_1} = ss_{\vartheta_0+1}s_{\eta_0}s = s\omega y_1^{-1}$ for some $y_1 \in W_0$. By Proposition 4.11, there exist $h_0, h_1 \in H_x \cap \mathbb{J}_x$ such that $g\omega h_0I \sim_{\lambda,b} gI \sim_{\lambda,b} gs\omega h_1I$. Thus, $\omega h_0, s\omega h_1 \in Q$ and

$$s\omega h_1h_0^{-1}\omega^{-1} \in Q. \tag{c}$$

As $h_0h_1^{-1} \in H_x \cap \mathbb{J}_x \subseteq I_{M_J}\{1,s\}I_{M_J}$ and $\omega s\omega^{-1} = s'$, by part (c) we have $s\omega h_0h_1^{-1}\omega^{-1} \in I\{s, ss'\}I$ and $s \in \mathrm{supp}^x(s\omega h_1h_0^{-1}\omega^{-1})$. By Lemma 6.2, we have $s \in Q$. Noting that $(W_{J_1}^aW_{J_0}) \cap \mathbb{J}_x = \{1,s\}$, we have $H_x \cap \mathbb{J}_x \subseteq Q$, $\omega \in Q$, and $s' = \omega s\omega^{-1} \in Q$ as desired. □

7. Proof of Proposition 2.6

In this section we show that $(\mathbb{Z}\Phi^\vee/\mathbb{Z}\Phi_J^\vee)^\sigma$ acts trivially on $\pi_0(X(\lambda,b))$. We follow closely the strategy of [CKV15, § 4]. This is based on the connecting algorithm in the previous section and together with a delicate analysis on the sets $C_{\lambda,b,x}$ for $x \in \mathcal{S}_{\lambda,b}^+$ introduced in the following.

Assume that (λ,b) is Hodge–Newton irreducible. Let $J = J_{\nu_G(b)}$ and let w_J denote the longest element of W_J. Let $\mathcal{O}$ be a σ-orbit of Φ^+. We set

$$\omega_{\mathcal{O}} = \sum_{\alpha\in\mathcal{O}} \alpha^\vee \in \pi_1(M_J)^\sigma \cong \Omega_J^\sigma \subseteq \Omega_J \cap \mathbb{J}_b.$$

Let $\Psi = \Phi \cap \mathbb{Z}(\mathcal{O} \cup J)$. We say $\mathcal{O}$ is of type I (respectively, type II; respectively, type III) if $|\mathcal{O}|$ equals n (respectively, $2n$; respectively, $3n$). Here $n \in \{d, 2d, 3d\}$ is the minimal positive integer such that $\alpha, \sigma^n(\alpha)$ are in the same connected component of Ψ for some/any $\alpha \in \mathcal{O}$. If $\mathcal{O}$ is of type II or III, then $n = d$, Φ is simply-laced, and $\mathcal{O} \cup J$ is a set of simple roots for Ψ. In this case, for $\alpha \in \mathcal{O}$ we denote by $\vartheta_\alpha \in \Phi^+$ the sum of simple roots in the (unique) minimal σ^n-stable connected subset of $\mathcal{O} \cup J$ which contains α, see [CKV15, § 4.7].

Let $x \in \mathcal{S}^+_{\lambda,b}$. Following [CKV15, Proposition 4.19] and [Nie18, Lemma 7.1] we define

$$C_{\lambda,b,x} = \{\alpha \in \Phi^+ \setminus \Phi_J; \mu_x + \alpha^\vee \preceq \lambda, \alpha^\vee \text{ is } J\text{-anti-dominant and strongly } J\text{-minuscule}\},$$

where strongly J-minuscule coroots are defined in § 5.2. As in [Nie18] the sets $C_{\lambda,b,x}$ will be used to construct affine lines connecting gI and $g\omega_{\mathcal{O}}I$ for $g \in \mathbb{J}_{b,x}$ and various σ-orbits $\mathcal{O}$ of Φ^+.

Once affine lines are constructed, we will use the following result to detect elements in $\ker(\eta_G) \cap \Omega_J^\sigma$ that fix any/some connected components of $X(\lambda, b)$.

PROPOSITION 7.1. *Let $x \in \mathcal{S}^+_{\lambda,b}$, $g \in \mathbb{J}_{b,x}$, and $y \in \tilde{W}$ such that $gI \sim_{\lambda,b} gy^{-1}I$. Then we have $gI \sim_{\lambda,b} gh\omega z^{-1}I \sim_{\lambda,b} g\omega I$, where $z \in W_0^J$ and $\omega \in \Omega_J$ such that $y \in z\omega^{-1}W_J^a$.*

Proof. The proof follows from Propositions 4.11 and 2.5. □

7.1 Computation of stabilizers

Fix a σ-orbit $\mathcal{O}$ of roots in $\Phi^+ \setminus \Phi_J$ which are J-anti-dominant and J-minuscule.

LEMMA 7.2. *Assume $x \overset{(\gamma,r)}{\to} x'$ with $x' = x - \gamma^\vee + \sigma^r(\gamma)^\vee \in \mathcal{S}^+_{\lambda,b}$ for some $\gamma \in \mathcal{O}$ and $1 \leqslant r \leqslant n$. Let $\omega = \gamma^\vee + \cdots + \sigma^{r-1}(\gamma)^\vee \in \pi_1(M_J) \cong \Omega_J$. If $U_{-w_J(\gamma)-1}xU_{-w_J\sigma^r(\gamma)-1} \subseteq I\mathrm{Adm}(\lambda)I$, then $gI \sim_{\lambda,b} g\omega I$ for $g \in \mathbb{J}_{b,x}$. Recall that w_J is the longest element of W_J.*

Moreover, if $\mathcal{O} = \mathcal{O}_\alpha$ for some $\alpha \in C_{\lambda,b,x}$, the inclusion condition above holds if: (1) $1 \leqslant r \leqslant n-1$; (2) $x = x'$; or (3) $\mu_x + \vartheta_\gamma^\vee \npreceq \lambda$ when $\mathcal{O}$ is of type II and $r = n$.

Proof. Let $\tilde{\theta} = w_J\sigma^{r-1}(\gamma) + 1 \in \tilde{\Phi}^+$. Suppose we have

(a) $$U_{-\sigma^{1-r}(\tilde{\theta})}xU_{-\sigma(\tilde{\theta})} = U_{-w_J(\gamma)-1}xU_{-w_J\sigma^r(\gamma)-1} \subseteq I\mathrm{Adm}(\lambda)I.$$

By [Nie18, Lemma 6.5] and that $1 \leqslant r \leqslant n$, we can assume further that

(b) $$x \overset{(\gamma,r)}{\rightarrowtail} x', \text{ and, hence, } (x\sigma)^i(\tilde{\theta}) = \sigma^i(\tilde{\theta}) \text{ for } 1 - r \leqslant i \leqslant 0.$$

Define $\mathrm{g} = \mathrm{g}_{g,-\tilde{\theta},x,r}$ for $g \in \mathbb{J}_{b,x}$. By parts (a) and (b) we have

$$\mathrm{g}^{-1}b\sigma(\mathrm{g}) \subseteq U_{-\sigma^{1-r}(\tilde{\theta})}xU_{-\sigma(\tilde{\theta})} \subseteq I\mathrm{Adm}(\lambda)I,$$

which means

$$gI = \mathrm{g}(0) \sim_{\lambda,b} \mathrm{g}(\infty) = gs_{\tilde{\theta}} \cdots s_{\sigma^{r-1}(\tilde{\theta})}I = g\omega u^{-1}I$$

for some $u \in W_0 = W_0^J W_J$. So $gI \sim_{\lambda,b} g\omega u^{-1}I \sim_{\lambda,b} g\omega I$ by Proposition 7.1.

If $\mathcal{O}$ is of type II and $r = n$, then $\vartheta_\gamma^\vee$ is J-anti-dominant and J-minuscule, which means $\mu_x + \vartheta_\gamma^\vee$ is J-minuscule and, hence,

$$\mu_x + \vartheta_\gamma^\vee \preceq \mu_x + (w_J(\gamma) + p(x)w_J\sigma^r(\gamma))^\vee.$$

Thus, the second statement follows from Lemma 5.4(∗) by noting that $\langle w_J(\gamma), p(x)w_J\sigma^r(\gamma^\vee)\rangle = 0$ if $1 \leqslant r \leqslant n-1$. □

Let $\mathcal{A}_{\lambda,b}$ be the group of elements $\omega \in \pi_1(M_J)^\sigma \cong \Omega_J^\sigma$ which fix some/any connected component of $X(\lambda, b)$.

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

Lemma 7.3. *Suppose $\mathcal{O} = \mathcal{O}_\xi$ for some $\xi \in C_{\lambda,b,x}$. If $\mathcal{O}$ is of type I, then there exist $\gamma \in \mathcal{O}$, $1 \leqslant r \leqslant n$, and $x' \in \mathcal{S}^+_{\lambda,b}$ such that $x \overset{(\gamma,r)}{\to} x'$. Moreover, $\omega_{\mathcal{O}} \in \mathcal{A}_{\lambda,b}$.*

Proof. Note that $\mu + \alpha^\vee \preceq \lambda$. If $\langle w_J\sigma^r(\alpha), \mu_x\rangle \geqslant 1$ for some $1 \leqslant r \leqslant n-1$, then $\langle w_J\sigma^r(\alpha), \alpha^\vee\rangle = 0$, which means $x \overset{(\sigma^r(\alpha), n-r)}{\to} x' \overset{(\alpha,r)}{\to} x$ with $x' = x - \sigma^r(\alpha^\vee) + \alpha^\vee \in \mathcal{S}^+_{\lambda,b}$. Otherwise, $\langle w_J\sigma^i(\alpha), \mu_x\rangle \leqslant 0$ for $1 \leqslant i \leqslant n-1$, which means $\langle w_J(\alpha), \mu_x\rangle \geqslant 1$ by Lemma 5.1. Thus, $x \overset{(\alpha,n)}{\to} x$ and the first statement follows. As $\mathcal{O}$ is of type I, the second statement follows from Proposition 7.1 and Lemma 7.2(1) (respectively, Lemma 7.2(2)) if $r \neq n$ (respectively, $r = n$). □

Lemma 7.4. *Suppose $\mathcal{O}$ is of type II. Assume $\mu_{x''} + \vartheta^\vee_\beta \npreceq \lambda$ for any $x'' \in \mathcal{S}^+_{\lambda,b}$ and $\beta \in \mathcal{O}$. If there exist $\gamma \in \mathcal{O}$, $n+1 \leqslant r \leqslant 2n-1$, and $x' \in \mathcal{S}^+_{\lambda,b}$ such that $x \overset{(\gamma,r)}{\rightarrowtail} x'$, then:*

(1) *$\langle \sigma^i(\gamma), \mu_x\rangle = 0$, $p(x)\sigma^i(\gamma) = \sigma^i(\gamma)$ for $1 \leqslant i \neq r-n \leqslant r-1$;*
(2) *$p(x)\sigma^{r-n}(\gamma) = \sigma^{r-n}(\vartheta_\gamma - \sigma^n(\gamma))$ and $\langle p(x)\sigma^{r-n}(\gamma), \mu_x\rangle = 1$;*
(3) *$\langle p(x)(\vartheta_\gamma - \sigma^n(\gamma)), \mu_x\rangle \geqslant 1$.*

Moreover, $gI \sim_{\lambda,b} g\omega I$ for $g \in \mathbb{J}_{b,x}$, where $\omega = \gamma^\vee + \cdots \sigma^{r-1}(\gamma^\vee) \in \pi_1(M_J) \cong \Omega_J$.

