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On Constructing Ergodic Hyperfinite Equivalence Relations of Non-Product Type

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Abstract. Product type equivalence relations are hyperfinite measured equivalence relations, which, up to orbit equivalence, are generated by product type odometer actions. We give a concrete example of a hyperfinite equivalence relation of non-product type, which is the tail equivalence on a Bratteli diagram. In order to show that the equivalence relation constructed is not of product type we will use a criterion called property A. This property, introduced by Krieger for non-singular transformations, is defined directly for hyperfinite equivalence relations in this paper.

1 Introduction

The theory of countable measured equivalence relations is closely related to the theory of von Neumann algebras. One can associate a von Neumann algebra with any countable measured equivalence relation(see [6]), and there is a one to one correspondence between ergodic hyperfinite measured equivalence relations (up to orbit equivalence) and approximately finite dimensional factors (up to isomorphism). The countable measured equivalence relations that, up to orbit equivalence, are generated by product type odometer actions are called of product type. They correspond to a special class of approximately finite dimensional factors, the ITPFI factors. An important result of Krieger [12] says that the orbit-equivalence classes of ergodic hyperfinite equivalence relations of type III are completely characterized by the conjugacy class of the associated flow. Among all hyperfinite equivalence relations, the product type equivalence relations correspond to the approximately transitive flows [2].

In order to show that there exist nonsingular transformations that are not orbit equivalent to any product type odometer, Krieger [11] introduced the so-called *property A*. He showed that any product type odometer of type III has this property and that property A is an invariant for orbit equivalence. He also constructed a nonsingular ergodic automorphism of type III, which does not have property A and therefore it is not orbit equivalent to any product type odometer. Trying to give a more explicit example of an automorphism of non-product type, Dooley and Hamachi [3] constructed a Markov odometer that does not satisfy Krieger's property A. More or less, for both examples, it is quite difficult to describe the transformations and the effect they have on the space (by this we mean the equivalence relations they produce). In [15], it was shown that property A is not a sufficient condition for a non-singular transformation to be of product type.

Given that a countable measured equivalence relation is hyperfinite if and only if it coincides up to a null set with an equivalence relation generated by a nonsingular

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automorphism (see [5,10]), property A can be defined directly for hyperfinite equivalence relations (Definition 2.2). In this setting, Krieger's result says that any ergodic equivalence relation of product type and of type III satisfies property A.

The purpose of this paper is to give an example of a hyperfinite measured equivalence relation that can de described explicitly and does not satisfy property A, and thus is not of product type.

Recently, by using matrix random walks, Giordano and Handelman [7] gave an example of an equivalence relation whose associated flow is not AT, and thus is not of product type. It would be interesting if one could prove that the equivalence relation constructed here is not of product type by using the techniques developed in [7].

The paper is organized as follows. In Section 2, we define property A for hyperfinite measured equivalence relations. In Section 3, we construct a hyperfinite measured equivalence relation $\mathcal{R}$ that is the tail equivalence relation on a Bratteli diagram. In Section 4, we show that $\mathcal{R}$ is ergodic and of type III, and in Section 5, we prove that $\mathcal{R}$ does not have property A, and therefore it is not of product type.

2 Definitions and Notations

Let $(X, \mathfrak{B}, \mu)$ be a Lebesgue space, and let $\mathcal{R}$ be an equivalence relation on X. We say that $\mathcal{R}$ is a *countable measured equivalence relation* if the equivalence classes $\mathcal{R}(x)$, $x \in X$ are countable, $\mathcal{R}$ is a measurable subset of $X \times X$, and the saturation of any set of measure zero has measure zero. $\mathcal{R}$ is called *ergodic* if any invariant set of positive measure has full measure. A countable measured equivalence relation $\mathcal{R}$ is *hyperfinite* if there exists an increasing sequence $(\mathcal{R}_n)_{n\geqslant 1}$ of equivalence relations with finite orbits such that $\mathcal{R}(x) = \cup_{n\geqslant 1}\mathcal{R}_n(x)$ for μ-a.e. $x \in X$. The measured equivalence relation $\mathcal{R}$ is said to be of *type III* if there is no σ-finite $\mathcal{R}$-invariant measure ν equivalent to μ. For $(x, y) \in \mathcal{R}$, let $\pi_l(x, y) = x$ be the left projection, and let $\pi_r(x, y) = y$ be the right projection of $\mathcal{R}$. The measures ν_l and ν_r on $\mathcal{R}$, defined by

$$\nu_l(C) = \int_X |\pi_l^{-1}(x) \cap C| d\mu(x) \quad \text{and} \quad \nu_r(C) = \int_X |\pi_r^{-1}(x) \cap C| d\mu(x)$$