Proof. Write $x' = x + \sigma^r(\gamma^\vee) - \sigma^{-r}(\sigma^r(\gamma^\vee))$. Then parts (1), (2), and (3) follow from [Nie18, Lemma 8.2] by using σ^{-1} instead of σ. Let $\tilde\theta = w_J\sigma^{r-1}(\gamma) + 1 \in \tilde\Phi^+$ and $\tilde\vartheta_\gamma = \vartheta_\gamma + 1 \in \tilde\Phi^+$. Note that $p(x)^{-1}w_J = w_J p(x)$ since $x \in \Omega_J$. By parts (1) and (2) we have $(x\sigma)^i(\tilde\theta) = \sigma^i(\tilde\theta) = w_J\sigma^{i+r-1}(\gamma) + 1$ for $1-n \leqslant i \leqslant 0$, and

$$(x\sigma)^i(\tilde\theta) = \sigma^{i+n-1}p(x)^{-1}w_J\sigma^{r-n}(\gamma) = \sigma^{i+n-1}w_Jp(x)\sigma^{r-n}(\gamma) = w_J\sigma^{i+r-1}(\vartheta_\gamma - \sigma^n(\gamma))$$

for $1-r \leqslant i \leqslant -n$. Define $g = g_{g,-\tilde\theta,x,r}$ for $g \in \mathbb{J}_{b,x}$. Then we have

$$g^{-1}b\sigma(g) \subseteq IU_{-w_J(\tilde\vartheta_\gamma)}xU_{-\sigma(\tilde\theta)}I \subseteq IxU_{-\sigma(\tilde\theta)}I \subseteq I\mathrm{Adm}(\lambda)I,$$

where the second inclusion follows from parts (1) and (3) that $\langle w_x(\vartheta_\gamma), \mu_x\rangle \geqslant 1$. Thus,

$$gI = g(0) \sim_{\lambda,b} g(\infty) = gs_{(x\sigma)^{1-r}(\tilde\theta)}\cdots s_{(x\sigma)^{-1}(\tilde\theta)}s_{\tilde\theta}I = g\omega u^{-1}I,$$

where $u \in W_0$ and $\omega = \gamma^\vee + \cdots + \sigma^{r-1}(\gamma^\vee) \in \pi_1(M_J) \cong \Omega_J$. It follows from Proposition 7.1 that $gI \sim_{\lambda,b} g\omega I$ as desired. □

Lemma 7.5. *Suppose $\mathcal{O} = \mathcal{O}_\xi$ for some $\xi \in C_{\lambda,b,x}$ and $\mathcal{O}$ is of type II. Assume $\mu_{x''} + \vartheta^\vee_\beta \npreceq \lambda$ for any $x'' \in \mathcal{S}^+_{\lambda,b}$ and $\beta \in \mathcal{O}$. If there do not exist $\gamma \in \mathcal{O}$, $1 \leqslant r \leqslant 2n-1$, and $x' \in \mathcal{S}^+_{\lambda,b}$ such that $x \overset{(\gamma,r)}{\rightarrowtail} x'$, then there exists $\alpha \in \mathcal{O}$ such that:*

(1) *$\langle \sigma^i(\alpha), \mu_x\rangle = 0$, $p(x)\sigma^i(\alpha) = \sigma^i(\alpha)$ for $1 \leqslant i \neq n \leqslant 2n-1$;*
(2) *$p(x)\sigma^n(\alpha) = \vartheta_\alpha - \alpha$ and $\langle w_J\sigma^n(\alpha), \mu_x\rangle = 1$;*
(3) *$\langle p(x)(\vartheta_\alpha), \mu_x + \alpha^\vee\rangle \geqslant 1$;*
(4) *$\langle p(x)(\vartheta_\alpha), \mu_x\rangle \geqslant 1$.*

As a consequence, $\omega_{\mathcal{O}} \in \mathcal{A}_{\lambda,b}$.

Proof. Statements (1), (2), and (3) follow from [Nie18, Lemmas 8.3 and 8.4]. Note that ϑ_α is J-anti-dominant. Thus, statement (4) follows from statement (1) and Lemma 5.1. By statements (3) and (4) we have

(a) $\quad \langle p(x)(\vartheta_\alpha), \mu_x\rangle \geqslant 1$, and either $\langle p(x)(\vartheta_\alpha), \mu_x\rangle \geqslant 2$ or $\langle p(x)(\vartheta_\alpha), \alpha^\vee\rangle \geqslant 0$.

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

Let $g \in \mathbb{J}_{b,x}$ and $\tilde{\theta} = w_J\sigma^{-1}(\alpha) + 1 \in \tilde{\Phi}^+$, and $\tilde{\vartheta} = w_J\sigma^{-1}(\vartheta_\alpha) + 1 \in \tilde{\Phi}^+$. By statements (1) and (2) we have $(x\sigma)^{1-n}(\tilde{\vartheta}) = \sigma^{1-n}(\tilde{\vartheta}) = w_J(\vartheta_\alpha)$ and

$$(x\sigma)^{-n}(\tilde{\theta}) = \sigma^{-1}p(x)^{-1}w_J\sigma^{-n}(\alpha) = \sigma^{-1}w_Jp(x)\sigma^{-n}(\alpha) = w_J\sigma^{-1}(\vartheta_\alpha - \alpha).$$

Define $g : \mathbb{P}^1 \to G(\breve{F})/I$ by

$$g(z) = gU_{-\tilde{\theta}}(z)\cdots {}^{(x\sigma)^{1-n}}U_{-\tilde{\theta}}(z)U_{-\tilde{\vartheta}}(cz^{1+q^{-n}})\cdots {}^{(x\sigma)^{1-n}}U_{-\tilde{\vartheta}}(cz^{1+q^{-n}})I,$$

where $c \in \mathcal{O}_{\breve{F}}^\times$ (as Φ is simply-laced) such that

$${}^{(x\sigma)^{-n}}U_{-\tilde{\theta}}(z)U_{-\tilde{\theta}}(z)U_{-\tilde{\vartheta}}(cz^{1+q^{-n}}) = U_{-\tilde{\theta}}(z)\,{}^{(x\sigma)^{-n}}U_{-\tilde{\theta}}(z).$$

Then by statement (1) we compute that

$$g^{-1}b\sigma(g) = U_{-w_J(\tilde{\vartheta}_\alpha)}xU_{-\sigma(\tilde{\theta})}I \subseteq IxU_{-\sigma(\tilde{\theta})}I \subseteq I\mathrm{Adm}(\lambda)I,$$

where the first inclusion follows from part (a) that

$${}^{x^{-1}}U_{-w_J(\tilde{\vartheta}_\alpha)}, [{}^{x^{-1}}U_{-w_J(\tilde{\vartheta}_\alpha)}, U_{-\sigma(\tilde{\theta})}] \subseteq I.$$

Thus, we have

$$gI = g(0) \sim_{\lambda,b} g(\infty) = g(s_{\tilde{\vartheta}}s_{\theta'})\cdots\sigma^{1-n}(s_{\tilde{\vartheta}}s_{\theta'})I = g\omega_{\mathcal{O}}u^{-1}I,$$

where $\theta' = (x\sigma)^{1-n}(\tilde{\vartheta}) \in \Phi$ and $u \in W_0$. By Proposition 7.1, $gI \sim_{\lambda,b} g\omega_{\mathcal{O}}I$ and $\omega_{\mathcal{O}} \in \mathcal{A}_{\lambda,b}$ as desired. □

7.2 The action of $\ker(\eta_G) \cap \mathbb{J}_b$

Now we have the following result.

PROPOSITION 7.6. *Let $\mathcal{O}$ be the σ-orbit of some element in $\cup_{x\in\mathcal{S}^+_{\lambda,b}} C_{\lambda,b,x}$. Then $\omega_{\mathcal{O}} \in \mathcal{A}_{\lambda,b}$.*

Proof. If $\mathcal{O}$ is of type I, the statement follows from Lemma 7.3. If $\mu_{x''} + \vartheta_\beta^\vee \preceq \lambda$ for some $x'' \in \mathcal{S}^+_{\lambda,b}$ and $\beta \in \mathcal{O}$, then we also have $\omega_{\mathcal{O}} = \omega_{\mathcal{O}_{\vartheta_\beta}} \in \mathcal{A}_{\lambda,b}$ since $\mathcal{O}_{\vartheta_\beta}$ is of type I. Assume $\mu_{x''} + \vartheta_\beta^\vee \npreceq \lambda$ for any $x'' \in \mathcal{S}^+_{\lambda,b}$ and $\beta \in \mathcal{O}$. If $\mathcal{O}$ is of type III, the statement is proved in § 8.2. Suppose $\mathcal{O}$ is of type II. By Lemma 7.5 we can assume that there exist $\gamma \in \mathcal{O}$, $1 \leqslant r \leqslant 2n-1$, and $x' \in \mathcal{S}^+_{\lambda,b}$ such that $x \overset{(\gamma,r)}{\to} x'$, and, hence, $x' \overset{(\sigma^r(\gamma),2n-r)}{\to} x$. If $n+1 \leqslant r \leqslant 2n-1$ (respectively, $1 \leqslant r \leqslant n$), we have $gI \sim_{\lambda,b} g\omega I$ by Lemma 7.4 (respectively, by Lemma 7.2(1) and (3)), where $\omega = \gamma^\vee + \cdots \sigma^{r-1}(\gamma^\vee) \in \pi_1(M_J) \cong \Omega_J$. Similarly, we have $g\omega \sim_{\lambda,b} g\omega\omega' I = g\omega_{\mathcal{O}}I$, where $\omega' = \sigma^r(\gamma^\vee) + \cdots + \sigma^{2n-1}(\gamma^\vee) \in \pi_1(M_J) \cong \Omega_J$. Thus, $gI \sim_{\lambda,b} g\omega_{\mathcal{O}}I$ and $\omega_{\mathcal{O}} \in \mathcal{A}_{\lambda,b}$ as desired. □

Proof of Proposition 2.6. First note that $(\mathbb{Z}\Phi^\vee/\mathbb{Z}\Phi_J^\vee)^\sigma$ is spanned by $\omega_{\mathcal{O}}$, where $\mathcal{O}$ ranges over σ-orbits of $\mathbb{S}_0$. Let $J \subseteq \mathbb{S}_0' \subseteq \mathbb{S}_0$ be such that $\omega_{\mathcal{O}} \in \mathcal{A}_{\lambda,b}$ for each σ-orbit of $\mathbb{S}_0'$. It suffices to show $\mathbb{S}_0' = \mathbb{S}_0$. Assume otherwise. Following the proof of [Nie18, Proposition 4.3, p. 1381], we can assume that Φ is simply-laced, and there exist $\alpha = \sigma^d(\alpha) \in \mathbb{S}_0 \setminus \mathbb{S}_0'$, $\vartheta = \sigma^d(\vartheta) \in \Phi^+$ such that $\vartheta^\vee - \alpha^\vee \in \mathbb{Z}\Phi^\vee_{\mathbb{S}_0'}$ and either: (b1) $\vartheta \in \cup_{x\in\mathcal{S}^+_{\lambda,b}} C_{\lambda,b,x}$; or (b2) $x \overset{(\beta,d)}{\to} x'$ and $x \overset{(\vartheta+\beta,d)}{\to} x'$ for some $x \in \mathcal{S}^+_{\lambda,b}$ and $\beta \in \Phi_{\mathbb{S}_0'} \setminus \Phi_J$ such that $x' = x - \beta^\vee + \sigma^d(\beta^\vee) \in \mathcal{S}^+_{\lambda,b}$ and $\vartheta + \beta \in \Phi^+$.

Note that $|\mathcal{O}_\alpha| = |\mathcal{O}_\vartheta| = d$ and $\omega_{\mathcal{O}_\alpha}^{-1}\omega_{\mathcal{O}_\vartheta} \in (\mathbb{Z}\Phi^\vee_{\mathbb{S}_0'}/\mathbb{Z}\Phi_J^\vee)^\sigma \subseteq \mathcal{A}_{\lambda,b}$. If part (b1) occurs, then $\omega_{\mathcal{O}_\vartheta} \in \mathcal{A}_{\lambda,b}$ by Proposition 7.6. Hence, $\omega_{\mathcal{O}} \in \mathcal{A}_{\lambda,b}$ and $\alpha \in \mathbb{S}_0'$, which is a contradiction. Suppose part (b2) occurs. Let $\omega = \beta^\vee + \cdots + \sigma^{d-1}(\beta^\vee) \in \pi_1(M_J) \cong \Omega_J$. Then

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

$\omega\omega_{\mathcal{O}_\vartheta} = (\beta+\vartheta)^\vee + \cdots + \sigma^{d-1}((\beta+\vartheta)^\vee) \in \pi_1(M_J) \cong \Omega_J$. We claim that

(a) $$g\omega I \sim_{\lambda,b} gI \sim_{\lambda,b} g\omega\omega_{\mathcal{O}_\vartheta} I \text{ for } g \in \mathbb{J}_{b,x}.$$

Given part (a) we have $g\omega I \sim_{\lambda,b} g\omega\omega_{\mathcal{O}_\vartheta} I$, and, hence, $\omega_{\mathcal{O}_\vartheta} \in \mathcal{A}_{\lambda,b}$, which is again a contradiction. Thus, $\mathbb{S}_0' = \mathbb{S}_0$ as desired.

It remains to show part (a). By symmetry, it suffices to show $gI \sim_{\lambda,b} g\omega I$. By switching x with x' we can assume $\beta \in \Phi^+ \setminus \Phi_J$ and β is J-anti-dominant and J-minuscule (see [Nie18, Lemma 6.6]). In particular, $\sigma^d(\beta) \in C_{\lambda,b,x}$. If $\mathcal{O}_\beta$ is of type I, it follows from Lemma 7.2. If $\mathcal{O}_\beta$ is of type III, it follows from Lemma 8.1. If $\mathcal{O}_\beta$ is of type II, by Lemmas 5.4 and 7.2 we have

$$\text{either } gI \sim_{\lambda,b} g\omega I \text{ or } g\omega I \sim_{\lambda,b} g\omega\omega' I = g\omega_{\mathcal{O}_\beta} I \text{ for } g \in \mathbb{J}_{b,x},$$

where $\omega' = \sigma^d(\beta^\vee) + \cdots + \sigma^{2d-1}(\beta^\vee) \in \pi_1(M_J)$. Note that $gI \sim_{\lambda,b} g\omega_{\mathcal{O}_\beta} I$ by Proposition 7.6. Thus, we always have $gI \sim_{\lambda,b} g\omega I$ as desired. Hence, part (a) is proved. □

8. The case when σ has order $3d$

In this section we handle the case when σ has order $3d$. We follow the strategy of [CKV15, § 4.7.7]. However, more details are involved. Note that in this case some/any connected component of $\mathbb{S}_0$ is of type D_4. Let $J = J_{\nu_G(b)}$.

8.1 Construction of affine lines

Let $\alpha, \beta \in \mathbb{S}_0$ such that $\langle \alpha, \beta^\vee \rangle = -1$ and $\beta = \sigma^d(\beta)$. Then the subset $\{\alpha, \sigma^d(\alpha), \sigma^{2d}(\alpha), \beta\}$ is a connected component of $\mathbb{S}_0$. In this subsection, we assume that $J = J_{\nu_G(b)} = \mathcal{O}_\beta$.

Let $x, x' \in \mathcal{S}_{\lambda,b}^+$ such that $x \overset{(\alpha,r)}{\to} x'$ for some J-anti-dominant root $\alpha \in \Phi^+ \setminus \Phi_J$ and $1 \leqslant r \leqslant 3d-1$. Let $\omega = \gamma^\vee + \cdots + \sigma^{r-1}(\gamma)^\vee \in \pi_1(M_J) \cong \Omega_J$.

Lemma 8.1. *If $1 \leqslant r \leqslant d$, then $gI \sim_{\lambda,b} gy^{-1}I$ for $g \in \mathbb{J}_{b,x}$ and some $y \in W_0^J \omega^{-1} W_J^a$.*

Proof. As in the proof Lemma 7.2, we can assume $x \overset{(\alpha,r)}{\rightarrowtail} x'$, and it suffices to show

$$U_{-(\alpha+\beta)-1} x U_{-\sigma^r(\alpha+\beta)-1} \subseteq I\mathrm{Adm}(\lambda)I.$$

Assume otherwise. Then $r = d$. Moreover, by Lemma 5.4(∗) we have $\langle \alpha+\beta, p(x)\sigma^d(\alpha+\beta)^\vee \rangle = -1$ (which implies $\langle \beta, \mu_x \rangle = 1$ and $p(x)\sigma^d(\alpha+\beta) = s_\beta(\sigma^d(\alpha)+\beta) = \sigma^d(\alpha)$) and

$$\langle \beta, \mu_x \rangle = \langle \alpha+\beta, \mu_x \rangle = -\langle \sigma^d(\alpha), \mu_x \rangle = 1, \text{ and } \mu_x \pm \delta^\vee \preceq \lambda,$$

where $\delta = \alpha + \beta + \sigma^d(\alpha)$. As δ is central for $J = \mathcal{O}_\beta$, by Lemma 5.3(2) we have

$$U_{-(\alpha+\beta)-1} x U_{-\sigma^r(\alpha+\beta)-1} \subseteq IU_{-(\delta+1)} xI \subseteq I\{s_{\delta+1}x, x\}I \subseteq I\mathrm{Adm}(\lambda)I,$$

which is a contradiction. □

Lemma 8.2. *Suppose $2d \leqslant r \leqslant 3d-1$ and the following conditions hold:*

(1) $\langle \alpha, \mu_x \rangle \geqslant 1$;
(2) *if $r = 2d$, then $\langle \sigma^d(\alpha), \mu_x \rangle = 0$;*
(3) *if $2d+1 \leqslant r \leqslant 3d-1$, then $\langle \sigma^r(\beta), \mu_x \rangle = 1$, $\langle \beta, \mu_x \rangle = 0$, and $\langle \sigma^i(\alpha), \mu_x \rangle = 0$ for $i \in \{r-d, r-2d, d, 2d\}$;*
(4) *$x\sigma^i(\alpha) = \sigma^i(\alpha)$ for $1 \leqslant i \leqslant r-1$ with $i \notin \{r-d, r-2d, d, 2d\}$.*

Then we have $gI \sim_{\lambda,b} gy^{-1}I$ for $g \in \mathbb{J}_{b,x}$ and some $y \in W_0^J \omega^{-1} W_J^a$.

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

Proof. Let $\tilde\theta = \sigma^{r-1}(\alpha+\beta)+1 \in \tilde\Phi^+$. Define $g = g_{g,-\tilde\theta,x,r}$ for $g \in \mathbb{J}_{b,x}$.

Case (1): $r = 2d$. By conditions (2) and (4) we have

$$g^{-1}b\sigma(g) \subseteq \begin{cases} IU_{-(\alpha+\beta+\sigma^d(\alpha))-1}xU_{-\sigma^r(\alpha+\beta)-1}I, & \text{if } \langle\beta,\mu_x\rangle = 1; \\ IU_{-(\alpha+\beta)-1}xU_{-\sigma^r(\alpha+\beta)-1}I, & \text{if } \langle\beta,\mu_x\rangle = 0; \end{cases}$$

By conditions (1) and (2), $\langle\alpha+\beta,\mu_x\rangle = \langle\alpha+\beta+\sigma^d(\alpha),\mu_x\rangle \geqslant \langle\beta,\mu_x\rangle + 1$, which means

$$g^{-1}b\sigma(g) \subseteq xU_{-\sigma^r(\alpha+\beta)-1}I \subseteq I\mathrm{Adm}(\lambda)I.$$

Thus, $g = g(0) \sim_{\lambda,b} g(\infty) = gsI$, where $s = \prod_{i=0}^{d-1} s_{\sigma^i(\alpha+\beta+\sigma^d(\alpha))+1}\prod_{i=0}^{d-1} s_{\sigma^i(\alpha)}$ if $\langle\beta,\mu_x\rangle = 1$, and $s = \prod_{i=0}^{2d-1} s_{\sigma^i(\alpha+\beta)+1}$ if $\langle\beta,\mu_x\rangle = 0$.

Case (2): $2d+1 \leqslant r \leqslant 3d-1$. Let $\vartheta = \alpha+\sigma^d(\alpha)+\sigma^{2d}(\alpha)+2\beta$. By conditions (3) and (4),

$$g^{-1}b\sigma(g) \subseteq IU_{-\vartheta-1}xU_{-\sigma^r(\alpha+\beta)-1}I \subseteq IxU_{-\sigma^r(\alpha+\beta)-1}I \subseteq I\mathrm{Adm}(\lambda)I,$$

which means $gI = g(0) \sim_{\lambda,b} g(\infty) = gsI$, where

$$s = \prod_{i=0}^{r-1} s_{\sigma^i(\vartheta)+1}s_{\sigma^i(\alpha+\beta)}s_{\sigma^{i+d}(\alpha)}\prod_{i=r}^{d-1} s_{\sigma^i(\alpha+\beta+\sigma^d(\alpha))+1}s_{\sigma^i(\alpha+\beta)}.$$

The proof is complete. □

The following two lemmas follow from the same construction as in Lemma 8.2.