are called the *left counting measure* and the *right counting measure* of μ. Recall that $\nu_l \sim \nu_r$ and that $\delta_\mu(x, y) = \frac{d\nu_l}{d\nu_r}(x, y)$ is the *Radon-Nikodym cocycle* of μ with respect to $\mathcal{R}$. A *partial Borel isomorphism* on X will be a Borel isomorphism ϕ defined on some $A \in \mathfrak{B}$ with range some $B \in \mathfrak{B}$. For a partial Borel isomorphism $\phi\colon A \to B$, the set $\operatorname{Graph}(\phi) = \{(x, \phi(x)), x \in A\}$ is called the *graph* of ϕ. If ϕ is a partial isomorphism with $\operatorname{Graph}(\phi) \subseteq \mathcal{R}$, then $\frac{d\mu\circ\phi}{d\mu}(x) = \delta_\mu(\phi(x), x)$ for μ-a.e $x \in \operatorname{Dom}(\phi)$. The *full group* $[\mathcal{R}]$ of $\mathcal{R}$ is the group of all nonsingular automorphisms V of $(X, \mathfrak{B}, \mu)$ with $(x, Vx) \in \mathcal{R}$ for μ-a.e. $x \in X$. For further details, see [5].

Two countable measured equivalence relations $\mathcal{R}$ and $\mathcal{R}'$ on $(X, \mathfrak{B}, \mu)$ and on $(X', \mathfrak{B}', \mu')$, respectively, are called *orbit equivalent* if there exists an isomorphism of measured spaces, $S\colon (X, \mathfrak{B}, \mu) \to (X', \mathfrak{B}', \mu')$, such that

$$S(\mathcal{R}(x)) = \mathcal{R}'(Sx) \text{ for } \mu\text{-a.e. } x \in X.$$

We recall that Krieger's defined property A for a nonsingular automorphism as follows.

Definition 2.1 [11] A nonsingular automorphism T on $(X, \mathfrak{B}, \mu)$ is said to satisfy *property A* if there exist constants $\eta, \delta > 0$ and a σ-finite measure $\nu \sim \mu$ such that every set A of positive measure contains a measurable subset B of positive measure with

$$\limsup_{s\to\infty} \nu(K_{\nu,T}(B, s, \delta)) > \eta \cdot \nu(B),$$

where

$$K_{\nu,T}(B, s, \delta) = \{x \in B : \exists \phi \in [T], \text{ such that } \phi(x) \in B \text{ and } \\ \log \frac{d\nu \circ \phi}{d\nu}(x) \in (e^{s-\delta}, e^{s+\delta}) \cup (-e^{s+\delta}, -e^{s-\delta})\},$$

and $[T]$ is the full group of T.

Let $\mathcal{R}$ be an ergodic hyperfinite measured equivalence relation $\mathcal{R}$ on $(X, \mathcal{B}, \mu)$. If ν is a measure on X, equivalent to μ, we denote by δ_ν the Radon–Nikodym cocycle of ν. For $x \in A$, we define

$$\begin{aligned} \Lambda_{\nu,A,\mathcal{R}}(x) &= \{\log \delta_\nu(y, x) : (x, y) \in \mathcal{R} \text{ and } y \in A\} \\ &= \Big\{ \log \frac{d\nu \circ \phi}{d\nu}(x) : \phi \in [\mathcal{R}] \Big\}. \end{aligned}$$

For a σ-finite measure $\nu \sim \mu$, $A \in \mathfrak{B}$ of positive measure and $s, \delta > 0$, we set

$$\begin{aligned} K_{\nu,\mathcal{R}}(A, s, \delta) &= \{x \in A : (e^{s-\delta}, e^{s+\delta}) \cap \Lambda_{\nu,A,\mathcal{R}}(x) \neq \varnothing\} \\ &\quad \cup \ \{x \in A : (-e^{s+\delta}, -e^{s-\delta}) \cap \Lambda_{\nu,A,\mathcal{R}}(x) \neq \varnothing\} \\ &= \{x \in A : \ \exists y \in A \text{ with } (x, y) \in \mathcal{R} \text{ and } |\log \delta_\nu(y, x)| \in (e^{s-\delta}, e^{s+\delta})\}. \end{aligned}$$

Definition 2.2 Let $(X, \mathfrak{B}, \mu)$ and $\mathcal{R}$ be a hyperfinite ergodic countable equivalence relation. Then $\mathcal{R}$ has *property A* if there exists a measure $\nu \sim \mu$ and $\eta, \delta > 0$ such that every measurable set A of positive measure contains a measurable set B of positive measure such that

$$\limsup_{s\to\infty} K_{\nu,\mathcal{R}}(B, s, \delta) > \eta \cdot \nu(B).$$

Given that $\mathcal{R}$ is hyperfinite, there exists a nonsingular automorphism T on $(X, \mathcal{B}, \mu)$ such that, up to a set of measure zero, $\mathcal{R}$ is equal to the equivalence relation $\mathcal{R}_T = \{(T^n x, x), x \in X, n \in \mathbb{Z}\}$ generated by T, that is, $\mathcal{R}(x) = \{T^n x, n \in \mathbb{Z}\}$, for μ-a.e. $x \in X$ (see [5, 10]). It follows that $[\mathcal{R}] = [T]$ and then, as can immediately be observed, $\mathcal{R}$ has property A if and only if T has property A in the sense of Krieger. Hence, it follows from [11] that property A is an invariant for orbit equivalence of hyperfinite equivalence relations and [11, Lemma 2.2] can be reformulated as follows.