LEMMA 8.3. *Assume $d+1 \leqslant r \leqslant 2d-1$ and the following conditions hold:*

(1) $\langle\beta,\mu_x\rangle = 0$ *and* $\langle\sigma^r(\beta),\mu_x\rangle \in \{0,1\}$;
(2) $\langle\sigma^d(\alpha),\mu_x\rangle = \langle\sigma^{r-d}(\alpha),\mu_x\rangle = 0$, *and* $\langle\alpha,\mu_x\rangle \geqslant 1$;
(3) $x\sigma^i(\alpha) = \sigma^i(\alpha)$ *for* $1 \leqslant i \leqslant r-1$ *with* $i \notin \{r-d, d\}$.

Then we have $gI \sim_{\lambda,b} gy^{-1}I$ for $g \in \mathbb{J}_{b,x}$ and some $y \in W_0^J\omega^{-1}W_J^a$.

LEMMA 8.4 [Nie18, Lemma 8.6]. *If $\langle\beta,\mu_x\rangle = 1$, $\langle\sigma^d(\alpha),\mu_x\rangle = \langle\sigma^{2d}(\alpha),\mu_x\rangle = 0$, $\langle\alpha,\mu_x\rangle \geqslant -1$, and $x\sigma^i(\alpha) = \alpha$ for $i \in \mathbb{Z}\setminus d\mathbb{Z}$, then $gI \sim_{\lambda,b} gy^{-1}I$ for $g \in \mathbb{J}_{b,x}$ and some $y \in W_0\omega_{\mathcal{O}_\alpha}^{-1}W_J^a$. Here $\omega_{\mathcal{O}_\alpha} = \alpha^\vee + \cdots \sigma^{3d-1}(\alpha^\vee) \in \pi_1(M_J) \cong \Omega_J$.*

LEMMA 8.5. *Let $x_1, x_2 \in \mathcal{S}_{\lambda,b}^+$, $\delta = \alpha+\beta+\sigma^{2d}(\alpha)$, and $1 \leqslant k \leqslant 3d-1$ such that $x_1 \overset{(\delta,k)}{\rightarrow} x_2$. Then we have $gI \sim_{\lambda,b} gy^{-1}I$ for $g \in \mathbb{J}_{b,x_1}$ and some $y \in W_0^J\omega^{-1}W_J^a$. Here $\omega = \delta^\vee + \cdots \sigma^{k-1}(\delta^\vee) \in \pi_1(M_J) \cong \Omega_J$.*

Proof. It follows from Lemma 7.3 by noticing that $\mathcal{O}_\delta$ is of type I. □

LEMMA 8.6. *Assume $d+1 \leqslant r \leqslant 2d-1$ and the following conditions hold:*

(1) $\langle\beta,\mu_x\rangle = 1$ *and* $\langle\sigma^r(\beta),\mu_x\rangle = 0$;
(2) $\langle\sigma^d(\alpha),\mu_x\rangle = -1$, $\langle\sigma^{r-d}(\alpha),\mu_x\rangle = 0$, $\langle\alpha,\mu_x\rangle \leqslant 0$, *and* $\langle\sigma^r(\alpha),\mu_x\rangle \leqslant -1$;
(3) $x\sigma^i(\alpha) = \sigma^i(\alpha)$ *for* $1 \leqslant i \leqslant r-1$ *with* $i \notin \{r-d, d\}$.

Then we have $\mathbb{J}_{b,x} \sim_{\lambda,b} \mathbb{J}_{b,x'}$.

Proof. Let $\delta = \alpha + \beta + \sigma^{2d}(\alpha)$. Assume $\mu_x - \delta^\vee \preceq \lambda$. By condition (2) we have

$$x \overset{(\delta,r)}{\rightarrow} x'' := x - \delta^\vee + \sigma^r(\delta^\vee) \overset{(\sigma^{r-d}(\alpha),3d-r)}{\rightarrow} x'.$$

Thus, $\mathbb{J}_{b,x} \sim_{\lambda,b} \mathbb{J}_{b,x''}$ by Lemma 8.5. It suffices to show $\mathbb{J}_{b,x''} \sim_{\lambda,b} \mathbb{J}_{b,x'}$. If $\langle \sigma^r(\sigma), \mu_{x''}\rangle \leqslant -1$, then

$$x'' \overset{(\sigma^{r-d},d)}{\rightarrow} x'' - \sigma^{r-d}(\alpha^\vee) + \sigma^r(\alpha^\vee) \overset{(\sigma^r(\alpha),2d-r)}{\rightarrow} x',$$

and the statement follows from Lemma 8.1 that $\mathbb{J}_{b,x''} \sim_{\lambda,b} \mathbb{J}_{b,x'}$. Otherwise, by condition (2) we have $\langle \sigma^r(\alpha), \mu_x\rangle = -1$, that is, $\langle \sigma^r(\sigma), \mu_{x''}\rangle = 0$. The statement follows from Lemma 8.3 that $\mathbb{J}_{b,x''} \sim_{\lambda,b} \mathbb{J}_{b,x'}$. Let $l = \min\{r+1 \leqslant i \leqslant 2d-1; \langle \sigma^i(\alpha), \mu_x\rangle \neq 0\}$. If $\langle \sigma^l(\alpha), \mu_x\rangle \geqslant 1$, then

$$x'' \overset{(\sigma^l(\alpha),2d-1)}{\rightarrow} x'' - \sigma^l(\alpha^\vee) + \sigma^{2d}(\alpha^\vee) \overset{(\sigma^{r-d}(\alpha),l+d-r)}{\rightarrow} x',$$

and the statement follows from Lemmas 8.1 and 8.3. If $\langle \sigma^l(\alpha), \mu_x\rangle \leqslant -1$, then

$$x'' \overset{(\sigma^{r-d}(\alpha),k+d-r)}{\rightarrow} x'' - \sigma^{r-d}(\alpha^\vee) + \sigma^l(\alpha^\vee) \overset{(\sigma^l(\alpha),2d-l)}{\rightarrow} x',$$

and the statement also follows from Lemmas 8.1 and 8.3.

Now we assume $\mu_x - \delta^\vee \npreceq \lambda$, which means (as $\mu_x - \alpha^\vee - \beta^\vee = \mu_{x-\alpha^\vee} \preceq \lambda$) that

(a) $$\langle \sigma^{2d}(\alpha), \mu_x\rangle \leqslant -1.$$

If $\langle \sigma^{r+d}(\alpha), \mu_x\rangle \geqslant 1$, then we have

$$x \overset{(\sigma^{r-d}(\delta),d)}{\rightarrow} x - \sigma^{r+d}(\alpha)^\vee + \sigma^r(\alpha)^\vee \overset{(\sigma^{r+d}(\alpha),2d-r)}{\leftarrow} x',$$

and the statement follows from Lemmas 8.5 and 8.1. Thus, we assume

(b) $$\langle \sigma^{r+d}(\alpha), \mu_x\rangle \leqslant 0.$$

By parts (a), (b), (1), and (2), we have

$$\sum_{i\in\{r-d,r,r+d,0,d,2d\}} \langle \sigma^i(\alpha), \mathrm{pr}_J(\mu_x)\rangle < 0.$$

By Lemma 5.1, there exists $r+1 \leqslant k \leqslant 3d-1$ with $k \notin \{2d, r+d\}$ such that

(c) $$k = \min\{r+1 \leqslant i \leqslant 3d-1; \langle \sigma^i(\alpha), \mu_x\rangle \geqslant 1\}.$$

Suppose $\langle \sigma^j(\alpha), \mu_x\rangle \leqslant -1$ for some $r+1 \leqslant j \leqslant 3d-1$ with $j \notin \{2d, k+d, k-d, r+d\}$. Let

$$z = x - \sigma^{k_1}(\delta)^\vee + \sigma^{j_1}(\delta)^\vee,\ z' = x' - \sigma^{k_1}(\delta)^\vee + \sigma^{j_1}(\delta)^\vee \in \mathcal{S}^+_{\lambda,b},$$

where $k_1 = k+d$ if $k > 2d$ and $k_1 = k$ otherwise, and j_1 is defined in the same way. By Lemma 8.5, we have $\mathbb{J}_{b,x} \sim_{\lambda,b} \mathbb{J}_{b,z}$ and $\mathbb{J}_{b,x'} \sim_{\lambda,b} \mathbb{J}_{b,z'}$. Moreover, there exist $z_1, z_2 \in \mathcal{S}^+_{\lambda,b}$ such that

$$z \overset{(\alpha,k-2d)}{\rightarrow} z_1 \overset{(\sigma^{k-2d}(\alpha),2d+r-k)}{\rightarrow} z' \text{ if } r+d+1 \leqslant k \leqslant 3d-1;$$

$$z \overset{(\alpha,k-d)}{\rightarrow} z_1 \overset{(\sigma^{k-2d}(\alpha),d+r-k)}{\rightarrow} z',\ \text{if } r+1 \leqslant k \leqslant 2d-1;$$

$$z \overset{(\alpha,k-2d)}{\rightarrow} z_1 \overset{(\sigma^{k-d}(\alpha),d+r-k)}{\rightarrow} z_2 \overset{(\sigma^{k-2d}(\alpha),d)}{\rightarrow} z',\ \text{if } 2d+1 \leqslant k \leqslant r+d-1.$$

By Lemma 8.1, $\mathbb{J}_{b,z} \sim_{\lambda,b} \mathbb{J}_{b,z'}$ and the statement follows. Thus, we can assume

(d) $$\langle \sigma^i(\alpha), \mu_x\rangle = 0 \text{ for } 1 \leqslant i \leqslant k-1 \text{ with } i \notin \{r-d, r, r+d, d, 2d\}.$$

As $\langle \sigma^{r-d}(\alpha), \mu_{x'}\rangle = -1$, we have $y := x' + \sigma^{r-d}(\alpha)^\vee - \sigma^k(\alpha)^\vee \in \mathcal{S}^+_{\lambda,b}$.

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

Case (1): $r+1 \leqslant k \leqslant 2d-1$. Then

$$x \overset{(\sigma^k(\delta),r-k)}{\to} x - \sigma^k(\delta)^\vee + \sigma^r(\delta)^\vee \overset{(\alpha,k-d)}{\to} y \overset{(\sigma^{r-d}(\alpha),k-r+d)}{\to} x'.$$

By Lemma 8.1, it suffices to show $\mathbb{J}_{b,y} \sim_{\lambda,b} \mathbb{J}_{b,x'}$. If $\langle \sigma^r(\alpha), \mu_x\rangle \leqslant -2$, that is, $\langle \sigma^r(\alpha), \mu_y\rangle \leqslant -1$, it follows from that

$$y \overset{(\sigma^{r-d}(\alpha),d)}{\to} x' + \sigma^r(\alpha)^\vee - \sigma^k(\alpha)^\vee \overset{(\sigma^r(\alpha),k-r)}{\to} x'.$$

Otherwise, we have $\langle \sigma^r(\alpha), \mu_x\rangle = -1$ by (2), that is, $\langle \sigma^r(\alpha), \mu_y\rangle = 0$. Then the statement follows from Lemma 8.3.

Case (2): $2d+1 \leqslant k \leqslant 3d-1$. Then we have

$$x \overset{(\sigma^{k+d}(\delta),r-k-d)}{\to} x - \sigma^{k+d}(\delta)^\vee + \sigma^r(\delta)^\vee \overset{(\alpha,k-2d)}{\to} y \overset{(\sigma^{r-d}(\alpha),k-r+d)}{\to} x'.$$

Again, it suffices to show $\mathbb{J}_{b,y} \sim_{\lambda,b} \mathbb{J}_{b,x'}$. If $k \leqslant r+d-1$, it follows similarly as in case (1). Otherwise, it follows from that

$$\begin{aligned} y &\overset{(\sigma^{k-d}(\alpha),r+2d-k)}{\to} y - \sigma^{k-d}(\alpha)^\vee + \sigma^{r+d}(\alpha)^\vee \overset{(\sigma^{k-2d}(\alpha),d)}{\to} y - \sigma^{k-2d}(\alpha)^\vee + \sigma^{r+d}(\alpha)^\vee \\ &\overset{(\sigma^{r-d}(\alpha),k-r-d)}{\to} y - \sigma^{r-d}(\alpha)^\vee + \sigma^{r+d}(\alpha)^\vee \overset{(\sigma^{r+d}(\alpha),k-r-d)}{\to} x', \end{aligned}$$

where the first arrow follows from part (b) that $\langle \sigma^{r+d}(\alpha), \mu_y\rangle = \langle \sigma^{r+d}(\alpha), \mu_x\rangle - 1 \leqslant -1$. □

8.2 Proofs of the main results

Recall that $J = J_{\nu_G(b)}$. We are ready to finish the proofs when σ is of order $3d$.

Proof of Proposition 2.4. Let $x, x' \in \mathcal{S}^+_{\lambda,b}$. To show $\mathbb{J}_{b,x} \sim_{\lambda,b} \mathbb{J}_{b,x'}$, by Proposition 5.6 we can assume $x \overset{(\gamma,r)}{\rightarrowtail} x'$ for some $1 \leqslant r \leqslant 2d-1$ and $\gamma \in \Phi^+ \setminus \Phi_J$ with $\gamma^\vee$ is J-anti-dominant and J-minuscule. In particular, $\sigma^r(\gamma) \in C_{\lambda,b,x}$. If $\mathcal{O}_\gamma$ is of type I, the statement follows from Lemma 7.2 and Proposition 4.11. Otherwise, we can assume $J = \mathcal{O}_\beta$ and $\gamma = \alpha$ as in § 8.1. If $1 \leqslant r \leqslant d$, the statement follows from Lemma 8.1 and Proposition 4.11. Otherwise, by the proof of [Nie18, Proposition 6.8, p. 1378, Case 2], either Lemma 8.3 or Lemma 8.6 applies. Thus, the statement also follows. □

Proof of Proposition 7.6. As $\mathcal{O}$ is of type III, we can assume $\mathcal{O} = \mathcal{O}_\alpha$ and $J = \mathcal{O}_\beta$, where α, β are as in § 8.1. Again we can assume that $\mu_{x''} + \vartheta^\vee_\gamma \npreceq \lambda$ for any $x'' \in \mathcal{S}^+_{\lambda,b}$ and $\gamma \in \mathcal{O}$. If there do not exist $\gamma \in \mathcal{O}$, $1 \leqslant r \leqslant 3d-1$, and $x' \in \mathcal{S}^+_{\lambda,b}$ such that $x \overset{(\gamma,r)}{\rightarrowtail} x'$, by [Nie18, Lemma 8.6] the statement follows from Lemma 8.2 and Proposition 7.1. Assume otherwise. Then there exists $x_i \in \mathcal{S}^+_{\lambda,b}$, $\gamma_i \in \mathcal{O}$, and $1 \leqslant r_i \leqslant 3d-1$ for $1 \leqslant i \leqslant m$ such that $\omega_{\mathcal{O}} = \sum_{i=1}^m \sum_{j=0}^{r_i-1} \sigma^j(\gamma_i^\vee) \in \pi_1(M_J)$ and

$$x = x_0 \overset{(\gamma_1,r_1)}{\to} x_1 \overset{(\gamma_2,r_1)}{\to} \cdots \overset{(\gamma_m,r_m)}{\to} x_m = x.$$

If $d+1 \leqslant r_i \leqslant 2d-1$, then either Lemma 8.3 or Lemma 8.6 occurs. If for each $1 \leqslant i \leqslant m$ we have either $r_i \leqslant d$ or $2d \leqslant r_i \leqslant 3d-1$ or Lemma 8.3 (for $(x, x', \alpha, r) = (x_{i-1}, x_i, \gamma_i, r_i)$) occurs, it follows that $\omega_{\mathcal{O}} \in \mathcal{A}_{\lambda,b}$ by Lemmas 8.1, 8.2, and 8.3 and Proposition 7.1. Otherwise, by the proof of [Nie18, Proposition 6.8, p. 1378, Case 2], there exists $1 \leqslant i \leqslant m$ such that the situation of Lemma 8.6 occurs (for $(x, x', \alpha, r) = (x_{i-1}, x_i, \gamma_i, r_i)$).

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

Let x, x', α, r be as in Lemma 8.6. If $\langle \sigma^{r+d}(\alpha), \mu_x\rangle \leqslant 0$, then we have $\langle \sigma^r(\vartheta_\alpha), \mu_x\rangle \leqslant -1$, which contradicts our assumption. Thus, $\langle \sigma^{r+d}(\alpha), \mu_x\rangle \geqslant 1$, and, hence,

$$x \overset{(\sigma^{d+r}(\alpha), 3d-r)}{\rightarrow} y := x - \sigma^{r+d}(\alpha)^\vee + \sigma^d(\alpha)^\vee \overset{(\sigma^d(\alpha), r)}{\rightarrow} x.$$

Then it suffices to show that

(a) $$g_2 I \sim_{\lambda,b} g_2\omega_2 I \text{ for } g_2 \in \mathbb{J}_{b,y};$$

(b) $$g_1 I \sim_{\lambda,b} g_1\omega_1 I \text{ for } g_1 \in \mathbb{J}_{b,x},$$

where $\omega_1 = \sigma^{r+d}(\alpha)^\vee + \cdots + \sigma^{4d-1}(\alpha)^\vee, \omega_2 = \sigma^d(\alpha)^\vee + \cdots + \sigma^{r+d-1}(\alpha)^\vee \in \pi_1(M_J) \cong \Omega_J$.

First we show part (a). Note that $\langle \sigma^r(\alpha), \mu_y\rangle = \langle \sigma^r(\alpha), \mu_x\rangle \leqslant -1$. We have

$$y \overset{(\sigma^d(\alpha), r-d)}{\rightarrow} y - \sigma^d(\alpha)^\vee + \sigma^r(\alpha)^\vee \overset{(\sigma^r(\alpha), d)}{\rightarrow} x,$$

and part (a) follows from Lemma 8.1 and Proposition 7.1.

Now we show part (b). If $\langle \alpha, \mu_x\rangle \leqslant -1$, the statement follows from that

$$x \overset{(\sigma^{r+d}(\alpha), 2d-r)}{\rightarrow} x - \sigma^{r+d}(\alpha) + \alpha^\vee \overset{(\alpha, d)}{\rightarrow} y.$$

Thus, we can assume $\langle \alpha, \mu_x\rangle = 0$. If $\langle \sigma^i(\alpha), \mu_x\rangle = 0$ for $r + d + 1 \leqslant i \leqslant 3d - 1$, it follows from Lemma 8.3. Otherwise, let

$$k = \max\{r + d + 1 \leqslant i \leqslant 3d - 1; \langle \sigma^i(\alpha), \mu_x\rangle \neq 0\}.$$

If $\langle \sigma^k(\alpha), \mu_x\rangle = -1$, then $\langle \sigma^{k-d}(\alpha), \mu_x\rangle \geqslant 1$ since $\langle \sigma^k(\vartheta_\alpha), \mu_x\rangle \geqslant 0$, which means

$$x \overset{(\sigma^{k-d}(\delta), 2d)}{\rightarrow} x_1 := x + \sigma^k(\alpha)^\vee - \sigma^{k-d}(\alpha)^\vee$$

$$y \overset{(\sigma^{k-d}(\delta), 2d)}{\rightarrow} y_1 := y + \sigma^k(\alpha)^\vee - \sigma^{k-d}(\alpha)^\vee.$$

By Lemma 7.3, we have

$$g_1 I \sim_{\lambda,b} g_1\omega' I \text{ for } g_1 \in \mathbb{J}_{b,x}, \quad g_2 I \sim_{\lambda,b} g_2\omega' I \text{ for } g_2 \in \mathbb{J}_{b,y},$$

where $\omega' = \sigma^{k-d}(\delta^\vee) + \cdots + \sigma^{k+d}(\delta^\vee) \in \pi_1(M_J) \cong \Omega_J$. Thus, we can replace the pair (x, y) with (x_1, y_1) so that $\langle \sigma^k(\alpha), \mu_x\rangle \geqslant 1$. Then

$$x \overset{(\sigma^k(\alpha), 4d-k)}{\rightarrow} x - \sigma^k(\alpha)^\vee + \sigma^d(\alpha)^\vee \overset{(\sigma^{r+d}(\alpha), k-r-d)}{\rightarrow} y,$$

and part (b) follows from Lemmas 8.3 and 8.1 and Proposition 7.1. □

Acknowledgements

We are grateful to X. He for his detailed comments which greatly improved the exposition of the paper. We would like to thank M. Rapoport for bringing his joint work [PR21] with G. Pappas to our attention. We are also grateful to R. Zhou for pointing out an application (Corollary 0.5) to the Langlands–Rapoport conjecture. Finally, we thank the anonymous reviewer for his/her careful reading of the paper and many helpful comments and suggestions.

The author is partially supported by National Key R&D Program of China, No. 2020YFA0712600, CAS Project for Young Scientists in Basic Research, Grant No. YSBR-003, and NSFC grants Nos. 11922119, 12288201 and 12231001.

Conflicts of Interest

None.

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

Appendix A. Distinct elements in $\mathrm{Adm}(\lambda)$

In this appendix, we study the distinct elements defined in § 3.

A.1

First we recall the following lemmas.