Proposition 2.3 *Assume that $\mathcal{R}$ has property A. Then there exist $\eta, \delta > 0$ such that for all $\lambda \sim \mu$ and all $\epsilon > 0$, every measurable set A of positive measure contains a measurable set B of positive measure with*

$$\limsup_{s\to\infty} K_{\lambda,\mathcal{R}}(B, s, \delta + \epsilon) > e^{-\epsilon}\eta \cdot \lambda(B)$$

Consider $(k_n)_{n\geqslant 1}$ a sequence of positive integers with $k_n \geqslant 2$. Let

$$X = \prod_{n\geqslant 1} \{0, 1, \dots k_n - 1\}$$

endowed with the product topology, the corresponding Borel structure, and a non-atomic product measure $\mu = \bigotimes_{n\geqslant 1} \mu_n$, where μ_n are probability measures on $\{0, 1, \dots k_n - 1\}$ such that the mass of every point is positive. We define T on X by setting $T(1, 1, 1, \dots) = (0, 0, 0, \dots)$, and if $x \neq (1, 1, 1, \dots)$,

$$(Tx)_n = \begin{cases} 0 & \text{if } n < N(x), \\ x_n + 1 & \text{if } n = N(x), \\ x_n & \text{if } n > N(x), \end{cases}$$

where $N(x) = \min\{n \geqslant 1 : x_n < k_n - 1\}$. The product type odometer on (X, μ) is called T. Notice that T is a non-singular and ergodic transformation with respect to μ.

The *tail equivalence* relation $\mathcal{T}$ is defined for $x = (x_n)_{n\geqslant 1}$ and $y = (y_n)_{n\geqslant 1}$ by $x\mathcal{T}y$ if and only if there exists $n \geqslant 1$ such that $x_i = y_i$ for all $i > n$.

Clearly, the tail equivalence relation $\mathcal{T}$ is orbit equivalent with the equivalence relation induced by the $\mathbb{Z}$-action of the product odometer on X.

Definition 2.4 An equivalence relation is said of *product type* if it is orbit equivalent to an equivalence relation induced by the $\mathbb{Z}$-action of a product type odometer.

Now, Krieger's result from [12] can be reformulated in the following way.

Theorem 2.5 *Any equivalence relation of product type and of type III satisfies property A.*

3 Construction of an Equivalence Relation

In this section we construct a hyperfinite equivalence equivalence relation $\mathcal{R}$ on a Lebesgue space $(X, \mathfrak{B}, \mu)$.

Definition 3.1 A *Bratteli diagram* $D = (V, E)$ is a graph with a set of vertices V and a set of edges E, with the following properties:

(i) V is the disjoint union of finite subsets V_n, $n \geqslant 0$;
(ii) E is the disjoint union of finite subsets E_n, $n \geqslant 1$, with each edge $e \in E_n$ connecting a vertex $s(e) \in V_{n-1}$ with a vertex $r(e) \in V_n$;

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(iii) For every vertex $v \in V$, there exists $e \in E$ with $s(e) = v$;
(iv) For every vertex $v \in V$, except for $v \in V_0$, there exists $e \in E$ with $r(e) = v$.

For simplicity, we assume that V_0 consists of a single vertex v_0. A *path* in D is defined as a sequence (e_k) of edges with $s(e_1) \in V_0$ and $s(e_k) = r(e_{k-1})$ for $k \geqslant 2$. We denote by X the space of paths of infinite length. To each path of length n, $f = (f_1, f_2, \dots f_n)$, we associate the set

$$[f_1, f_2, \dots f_n] = \{e \in X, e_k = f_k, 1 \leqslant k \leqslant n\},$$

which is called *cylinder* of length n. On X we consider the σ-algebra $\mathfrak{B}$ generated by all cylinder sets. The *tail equivalence* relation on X, denoted by $\mathcal{R}$, is defined by

$$e\mathcal{R}f \text{ if and only if for some } n,\ e_k = f_k \text{ for all } k \geqslant n.$$

Notice that $\mathcal{R}$ is the union of an increasing sequence of equivalence relations $\mathcal{R}_n$, where, for $n \geqslant 1$, $\mathcal{R}_n$ is the equivalence relation on X given by $e\mathcal{R}_n f$ if and only if $e_k = f_k$ for all $k \geqslant n$.

We recall (see for example [4]) that a *Markov measure* (or *AF measure*) μ_p on Ω is a measure determined by a system of transition probabilities p (*i.e.*, maps $p\colon E \to [0,1]$ with $p(e) > 0$ and $\sum_{\{e \in E, s(e)=v\}} p(e) = 1$ for every $v \in V$) given by

$$\mu_p\big([f_1, f_2, \dots f_n]\big) = \prod_{k=1}^{n} p(f_k),$$

for each cylinder $[f_1, f_2, \dots f_n]$.