Lemma A.1. *Let $s, s' \in \mathbb{S}^a$ and $\tilde{w} \in \tilde{W}$ such that $\ell(s\tilde{w}) = \ell(\tilde{w}s')$ and $\ell(s\tilde{w}s') = \ell(\tilde{w})$. Then $\tilde{w} = s\tilde{w}s'$.*

Lemma A.2 ([CN20, Lemmas 1.8 and 1.9] and [Hai01, Lemma 4.5]). *Let $s \in \mathbb{S}^a$ and $\tilde{w} \in \mathrm{Adm}(\lambda)$ with $\lambda \in Y$ such that $\tilde{w} < s\tilde{w}$. Then we have:*

(1) *$\tilde{w}s \in \mathrm{Adm}(\lambda)$ if $\tilde{w}s < s\tilde{w}s$;*
(2) *$\tilde{w}s = s\tilde{w}$ if $\tilde{w}s \notin \mathrm{Adm}(\lambda)$;*
(3) *$s\tilde{w}s \in \mathrm{Adm}(\lambda)$ if $\ell(s\tilde{w}s) = \ell(\tilde{w})$.*

Lemma A.3. *Let $\tilde{w} \notin \mathrm{Adm}(\lambda)$ and $s \in \mathbb{S}^a$ such that $\tilde{w}s > \tilde{w}$. Then $s\tilde{w}s \notin \mathrm{Adm}(\lambda)$.*

Proof. Assume $s\tilde{w}s \in \mathrm{Adm}(\lambda)$, then $s\tilde{w}s < \tilde{w}s$ and, hence, $\ell(s\tilde{w}s) = \ell(\tilde{w})$. By Lemma A.2(3), we have $\tilde{w} \in \mathrm{Adm}(\lambda)$, contradicting the assumption that $\tilde{w} \notin \mathrm{Adm}(\lambda)$. □

A.2

Fix $\lambda \in Y^+$. Let $R \subseteq \mathbb{S}_0$ and $\tilde{w} \in \mathrm{Adm}(\lambda)$. We say $\tilde{w}$ is left R-distinct (respectively, right R-distinct) if $s\tilde{w} \notin \mathrm{Adm}(\lambda)$ (respectively, $\tilde{w}s \notin \mathrm{Adm}(\lambda)$) for all $s \in R$. Let w_R denote the longest element of W_R.

For a reflection $s \in W_0$ we denote by $\alpha_s \in \Phi^+$ the corresponding simple root.

Lemma A.4. *Let $R \subseteq \mathbb{S}_0$ be commutative or of type A_2. Let $\tilde{w} \in \mathrm{Adm}(\lambda)$ be right (respectively, left) R-distinct. Let $u, u' \in W_R$ with $\ell(u') \leqslant \ell(u)$. Then $u'\tilde{w}u^{-1} \in \mathrm{Adm}(\lambda)$ (respectively, $u^{-1}\tilde{w}u'$) if and only if $u = u'$. As a consequence, $w_R\tilde{w}w_R \in \mathrm{Adm}(\lambda)$ is left (respectively, right) R-distinct.*

Proof. By symmetry, it suffices to handle the case when $\tilde{w}$ is right R-distinct. Suppose the 'only if' part is true, we show the 'if' part, that is, $u\tilde{w}u^{-1} \in \mathrm{Adm}(\lambda)$ for $u \in W_R$. We argue by induction on $\ell(u)$. If $u = 1$, the statement is true. Let $u = su_1 > u_1$ with $u_1 \in W_R$ and $s \in R$. We assume $u_1\tilde{w}u_1^{-1} \in \mathrm{Adm}(\lambda)$ by induction hypothesis. It remains to show that $u\tilde{w}u^{-1} \in \mathrm{Adm}(\lambda)$. Otherwise, we have $\ell(u\tilde{w}u^{-1}) = \ell(u_1\tilde{w}u_1^{-1}) + 2$ and $u_1\tilde{w}u^{-1} \in \mathrm{Adm}(\lambda)$ by Lemma A.2(1) and (3), which contradicts the 'only if' part.

Now we show the 'only if' part. Note that $\tilde{w} \in \tilde{W}^R$, see § 1.1.

Case (1): R is of type $A_1 \times A_1$ or A_2. Without loss of generality, we can assume $R = \{s_1, s_2\}$ is of type A_2. By symmetry, it suffices to consider the following cases.

Suppose $s\tilde{w}s' \in \mathrm{Adm}(\lambda)$. Then $s\tilde{w}s' < \tilde{w}s'$ and $s\tilde{w}s's \notin \mathrm{Adm}(\lambda)$ (see Lemma A.3). By Lemma A.2(2) we have $s\tilde{w}s'(\alpha_s) = \alpha_s$, that is, $\tilde{w}(\alpha_s + \alpha_{s'}) = -\alpha_s$. This is impossible since $\tilde{w} \in \tilde{W}^R$.

Suppose $s\tilde{w}ss' \in \mathrm{Adm}(\lambda)$. Then $s\tilde{w}ss' < s\tilde{w}s'$ (as $s\tilde{w}s' \notin \mathrm{Adm}(\lambda)$), that is, $s\tilde{w}s'(s'(\alpha_s)) = s\tilde{w}(\alpha_s) \in \tilde{\Phi}^+$. Since $\tilde{w}(\alpha_s) \in \tilde{\Phi}^-$ (as $\tilde{w} \in \tilde{W}^R$), we have $\tilde{w}(\alpha_s) = \alpha_s$. This means $s\tilde{w}ss' = \tilde{w}s' \notin \mathrm{Adm}(\lambda)$, a contradiction. Note that $s\tilde{w}s's \notin \mathrm{Adm}(\lambda)$ by Lemma A.3.

Suppose $ss'\tilde{w}ss' \in \mathrm{Adm}(\lambda)$. Then $ss'\tilde{w}ss' < s'\tilde{w}ss'$. If $s'\tilde{w}ss' < s'\tilde{w}ss's$, then $ss'\tilde{w}ss's \notin \mathrm{Adm}(\lambda)$ by Lemma A.3. Otherwise, by Lemma A.1 we have $s'\tilde{w}ss's = \tilde{w}ss'$ (since $\tilde{w}ss' < \tilde{w}ss's$) and hence $ss'\tilde{w}ss's = s\tilde{w}ss' \notin \mathrm{Adm}(\lambda)$. Thus, we always have $ss'\tilde{w}ss's \notin \mathrm{Adm}(\lambda)$. By Lemma A.2 we have $ss'\tilde{w}ss'(\alpha_s) = \alpha_s$, that is, $\tilde{w}(\alpha_{s'}) = -(\alpha_s + \alpha_{s'})$, which is impossible as $\tilde{w} \in \tilde{W}^R$.

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

Suppose $ss'\tilde{w}ss's \in \mathrm{Adm}(\lambda)$. Then $ss'\tilde{w}ss's < s'\tilde{w}ss's$. Since $ss'\tilde{w}ss' \notin \mathrm{Adm}(\lambda)$, by Lemma A.2 we have $ss'\tilde{w}ss's(\alpha_s) = \alpha_s$, that is, $\tilde{w}(\alpha_{s'}) = \alpha_s + \alpha_{s'}$. This means $ss'\tilde{w}ss's = s'\tilde{w}ss' \in \mathrm{Adm}(\lambda)$, a contradiction.

Case (2): R is commutative. We argue by induction on $|R|$ and $\ell(u')$. If $R = \emptyset$ or $u' = 1$, the statement is trivial. Assume $\ell(u') \geqslant 1$. Let $s \in R$ such that $su' < u'$. If $su < u$, then $s\tilde{w}s \in \mathrm{Adm}(\lambda)$ is right $(R \setminus \{s\})$-distinct by case (1), and, hence, the statement follows by induction hypothesis. Assume $su > u$. We need to show that $u'\tilde{w}u^{-1} \notin \mathrm{Adm}(\lambda)$. By the induction hypothesis and the previous discussion we have $su'\tilde{w}u^{-1}, u'\tilde{w}u^{-1}s \notin \mathrm{Adm}(\lambda)$. Applying Lemma A.2(2) we have $u'\tilde{w}u^{-1}(\alpha_s) = \alpha_s$, that is, $\tilde{w}(\alpha_s) = -\alpha_s$ (as R is commutative), which is impossible since $\tilde{w} \in \tilde{W}^R$. □

Lemma A.5. *Let $\tilde{w} \in \mathrm{Adm}(\lambda)$ and $s \in \mathbb{S}_0$ such that $s\tilde{w}s \in \mathrm{Adm}(\lambda)$ and $s\tilde{w} \notin \mathrm{Adm}(\lambda)$. Let $\alpha \in \Phi^+ \setminus \{\alpha_s\}$ such that $\tilde{w}s_\alpha \in \mathrm{Adm}(\lambda)$. Then $s\tilde{w}s_\alpha s \in \mathrm{Adm}(\lambda)$*

Proof. Suppose $s\tilde{w}s_\alpha s \notin \mathrm{Adm}(\lambda)$, then $s\tilde{w}s_\alpha \in \mathrm{Adm}(\lambda)$ by Lemma A.2. As $s\tilde{w} \notin \mathrm{Adm}(\lambda)$, we have $s\tilde{w}(\alpha) \in \tilde{\Phi}^+$. On the other hand, as $s(\alpha) \in \Phi^+$, $s\tilde{w}s_\alpha s \notin \mathrm{Adm}(\lambda)$ and $s\tilde{w}s \in \mathrm{Adm}(\lambda)$, we have $s\tilde{w}(\alpha) \in \tilde{\Phi}^-$, which is a contradiction. □

Corollary A.6. *Let R be as in Lemma A.4. Let $\tilde{w} \in \mathrm{Adm}(\lambda)$ be left R-distinct. Let $\alpha \in \Phi^+ \setminus \Phi_R$ such that $\tilde{w}s_\alpha \in \mathrm{Adm}(\lambda)$. Then $u\tilde{w}s_\alpha u^{-1} \in \mathrm{Adm}(\lambda)$ for $u \in W_R$.*

Proof. We argue by induction on $\ell(u)$. If $u = 1$, the statement follows by assumption. Supposing it is true for u_1, that is, $u_1\tilde{w}u_1^{-1}s_{u_1(\alpha)} = u_1\tilde{w}s_\alpha u_1^{-1} \in \mathrm{Adm}(\lambda)$, we show it is also true for $u = su_1 > u_1$ with $s \in R$. By Lemma A.4 we have $u_1\tilde{w}u_1^{-1}, su_1\tilde{w}u_1^{-1}s \in \mathrm{Adm}(\lambda)$ and $su_1\tilde{w}u_1^{-1} \notin \mathrm{Adm}(\lambda)$. Moreover, we have $u_1(\alpha) \neq \alpha_s$ since $\alpha \in \Phi^+ \setminus \Phi_R$. Thus, $u\tilde{w}s_\alpha u^{-1} = su_1\tilde{w}u_1^{-1}s_{u_1(\alpha)}s \in \mathrm{Adm}(\lambda)$ by Lemma A.5. □

Appendix B. Proof of Lemma 6.5

We start with a general lemma on root systems.

Lemma B.1. *Let $\mu \in Y$, $\lambda \in Y^+$ and $\alpha \in \Phi^+$ such that $\mu \preceq \lambda$, $\mu + \alpha^\vee \leqslant \lambda$, and $\mu + \alpha^\vee \npreceq \lambda$. Then there exists $\beta \in \Phi^+$ such that $\langle \beta, \mu + \alpha^\vee \rangle \leqslant -2$, and either $\mu + \beta^\vee \preceq \lambda$ or $\mu + \alpha^\vee + \beta^\vee \leqslant \lambda$.*

Proof. We argue by induction on $\mu + \alpha^\vee$ via the partial order $\leqslant$. If $\mu + \alpha^\vee \in Y^+$, then $\mu + \alpha^\vee \preceq \lambda$, contradicting our assumption. Thus, there exists $\beta \in \mathbb{S}_0$ such that $\langle \beta, \mu + \alpha^\vee \rangle \leqslant -1$ and, hence, $\mu + \alpha^\vee + \beta^\vee \leqslant \lambda$ (by [Gas10, Proposition 2.2]). If $\langle \beta, \mu + \alpha^\vee \rangle \leqslant -2$, the statement follows. Assume $\langle \beta, \mu + \alpha^\vee \rangle = -1$. Then $\mu + \alpha^\vee < s_\beta(\mu + \alpha^\vee) \npreceq \lambda$. If $\beta = \alpha$, then $\langle \alpha, \mu \rangle = -3$ and $\mu + \alpha^\vee \preceq \mu \preceq \lambda$, a contradiction. Thus, $\beta \neq \alpha$ and $s_\beta(\alpha) \in \Phi^+$. By the induction hypothesis, for the pair $(s_\beta(\mu), s_\beta(\alpha))$ there exists $\gamma \in \Phi^+$ such that

$$\langle \gamma, s_\beta(\mu + \alpha^\vee) \rangle = \langle s_\beta(\gamma), \mu + \alpha^\vee \rangle \leqslant -2$$

(which means $\beta \neq \gamma$ and $s_\beta(\gamma) \in \Phi^+$), and either $s_\beta(\mu) + \gamma^\vee \preceq \lambda$ or $s_\beta(\mu + \alpha^\vee) + \gamma^\vee \leqslant \lambda$. If the former case occurs, we have $\mu + s_\beta(\gamma^\vee) \preceq \lambda$, and the statement follows. Otherwise, $\langle s_\beta(\gamma), \mu \rangle \geqslant 0$ and the latter case occurs. In particular, $\langle s_\beta(\gamma), \alpha^\vee \rangle \leqslant -2$, and, hence, means γ is a long root. Thus, we have

$$\mu + \alpha^\vee + s_\beta(\gamma^\vee) \leqslant \mu + \alpha^\vee + \gamma^\vee + \beta^\vee = s_\beta(\mu + \alpha^\vee) + \gamma^\vee \leqslant \lambda,$$

and the statement also follows. □

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

Proof of Lemma 6.5. By [Nie18, Lemma 3.3], there exists $x \in \mathcal{S}_{\lambda,b}^+$ such that μ_x is weakly dominant, that is, $\langle \delta, \mu_x \rangle \geqslant -1$ for $\delta \in \Phi^+$. As (λ, b) is Hodge–Newton irreducible, $\lambda^\diamond - \nu_x \in \sum_{\alpha \in \mathbb{S}_0} \mathbb{R}_{>0} \alpha^\vee$. As $p(x) \in W_{J_1}$, we have $\mu_x^\diamond - \nu_x \in \mathbb{R}\Phi_{J_1}^\vee$. Note that $\mu_x \leqslant \lambda$. Thus, there exists $\alpha \in K = \sigma(K) \subseteq J_0$ such that $\mu_x + \alpha^\vee \leqslant \lambda$. We show that

(a) (a1) there is $\xi \in \Phi^+ \setminus \Phi_J$ such that $\langle \alpha, \xi^\vee \rangle \leqslant -1$ and $\mu + \xi^\vee \preceq \lambda$;

(a2) if, moreover, Φ is simply laced then, $\langle \xi, \mu_x \rangle = -1$.

By assumption, $\mu_x + \alpha^\vee \npreceq \lambda$. By Lemma B.1, there exists $\zeta \in \Phi^+$ such that $\langle \zeta, \mu_x + \alpha^\vee \rangle \leqslant -2$, and either $\mu_x + \zeta^\vee \preceq \lambda$ or $\mu_x + \alpha^\vee + \zeta^\vee \leqslant \lambda$. As μ_x is weakly dominant, we have: (i) $\langle \zeta, \alpha^\vee \rangle \leqslant \langle \zeta, \mu_x \rangle = -1$; (ii) $\langle \zeta, \alpha^\vee \rangle \leqslant -2$ and $\langle \zeta, \mu_x \rangle = 0$; or (iii) $\langle \zeta, \alpha^\vee \rangle = -3$ and $\langle \zeta, \mu_x \rangle = 1$. Take $\xi = \zeta$ if choice (i) occurs. Assume choice (ii) or (iii) occurs. Then Φ is non-simply-laced and $\langle \alpha, \zeta^\vee \rangle = -1$. If $\mu_x + \zeta^\vee \preceq \lambda$, take $\xi = \zeta$. Otherwise, $\mu_x + \zeta^\vee \leqslant \lambda$ is not weakly dominant (by [Gas10, Proposition 2.2]). Thus, there exists $\gamma \in \Phi^+$ such that $\langle \gamma, \mu_x + \zeta^\vee \rangle \leqslant -2$, which means $\langle \gamma, \zeta^\vee \rangle = \langle \gamma, \mu_x \rangle = -1$ since μ_x is weakly dominant and ζ is a long root. Then $\gamma \in \Phi^+ \setminus \Phi_J$ and $\mu_x + \gamma^\vee \preceq \lambda$. Note that α is a short root and $\langle \alpha, \mu_x \rangle = 0$. If $\langle \alpha, \gamma^\vee \rangle = -1$, we take $\xi = \gamma$. If $\langle \alpha, \gamma^\vee \rangle = 0$, then choice (ii) occurs (since if choice (iii) occurs, then $\gamma = -3\alpha - 2\zeta$, contradicting that $\langle \gamma, \mu_x \rangle = -1$), which means $\mu_x + \gamma^\vee + \zeta^\vee \preceq \lambda$. Thus, we take $\xi = s_\gamma(\zeta)$. If $\langle \alpha, \gamma^\vee \rangle = 1$, we take $\xi = s_\alpha(\gamma)$. It remains to show $\xi \in \Phi^+ \setminus \Phi_J^+$. Otherwise, $\xi \in \Phi_K$ since $\langle \alpha, \xi^\vee \rangle \neq 0$, contradicting our assumption that $\mu + \xi^\vee \npreceq \lambda$. Thus, part (a) is proved.

Let β be the J-anti-dominant conjugate of ξ under W_J. Let $K_0 \subseteq \Psi_\beta$ be the connected component of K containing α. By part (a) we have

(b) $\langle \beta, \mu_x \rangle = -1$ if Φ is simply laced;

(c) $\mu_x + \beta^\vee \preceq \lambda$;

(d) $\beta^\vee$ is non-central on K_0.

We claim that

(e) $\beta^\vee$ is K-minuscule.

Otherwise, $\langle \theta, \beta^\vee \rangle \leqslant -2$ for some $\theta \in \Phi_K^+$. Then $\mu_x + \beta^\vee + \theta^\vee \preceq \lambda$. If $\langle \beta, \mu_x \rangle \geqslant 0$, then $\langle \beta, \mu_x + \beta^\vee + \theta^\vee \rangle \geqslant 1$ and $\mu_x + \theta^\vee \preceq \lambda$, contradicting our assumption. Otherwise, we have

$$\langle \beta, \mu_x \rangle = -1 \text{ and } \langle s_\beta(\theta), \mu_x \rangle = -\langle \theta, \beta^\vee \rangle \langle \beta, \mu_x \rangle \leqslant -2,$$

which contradicts that μ_x is weakly dominant. Thus, part (e) follows.

Applying [Nie18, Lemma 6.6] we can assume furthermore that $\beta^\vee$ is J-anti-dominant and J-minuscule. Hence, Lemma 6.5(1) is proved.

If $\langle p(x)\sigma^i(\beta), \mu_x \rangle \geqslant 1$ for some $i \in \mathbb{Z} \setminus n\mathbb{Z}$, then

$$\mu_1 := \mu_x + \beta^\vee - p(x)\sigma^i(\beta)^\vee \preceq \lambda \text{ and hence } x_1 := x + \beta^\vee - \sigma^i(\beta)^\vee \in \mathcal{S}_{\lambda,b}^+.$$

By part (d), μ_1 is non-central on K_0. As μ_{x_1}, μ_1 are conjugate by W_J (see Lemma 5.3), μ_{x_1} is also non-central on K_0, contradicting that $K_0 \subseteq J_0$. Thus, $\langle p(x)\sigma^i(\beta), \mu_x \rangle \leqslant 0$ for $i \in \mathbb{Z} \setminus n\mathbb{Z}$. If $\langle \sigma^i(\beta), \mu_x \rangle \leqslant -1$ for some $i \in \mathbb{Z} \setminus n\mathbb{Z}$, by Lemma 5.1 there exists $j \in n\mathbb{Z}$ such that $\langle p(x)\sigma^j(\beta), \mu_x \rangle \geqslant 1$. Then

$$\mu_2 := \mu_x - p(x)\sigma^j(\beta)^\vee + \sigma^i(\beta)^\vee \preceq \lambda \text{ and, hence, } x_2 := x - \sigma^j(\beta)^\vee + \sigma^i(\beta)^\vee \in \mathcal{S}_{\lambda,b}^+,$$

which is also impossible since μ_2 is non-central on $\sigma^j(K_0)$. So $\langle \sigma^i(\beta), \mu_x \rangle = \langle p(x)\sigma^i(\beta), \mu_x \rangle = 0$ for $i \in \mathbb{Z} \setminus n\mathbb{Z}$ and Lemma 6.5(2) is proved.

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

If $\sigma^{2n}(\beta) \neq \beta$, then $\Phi = \Psi$ and Ψ_β is of type D_4, whose simple roots are $\beta, \sigma^n(\beta), \sigma^{2n}(\beta), \alpha$ with $\sigma^n(\alpha) = \alpha$. Moreover, $J = J_0 = \mathcal{O}_\alpha$. By Lemma 6.5(2), we have $\mu_x|_{\Psi \setminus \Psi_\beta} = 0$. Thus, $\sum_{i=0}^n \langle \sigma^i(\beta), \mu_x \rangle \geqslant 1$ by Lemma 5.1. If $\langle \sigma^n(\beta), \mu_x \rangle \geqslant 1$, then part (3) follows. If $\langle \sigma^n(\beta), \mu_x \rangle \leqslant -1$, it follows by replacing β with $\sigma^n(\beta)$. If $\langle \sigma^n(\beta), \mu_x \rangle = 0$, it follows by replacing x with $x - \sigma^{2n}(\beta)^\vee + \sigma^n(\beta)^\vee \in \mathcal{S}^+_{\lambda,b}$.