We construct a Bratteli diagram $D = (V, E)$ as follows. Let us start by considering $(r_n)_{n \geqslant 1}$, the sequence of positive integers given by $r_n = 2^{n!}$, for $n \geqslant 1$. The set of vertices is $V = \cup_{n \geqslant 0} V_n$, where $V_n = \{(n, 0), (n, 1), \dots, (n, n)\}$, $n \geqslant 0$. Let $\psi\colon \mathbb{N}^* \to V \setminus \{(n, 0); n \geqslant 0\}$ be the bijection given by $\psi(m) = (n, k)$, where $n \geqslant 1$ and $1 \leqslant k \leqslant n$ satisfy $0 + 1 + 2 + \cdots + n - 1 + k = m$. and For example, $\psi(1) = (1, 1)$, $\psi(2) = (2, 1)$, $\psi(3) = (2, 2)$, $\psi(4) = (3, 1)$, $\psi(5) = (3, 2)$, and so on.

Define

$$\lambda_{n,k} = r_{\psi^{-1}(n,k)} \text{ for } n \geqslant 1 \text{ and } 1 \leqslant k \leqslant n.$$

The set of edges of the Bratteli diagram is $E = \cup_{n \geqslant 1} E_n$, where

$$E_n = \big\{ (n, k, k+1, j); 1 \leqslant j \leqslant \lambda_{n,k+1}, 0 \leqslant k \leqslant n-1 \} \cup \{ (n, k, k, 0); 0 \leqslant k \leqslant n-1 \big\}.$$

The edge $(n, k, k, 0)$ goes from the vertex $(n-1, k)$ to the vertex (n, k), and for $1 \leqslant j \leqslant \lambda_{n,k+1}$, the edge $(n, k, k+1, j)$ goes from the vertex $(n-1, k)$ to the vertex $(n, k+1)$. Then the space X of infinite paths consists of all

$$x = \big((1, k_0, k_1, i_1), (2, k_1, k_2, i_2), \dots, (n, k_{n-1}, k_n, i_n) \dots \big),$$

where $0 \leqslant k_n \leqslant n$, and either $k_n = k_{n+1}$ and $i_n = 0$, or $k_n + 1 = k_{n+1}$ and $1 \leqslant i_n \leqslant \lambda_{n,k_{n+1}}$. Notice that $k_0 = 0$, for any $x \in X$. When necessary, a path $x \in X$ is

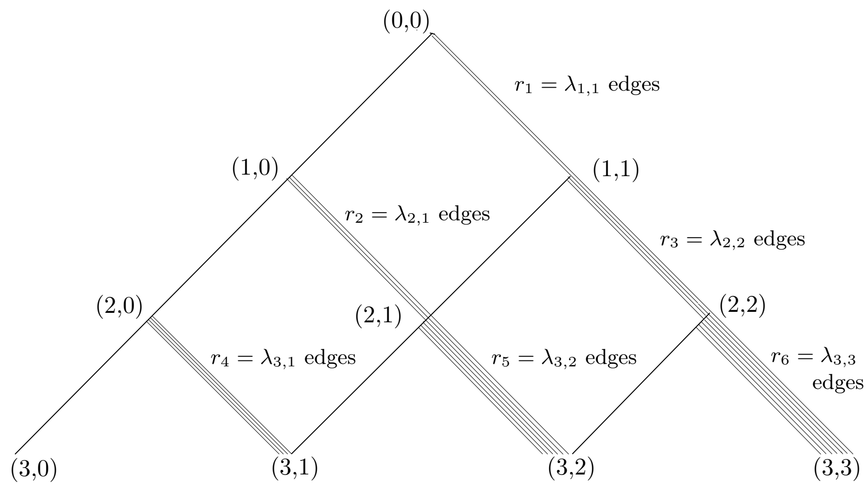

Figure 3.1

denoted by $x = (x_1, x_2, \dots, x_n, \dots)$, where $x_n \in E_n$ and $r(x_n) = s(x_{n+1})$. For a path $x = (x_1, x_2, \dots, x_n, \dots) \in X$, we denote the cylinder set $[x_1, x_2, \dots, x_n]$ with E_x^n.

This Bratteli diagram is described by a sequence $(A_n)_{n\geqslant 1}$ of transition matrices. For $n \geqslant 1$, $A_n = \{a_n^{i,j}, 0 \leqslant i \leqslant n-1, 0 \leqslant j \leqslant n\}$, where $a_n^{i,j}$ is the number of edges going from the vertex $(n-1, i)$ to the vertex (n, j). Hence, $a_n^{k,k} = 1$, $a_n^{k,k+1} = \lambda_{n,k+1}$, for $k = 1, \dots, n-1$, and otherwise $a_n^{i,j} = 0$. The first three levels of this Bratteli diagram are shown in Figure 3.1.

We consider on E a system p of transition probabilities, defined as follows. Let $p_1 \colon E_1 \to [0,1]$, where

$$p_1\big((1,0,0,0)\big) = \frac{1}{2}, \quad p_1\big((1,0,1,j)\big) = \frac{1}{2\cdot a_1^{01}}, \text{ for } 1 \leq j \leq a_1^{01}.$$

For $n \geq 2$, we define $p_n \colon E_n \to [0,1]$ by setting

$$p_n((n,k,k,0)) = \frac{1}{2}, \quad p_n\big((n,k,k+1,j)\big) = \frac{1}{2\cdot a_n^{k,k+1}}$$

for $0 \leq k \leq n-1, 1 \leq j \leq a_n^{k,k+1}$.