Now we assume $\sigma^{2n}(\beta) = \beta$. By Lemmas 6.5(2) and 5.1,

$$\text{(f)} \qquad \langle \beta + \sigma^n(\beta), \mathrm{pr}_J(\mu_x) \rangle = \langle \beta + \sigma^n(\beta), \mathrm{pr}_{J_1}(\mu_x) \rangle > 0.$$

Thus, Lemma 6.5(3) follows if $\beta = \sigma^n(\beta)$. Assume $\beta \neq \sigma^n(\beta)$. Then Φ is simply-laced, and, hence, $\langle \beta, \mu_x \rangle = -1$ by part (b). Moreover, $\mathcal{O}_\beta \cup J$ is a set of simple roots of Ψ by [CKV15, Proposition 4.2.11]. As β is a neighbor of K_0 in Ψ_β and $\langle \beta, \mu_x \rangle = -1$, one checks (on the type of Ψ_β) that $\langle \beta, \mathrm{pr}_{J_1}(\mu_x) \rangle < 0$. By part (f), we have $\langle p(x)\sigma^n(\beta), \mu_x \rangle \geqslant 1$ and Lemma 6.5(3) follows.

Assume σ^n does not act trivially on $\Psi_\beta \cap J_0$. Then Φ is simply-laced and $\langle \beta, \mu_x \rangle = -1$. We may assume σ^n does not fix each point of K_0. Let $\alpha \in K_0$ such that $\langle \beta, \alpha^\vee \rangle = -1$. If $\sigma^n(\beta) = \beta$, then one checks directly (on the type of Ψ_β and using the assumption on K_0) that $\langle \beta, \mathrm{pr}_J(\mu_x) \rangle < 0$, which contradicts part (f). Thus, $\beta \neq \sigma^n(\beta) \in \Psi_\beta$. Let $x_3 = x + \beta^\vee - \sigma^n(\beta)^\vee \in \pi_1(M_J)$. If $\beta, \sigma^n(\beta)$ are in distinct connected components of $\mathcal{O}_\beta \cup J \setminus \{\alpha, \sigma^n(\alpha)\} \supseteq \mathcal{O}_\beta \cup J_1$, then $x_3 \in \mathcal{S}^+_{\lambda,b}$ by Lemma 6.5(2) that $\langle p(x)\sigma^n(\beta), \mu_x \rangle \geqslant 1$. As $\langle \alpha, \mu_{x_3} \rangle = 0$, we deduce that $\alpha = \sigma^n(\alpha)$ is the common neighbor of $\beta, \sigma^n(\beta)$ in Ψ_β, which implies that σ^n fixes each point of K_0, contradicting our assumption. Thus, $\beta, \sigma^n(\beta)$ are connected in $\mathcal{O}_\beta \cup J \setminus \{\alpha, \sigma^n(\alpha)\}$. Then $\alpha \neq \sigma^n(\alpha)$, and it follows from part (f) that either $\langle p(x)\sigma^n(\beta), \mu_x \rangle \geqslant 2$ or the case in Lemma 6.5(4) occurs. The former case does not occur since $x_3 \in \mathcal{S}^+_{\lambda,b}$ but μ_{x_3} is non-central on K_0. Thus, Lemma 6.5(4) follows. □

References

BS17 B. Bhatt and P. Scholze, *Projectivity of the Witt vector Grassmannian*, Invent. Math. **209** (2017), 329–423.

CS17 A. Caraiani and P. Scholze, *On the generic part of the cohomology of compact unitary Shimura varieties*, Ann. of Math. (2) **186** (2017), 649–766.

CKV15 M. Chen, M. Kisin and E. Viehmann, *Connected components of affine Deligne–Lusztig varieties in mixed characteristic*, Compos. Math. **151** (2015), 1697–1762.

CN19 L. Chen and S. Nie, *Connected components of closed affine Deligne–Lusztig varieties*, Math. Ann. **375** (2019), 1355–1392.

CN20 L. Chen and S. Nie, *Connected components of closed affine Deligne–Lusztig varieties for* $\mathrm{Res}_{E/F}\mathrm{GL}_n$, J. Algebra **546** (2020), 1–26.

Gas10 Q. Gashi, *On a conjecture of Kottwitz and Rapoport*, Ann. Sci. Éc. Norm. Supér. (4) **43** (2010), 1017–1038.

GHN15 U. Görtz, X. He and S. Nie, *P-alcoves and nonemptiness of affine Deligne–Lusztig varieties*, Ann. Sci. Éc. Norm. Supér. (4) **48** (2015), 647–665.

GHN19 U. Görtz, X. He and S. Nie, *Fully Hodge–Newton decomposable Shimura varieties*, Peking Math. J. **2** (2019), 99–154.

Hai01 T. Haines, *The combinatorics of Bernstein functions*, Trans. Amer. Math. Soc. **353** (2001), 1251–1278.

HH17 T. Haines and X. He, *Vertexwise criteria for admissibility of alcoves*, Amer. J. Math. **139** (2017), 769–784.

https://doi.org/10.1112/S0010437X23007339 Published online by Cambridge University Press

Ham19 P. Hamacher, *The product structure of Newton strata in the good reduction of Shimura varieties of Hodge type*, J. Algebraic Geom. **28** (2019), 721–749.

Ham17 P. Hamacher, *The almost product structure of Newton strata in the deformation space of a Barsotti–Tate group with crystalline Tate tensors*, Math. Z. **287** (2017), 1255–1277.

HK19 P. Hamacher and W. Kim, *ℓ-adic étale cohomology of Shimura varieties of Hodge type with non-trivial coefficients*, Math. Ann. **375** (2019), 973–1044.

He07 X. He, *Minimal length elements in some double cosets of Coxeter groups*, Adv. Math. **215** (2007), 469–503.

He10 X. He, *Minimal length elements in conjugacy classes of extended affine Weyl group*, Preprint (2010), arXiv:1004.4040.

He14 X. He, *Geometric and homological properties of affine Deligne–Lusztig varieties*, Ann. of Math. (2) **179** (2014), 367–404.

He16 X. He, *Kottwitz–Rapoport conjecture on unions of affine Deligne–Lusztig varieties*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), 1125–1141.

HN14 X. He and S. Nie, *Minimal length elements of extended affine Weyl groups*, Compos. Math. **150** (2014), 1903–1927.

HN17 X. He and S. Nie, *On the μ-ordinary locus of a Shimura variety*, Adv. Math. **321** (2017), 513–528.

HR17 X. He and M. Rapoport, *Stratifications in the reduction of Shimura varieties*, Manuscripta Math. **152** (2017), 317–343.

HZ20 X. He and R. Zhou, *On the connected component of affine Deligne–Lusztig varieties*, Duke Math. J. **169** (2020), 2697–2765.

Hum72 J. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9 (Springer, New York, Berlin, 1972).

Kim19 W. Kim, *On central leaves of Hodge-type Shimura varieties with parahoric level structure*, Math. Z. **291** (2019), 329–363.

Kis10 M. Kisin, *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. **23** (2010), 967–1012.

Kis17 M. Kisin, *Mod p points on Shimura varieties of abelian type*, J. Amer. Math. Soc. **30** (2017), 819–914.

KMS22 M. Kisin, K. Madapusi Pera and S. Shin, *Honda–Tate theory for Shimura varieties*, Duke Math. J. **171** (2022), 1559–1614.

KP18 M. Kisin and G. Pappas, *Integral models of Shimura varieties with parahoric level structure*, Publ. Math. Inst. Hautes Études Sci. **128** (2018), 121–218.

Kot85 R. Kottwitz, *Isocrystals with additional structure*, Compos. Math. **56** (1985), 201–220.

Kot92 R. Kottwitz, *Points on some Shimura varieties over finite fields*, J. Amer. Math. Soc. **5** (1992), 373–444.

Lan76 R. Langlands, *Some contemporary problems with origins in the Jugendtraum*, in *Mathematical developments arising from Hilbert problems*, Proceedings of Symposia in Pure Mathematics, vol. XXVIII, Part1 (American Mathematical Society, Providence, RI, 1976), 401–418.

LR87 R. Langlands and M. Rapoport, *Shimuravarietäten und Gerben*, J. Reine Angew. Math. **378** (1987), 113–220.

Man05 E. Mantovan, *On the cohomology of certain PEL-type Shimura varieties*, Duke Math. J. **129** (2005), 573–610.

Nie15 S. Nie, *Fundamental elements of an affine Weyl group*, Math. Ann. **362** (2015), 485–499.

Nie18 S. Nie, *Connected components of closed affine Deligne–Lusztig varieties in affine Grassmannians*, Amer. J. Math. **140** (2018), 1357–1397.

PR21 G. Pappas and M. Rapoport, *p-adic shtukas and the theory of global and local Shimura varieties*, Preprint (2021), arXiv:2106.08270.

Rap05 M. Rapoport, *A guide to the reduction modulo p of Shimura varieties*, Astérisque **298** (2005), 271–318.

RZ96 M. Rapoport and T. Zink, *Period spaces for p-divisible groups*, Ann. Math. Studies, vol. 141 (Princeton University Press, 1996).

SYZ21 X. Shen, C.-F. Yu and C. Zhang, *EKOR strata for Shimura varieties with parahoric level structure*, Duke Math. J. **170** (2021), 3111–3236.

vH20 P. van Hoften, *Mod p points on Shimura varieties of parahoric level (with an appendix by Rong Zhou)*, Preprint (2020), arXiv:2010.10496v2.

Vie08 E. Viehmann, *Connected components of affine Deligne–Lusztig varieties*, Math. Ann. **340** (2008), 315–333.

Zho20 R. Zhou, *Mod p isogeny classes on Shimura varieties with parahoric level structure*, Duke Math. J. **169** (2020), 2937–3031.

Zhu17 X. Zhu, *Affine Grassmannians and the geometric Satake in mixed characteristic*, Ann. of Math. (2) **185** (2017), 403–492.

Sian Nie niesian@amss.ac.cn

Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, PR China

and

School of Mathematical Sciences, University of Chinese Academy of Sciences, Chinese Academy of Sciences, Beijing 100049, PR China

Compositio Mathematica is owned by the Foundation Compositio Mathematica and published by the London Mathematical Society in partnership with Cambridge University Press. All surplus income from the publication of *Compositio Mathematica* is returned to mathematics and higher education through the charitable activities of the Foundation, the London Mathematical Society and Cambridge University Press.