Let μ be the Markov measure on X, which, for any cylinder

$$C = [(1,0,k_1,i_1),(2,k_1,k_2,i_2),\dots,(n,k_{n-1},k_n,i_n)],$$

is given by

$$\begin{aligned}\mu(C) &= p_1(1,0,k_1,i_1)\cdot p_2(2,k_1,k_2,i_2)\cdots p_n(n,k_{n-1},k_n,i_n)\\ &= \frac{1}{2^n}\frac{1}{a_1^{0k_1}a_2^{k_1k_2}\cdots a_n^{k_{n-1}k_n}}.\end{aligned}$$

Note that $\mathcal{R}$ is a hyperfinite measured equivalence relation. For the rest of this paper, the Radon–Nikodym cocycle of μ with respect to $\mathcal{R}$ will simply be denoted by δ. Let $(x, y) \in \mathcal{R}$. As x and y are tail equivalent, we can write

$$x = \big((1, 0, k_1, i_1), (2, k_1, k_2, i_2), \dots, (n, k_{n-1}, k_n, i_n), (n+1, k_n, k_{n+1}, i_{n+1}), \dots\big)$$
$$y = \big((1, 0, j_1, l_1), (2, j_1, j_2, l_2), \dots, (n, j_{n-1}, k_n, l_n), (n+1, k_n, k_{n+1}, i_{n+1}), \dots\big).$$

It follows that
$$\delta(y, x) = \frac{a_1^{0k_1}}{a_1^{0j_1}} \frac{a_2^{k_1k_2}}{a_2^{j_1j_2}} \cdots \frac{a_n^{k_{n-1}k_n}}{a_n^{j_{n-1}k_n}}.$$

4 The Equivalence Relation $\mathcal{R}$ is Ergodic and of Type III

In this section we show that $\mathcal{R}$ is ergodic with respect to the measure μ and that $\mathcal{R}$ is of type III.

Let $\widetilde{X} = \prod_{n=1}^{\infty}\{0, 1\}$. On $\widetilde{X}$ we consider the product measure $\widetilde{\mu} = \bigotimes_{n\geqslant 1} \widetilde{\mu}_n$ given by $\widetilde{\mu}_n(0) = \widetilde{\mu}_n(1) = \frac{1}{2}$, for $n \geqslant 1$. We define a map $\pi\colon X \to \widetilde{X}$ by setting

$$\pi\big((1, k_0, k_1, i_1), (2, k_1, k_2, i_2), (n, k_2, k, i_3) \dots\big) = (k_1, k_2 - k_1, k_3 - k_2, \dots).$$

Note that the measure $\widetilde{\mu}$ is the pushforward of the measure μ by the map π. We denote by $F_{n,k}$ the set of all paths in X that cross the vertex (n, k).

Lemma 4.1 $\lim_{m\to\infty} \mu(F_{\psi(m)}) = 0.$

Proof Let $m \geqslant 1$ and $(n, k) = \psi(m)$. Let $A_{n,k} = \{\widetilde{y} \in \widetilde{X},\ \sum_{i=1}^{n} \widetilde{y}_i = k\}$. We have

$$\widetilde{\mu}(A_{n,k}) = \frac{1}{2^n}\binom{n}{k}.$$

A path $x \in X$ crosses the vertex (n, k) if and only if $\sum_{i=1}^{n} \pi(x)_i = k$ or, equivalently, if and only if $x \in \pi^{-1}(A_{n,k})$. Then

$$\mu(F_{\psi(m)}) = \mu(F_{n,k}) = \mu \circ \pi^{-1}(A_{n,k}) = \widetilde{\mu}(A_{n,k}) = \frac{1}{2^n}\binom{n}{k}.$$

Let
$$a_n = \max\Big\{\frac{1}{2^n}\binom{n}{k}; 1 \leqslant k \leqslant n\Big\}.$$

It easily can be checked that $\lim\limits_{n\to\infty} a_n = 0$. From the definition of the function ψ, it results that $\lim\limits_{m\to\infty} F_{\psi(m)} = 0$. ■

Lemma 4.2 *If $A \subseteq X$ with $\mu(A) > 0$, then*

$$\lim_{n\to\infty} \frac{\mu(A \cap E_x^n)}{\mu(E_x^n)} = 1 \textit{ for } \mu\textit{-a.e. } x \in A.$$

Proof Let $\mathfrak{B}_n$ be the σ-algebra generated by the set of cylinders of length n. Then $\mathfrak{B}$, the σ-algebra generated by all cylinders of X, is the σ-algebra generated by all $\mathfrak{B}_n$, $n \geqslant 1$. Denoting with $E(\chi_A|B_n)$ the conditional expectation of χ_A with respect to the σ-algebra $\mathfrak{B}_n$, the martingale convergence theorem (see for example [1, Theorem 35.6]) implies that $E(\chi_A|B_n) \to E(\chi_A|B) = \chi_A$ almost everywhere, as $n \to \infty$. Hence,

$$\lim_{n\to\infty} \frac{\mu(A \cap E_x^n)}{\mu(E_x^n)} = \lim_{n\to\infty} E(\chi_A|B_n)(x) = 1 \text{ for } \mu - \text{a.e. } x \in A. \qquad \blacksquare$$

Proposition 4.3 *The equivalence relation $\mathcal{R}$ is ergodic with respect to μ.*

Proof Note that two paths $x = (x_1, x_2, \dots, x_n, \dots)$ and $y = (y_1, y_2, \dots, y_n, \dots)$ intersect if there exists $n \geqslant 1$ such that $r(x_n) = r(y_n)$. This is equivalent to saying that $\sum_{i=1}^n \widetilde{x}_i = \sum_{i=1}^n \widetilde{y}_i$, where $\widetilde{x} = \pi(x)$ and $\widetilde{y} = \pi(y)$. The space $(\prod_{n\geqslant 1}\{0,1\}, \otimes\widetilde{\mu}_n)$ can be identified with $([0,1], \lambda)$, where λ is the Lebesgue measure (see [14]). Then, [13, Lemma 17.2] implies that for $\widetilde{\mu} \times \widetilde{\mu}$-a.e. $(\widetilde{x}, \widetilde{y}) \in \widetilde{X} \times \widetilde{X}$, there exist infinitely many $n > 1$ such that $\sum_{i=1}^n \widetilde{x}_i = \sum_{i=1}^n \widetilde{y}_i$. As $\widetilde{\mu} = \mu \circ \pi^{-1}$, it follows that for $\mu \times \mu$-a.e. $(x, y) \in X \times X$, x and y intersect infinitely often.

Let A be an $\mathcal{R}-$invariant set with $0 < \mu(A) < 1$. From Lemma 4.2, we have

$$\frac{\mu(A \cap E_x^n)}{\mu(E_x^n)} \to 1, \quad \frac{\mu(A^c \cap E_y^n)}{\mu(E_y^n)} \to 1 \text{ for } \mu \times \mu\text{-a.e } (x, y) \in A \times A^c.$$

Since for $\mu \times \mu$-a.e. $(x, y) \in X \times X$, x and y intersect infinitely often, we can find $x = (x_1, x_2, \dots, x_n, \dots) \in A$, $y = (y_1, y_1, \dots, y_n, \dots) \in A^c$ and $m \geqslant 1$ such that

$$\frac{\mu(A \cap E_x^n)}{\mu(E_x^n)} > \frac{1}{2}, \quad \frac{\mu(A^c \cap E_y^n)}{\mu(E_y^n)} > \frac{1}{2},$$

and $r(x_m) = r(y_m)$. There exists a partial isomorphism ϕ with $\operatorname{Graph}(\phi) \subseteq \mathcal{R}$ and affecting only the first m coordinates such that $\phi E_x^n = E_y^n$. Thus,

$$\mu(A \cap E_y^n) = \int_{A\cap E_y^n} d\mu = \int_{A\cap E_x^n} \frac{d\mu \circ \phi}{d\mu}(x) d\mu(x) = \frac{\mu(E_y^n)}{\mu(E_x^n)} \mu(A \cap E_x^n) > \frac{1}{2}\mu(E_y^n).$$

Therefore,

$$1 = \frac{\mu(A \cap E_y^n)}{\mu(E_y^n)} + \frac{\mu(A^c \cap E_y^n)}{\mu(E_y^n)} > \frac{1}{2} + \frac{1}{2} = 1.$$

This is a contradiction. Hence $\mu(A) = 1$ or $\mu(A) = 0$, and so $\mathcal{R}$ is ergodic. $\blacksquare$

In order to show that $\mathcal{R}$ is of type III we use a criterion from [8] that says that a countable measured equivalence relation $\mathcal{R}$ on (X, μ) is of type III if and only if $\sup\{\log\delta(y, x); (x, y) \in \mathcal{R}\} = \infty$ and $\inf\{\log\delta(y, x); (x, y) \in \mathcal{R}\} = -\infty$, for μ-a.e. $x \in X$.

Proposition 4.4 *The equivalence relation $\mathcal{R}$ is of type III.*

Proof It can immediately be seen that the set

$$A = \{\widetilde{x} \in \widetilde{X}; \widetilde{x}_n = 0 \text{ and } \widetilde{x}_n = 1 \text{ for infinitely many } n\}$$

is conull in $\widetilde{X}$. Thus, $F = \pi^{-1}(A)$ is conull in X.

Consider $\alpha > 0$ arbitrary, and take $x \in F$,

$$x = (x_1, x_2, \dots, x_n, \dots) = \big((1, 0, k_1, i_1), (2, k_1, k_2, i_2), \dots, (n, k_{n-1}, k_n, i_n), \dots\big).$$

Let $m = \min\{n \geqslant 2; \pi(x)_n = 0\} = \min\{n \geqslant 2, k_n = k_{n-1}\}$. Choose $p > m$ large enough such that $\pi(x)_p = 1$ and $\log r_{N-1} > \alpha$, where $N = \psi^{-1}(p, k_p)$. Note that $k_p = k_{p-1} + 1$. We can find $y \in X$,

$$y = \big((y_1, y_2, \dots, y_n, \dots) = (1, 0, j_1, l_1), (2, j_1, j_2, l_2), \dots, (n, j_{n-1}, j_n, l_n), \dots\big)$$

such that $(x, y) \in \mathcal{R}$ and

$$\begin{cases} y_n = x_n & \text{if } n < m \text{ or } n > p, \\ \pi(y)_n = \pi(x)_n & \text{if } m < n < p, \\ \pi(y)_m = 1, & \pi(y)_p = 0. \end{cases}$$

Thus, $j_p = j_{p-1} = k_p$, $j_m = j_{m-1} + 1 = k_{m-1} + 1$, and

$$\delta(y, x) = \frac{a_m^{k_{m-1}k_m}}{a_m^{k_{m-1}j_m}} \frac{a_{m+1}^{k_m k_{m+1}}}{a_{m+1}^{j_m j_{m+1}}} \cdots \frac{a_p^{k_{p-1}k_p}}{a_p^{j_{p-1}k_p}}.$$

Since in the above product each a_n^{ij} is either 1 or certain r_k, it follows that $a_p^{k_{p-1}k_p} = \lambda_{p,k_p} = r_N$ and $a_p^{j_{p-1}k_p} = 1$. Hence, $\delta(y, x) = \prod_{i=1}^{N} r_i^{\beta_i}$, where $\beta_i \in \{0, 1, -1\}$ for $m \leqslant i < N$, and $\beta_N = 1$. Consequently,

$$\log \delta(y, x) \geqslant \log r_N - \sum_{i=1}^{N-1} \log r_i > (N-1)! \log 2 = \log r_{N-1} > \alpha.$$

This implies that $\sup\{\log \delta(y, x); (x, y) \in \mathcal{R}\} = \infty$, and similarly it can be shown that $\inf\{\log \delta(y, x); (x, y) \in \mathcal{R}\} = -\infty$. Therefore $\mathcal{R}$ is of type III. ■

5 The Equivalence Relation $\mathcal{R}$ is not of Product Type

In this section we prove the main result of this paper by showing that $\mathcal{R}$ does not satisfy property A.

Let $(L_n)_{n \geqslant 1}$ be the sequence of positive reals given by

$$L_n = \frac{\log r_n + \log r_{n+1}}{2}, \ n \geqslant 1,$$

and let us denote with I_n the intervals $I_n = [L_{n-1}, L_n)$, $n \geqslant 2$.

It is straightforward to check that

$$\log r_1 + \log r_2 + \cdots + \log r_n < L_n, \text{ for } n \geqslant 1 \tag{5.1}$$

and

$$\log r_n - (\log r_1 + \log r_2 + \cdots + \log r_{n-1}) > L_{n-1}, \text{ for } n \geqslant 5. \tag{5.2}$$

Lemma 5.1 *Let $m \geqslant 5$. If $(x, y) \in \mathcal{R}$ and $|\log \delta(y, x)| \in I_m$, then $x \in F_{\psi(m)}$.*

Proof Consider $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ such that $(x, y) \in \mathcal{R}$ and $L_{m-1} \leqslant |\log \delta(y, x)| < L_m$, where $m \geqslant 5$ and Let $s = \max\{i : x_i \neq y_i\}$ and $p = \max\{i : s(x_i) \neq s(y_i), r(x_i) = r(y_i)\}$. Clearly, $p \leqslant s$. Notice that $p \geqslant 2$, as otherwise $\log \delta(y, x) = 0$. Thus, beginning with the vertex $r(x_p) = r(y_p)$, x and y cross the same vertices of the Bratteli diagram. We write

$$\begin{aligned} x = (1, 0, k_1, i_1), (2, k_1, k_2, i_2), \dots, (p, k_{p-1}, k_p, i_p), (p+1, k_p, k_{p+1}, i_{p+1}), \dots \\ \dots, (s, k_{s-1}, k_s, i_s), (s+1, k_s, k_{s+1}, i_{s+1}), \dots, \\ y = ((1, 0, j_1, l_1), (2, j_1, j_2, l_2), \dots, (p, j_{p-1}, k_p, l_p), (p+1, k_p, k_{p+1}, l_{p+1}), \dots \\ \dots, (s, k_{s-1}, k_s, l_s), (s+1, k_s, k_{s+1}, i_{s+1}), \dots. \end{aligned}$$

Hence,

$$\delta(y, x) = \frac{a_1^{0k_1}}{a_1^{0j_1}} \frac{a_2^{k_1k_2}}{a_2^{j_1j_2}} \cdots \frac{a_p^{k_{p-1}k_p}}{a_p^{j_{p-1}k_p}}.$$

Notice that either $a_p^{j_{p-1}k_p} = \lambda_{p,k_p}$ and $a_p^{k_{p-1}k_p} = 1$ or $a_p^{j_{p-1}k_p} = 1$ and $a_p^{k_{p-1}k_p} = \lambda_{p,k_p}$. As each a_m^{ij} in the above product is either 1 or certain r_n, we have

$$\delta(y, x) = \prod_{i=1}^{\psi^{-1}(p,k_p)} r_i^{\beta_i},$$

where $\beta_i \in \{0, 1, -1\}$ for $1 \leqslant i < \psi^{-1}(p, k_p)$ and $\beta_{\psi^{-1}(p,k_p)} \neq 0$. Let $n = \psi^{-1}(p, k_p)$. It follows that

$$|\log \delta(y, x)| = \left| \sum_{i=1}^{n} \beta_i \log r_i \right| \leqslant \sum_{i=1}^{n} \log r_n < L_n.$$

Since $|\log \delta(y, x)| \geqslant L_4$, we have $n \geqslant 5$. We claim that $m = n$. Indeed, if $m > n$, using (5.1) we would have,

$$|\log \delta(y, x)| \leqslant \sum_{i=1}^{n} \log r_i \leqslant \sum_{i=1}^{m-1} \log r_i < L_{m-1},$$

which contradicts $|\log\delta(y,x)| > L_{m-1}$. If $m < n$, then, by (5.2),

$$|\log\delta(y,x)| \geqslant \log r_n - \sum_{i=1}^{n-1}\log r_i > L_{n-1} \geqslant L_m,$$

which contradicts $|\log\delta(y,x)| \leqslant L_m$. Therefore, $n = m$ and so $(p,k_p) = \psi(m)$. Hence, x crosses the vertex $\psi(m) = (p,k_p)$, that is $x \in F_{\psi(m)}$. ∎

The following proposition shows that for s large enough the interval $(e^{s-\delta}, e^{s+\delta})$ intersects at most two consecutive intervals I_m and I_{m+1}.

Proposition 5.2 *Let $\delta > 0$ and $M > e^{2\delta}$. Consider $s > 0$ and $m \geqslant 1$ such that $e^{s-\delta} > L_M$ and $I_m \cap (e^{s-\delta}, e^{s+\delta}) \neq \varnothing$. Then $I_j \cap (e^{s-\delta}, e^{s+\delta}) = \varnothing$ for $j \neq m, m-1$ or $I_j \cap (e^{s-\delta}, e^{s+\delta}) = \varnothing$ for $j \neq m, m+1$.*

Proof As $e^{s-\delta} > L_M$ it follows that $(e^{s-\delta}, e^{s+\delta})$ does not intersect any of the intervals I_j, $j = 1,2,\dots,M$, and so $m \geqslant M+1$. We have either $L_{m-1} \leqslant e^{s-\delta}$ or $L_{m-1} > e^{s+\delta}$.

Let us assume first that $L_{m-1} \leqslant e^{s-\delta}$. Clearly, $L_m > e^{s-\delta}$, as otherwise, $(e^{s-\delta}, e^{s+\delta}) \cap I_m \neq \varnothing$. It follows that

$$L_{m+1} = \frac{L_{m+1}}{L_m}L_m = \frac{(m+1)(m+3)}{m+2}L_m > M\cdot L_m > e^{2\delta}e^{s-\delta} = e^{s+\delta},$$

and so $I_j \cap (e^{s-\delta}, e^{s+\delta}) = \varnothing$ for $j \neq m, m+1$.

Assume now that $L_{m-1} > e^{s-\delta}$. Since $(e^{s-\delta}, e^{s+\delta}) \cap I_m \neq \varnothing$, we have $L_{m-1} < e^{s+\delta}$. Hence,

$$L_{m-2} = \frac{L_{m-2}}{L_{m-1}}L_{m-1} = \frac{m}{(m-1)(m+1)}L_{m-1} < \frac{L_{m-1}}{M} < \frac{e^{s+\delta}}{e^{2\delta}} = e^{s-\delta}$$

and

$$L_m = \frac{L_m}{L_{m-1}}L_{m-1} = \frac{m(m+2)}{m+1}L_{m-1} > M\cdot L_{m-1} > e^{2\delta}e^{s-\delta} = e^{s+\delta}.$$

Consequently, $I_j \cap (e^{s-\delta}, e^{s+\delta}) = \varnothing$ for $j \neq m, m-1$. ∎

Theorem 5.3 *$\mathcal{R}$ does not have property A and is not of product type.*

Proof From Lemmas 4.1 and 5.1, we get

$$\lim_{m\to\infty}\mu\Big(\big\{x \in X, \exists y \in X, (x,y) \in \mathcal{R} \text{ and } |\log\delta(y,x)| \in I_m\big\}\Big) \leqslant \mu(F_{\psi(m)}) = 0.$$

This and Lemma 5.2 imply that for any $\delta > 0$, we have

$$\lim_{s\to\infty}\mu\Big(\big\{x \in X, \exists y \in X, (x,y) \in \mathcal{R} \text{ and } |\log\delta(y,x)| \in (e^{s-\delta}, e^{s+\delta})\big\}\Big) = 0.$$

By using Proposition 2.3, we conclude that $\mathcal{R}$ does not have property A, and therefore it is not of product type. ∎

Notice that $M(X, \mu, \mathcal{R})$, the von Neumann algebra associated with $\mathcal{R}$ (see [6]), provides an explicit example of an approximately finite dimensional factor that is not an ITPFI factor.

Note also that the associated flow of this equivalence relation is, up to conjugacy, a flow built under a ceiling function and having as base automorphism the Pascal adic transformation. To obtain this realization of the flow we refer the reader to [8] or [9].

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