Canad. J. Math. 2024, pp. 1–29
http://dx.doi.org/10.4153/S0008414X24000580

Resolving dualities and applications to homological invariants

Hongxing Chen and Jiangsheng Hu

Abstract. Dualities of resolving subcategories of module categories over rings are introduced and characterized as dualities with respect to Wakamatsu tilting bimodules. By restriction of the dualities to smaller resolving subcategories, sufficient and necessary conditions for these bimodules to be tilting are provided. This leads to the Gorenstein version of both the Miyashita's duality and Huisgen-Zimmermann's correspondence. An application of resolving dualities is to show that higher algebraic K-groups and semi-derived Ringel–Hall algebras of finitely generated Gorenstein-projective modules over Artin algebras are preserved under tilting.

1 Introduction

In homological algebra and representation theory, equivalences of additive categories of different types, such as derived equivalences and stable equivalences of Morita type (see [2, 12, 22, 26, 38]), have been applied successfully to compare homological invariants or dimensions of relevant algebras and modules. For example, higher algebraic K-groups and finiteness of global and finitistic dimensions of rings are preserved under derived equivalences (see [16, 35]). As important subjects in tilting theory, tilting modules (see [10, 14, 22, 23, 33]) not only provide a class of derived equivalences of algebras but also have some special properties related to resolving subcategories of module categories. Classical results include the Brenner–Butler tilting theorem (see [10]) and a one-to-one correspondence between basic tilting modules and contravariant finite resolving subcategories of modules with finite projective dimension (see [4]).

However, in contrast to covariant equivalences, dualities of categories have received far less attention. A classical result, due to Morita [1, 34], characterizes dualities of full subcategories of module categories as dualities with respect to some faithfully balanced bimodules provided only that the categories contain the appropriate regular modules. Moreover, the bimodule defines a Morita duality if and only if it is an injective cogenerator as a one-sided module. In the literature, there have been two

Received by the editors July 24, 2023; revised June 9, 2024; accepted July 22, 2024.

This research was supported by the National Natural Science Foundation of China (Grant Nos. 12122112, 12031014, and 12171206). Also, the corresponding author J. S. Hu thanks Jiangsu 333 Project for partial support.

AMS subject classification: **16G10, 18G25, 16W22**, 16E30, 16E20, 18G80.

Keywords: Duality, Gorenstein-projective module, resolving subcategory, semi-derived Ringel–Hall algebra, Wakamatsu tilting module.

important generalizations of Morita dualities. On the one hand, for classical tilting modules of finite projective dimension over general rings, Miyashita established dualities of some resolving subcategories of finitely generated modules with finite projective dimension (see [33]). A converse of this result in the case of Artin algebras has been recently shown by Huisgen-Zimmermann in [24] (see also [25]): any dualities of resolving subcategories of finitely generated modules with finite projective dimension over Artin algebras are always obtained from tilting bimodules provided that the dualities are induced by strictly exact functors. On the other hand, for Wakamatsu tilting modules, dualities of some resolving subcategories of finitely generated modules with relative coresolutions of arbitrary lengths were constructed by Wakamatsu in [42] (see also [21]). In these dualities, (Wakamatsu) tilting modules play the role of relative Ext-injective cogenerators, similar to injective cogenerators in Morita dualities. Let us emphasize that Wakamatsu tilting modules are modules of possibly infinite projective dimension and were one of the many generalizations of tilting modules. They were first introduced by Wakamatsu in [41] and later developed by Mantese and Reiten in [31] to establish a close connection between Wakamatsu tilting modules and resolving subcategories. A close relationship between Wakamatsu tilting modules and tilting modules is predicted by the *Wakamatsu tilting conjecture* which predicts that Wakamatsu tilting modules of finite projective dimension over Artin algebras are tilting modules (see [9, Chapter IV]). This conjecture is of particular interest due to its strong connection with several long-standing homological conjectures such as the finitistic dimension conjecture, the generalized Nakayama conjecture and the Gorenstein symmetry conjecture (see [19, 31]).

In this article, we introduce the notion of resolving dualities between full subcategories of module categories, which unifies the abovementioned examples of dualities.

Definition 1.1 Let $\mathcal{A}$ and $\mathcal{B}$ be the categories of modules over associative rings with identity, and let $\mathcal{C}$ and $\mathcal{D}$ be full subcategories of $\mathcal{A}$ and $\mathcal{B}$, respectively. Two contravariant additive functors

$$F:\mathcal{C}\to\mathcal{D}\quad\text{and}\quad G:\mathcal{D}\to\mathcal{C}$$

are called *inverse resolving dualities* between $\mathcal{C}$ and $\mathcal{D}$ if the following conditions hold:

(1) $\mathcal{C}\subseteq\mathcal{A}$ and $\mathcal{D}\subseteq\mathcal{B}$ are resolving subcategories, that is, they contain all projective modules and are closed under isomorphisms, extensions, and kernels of epimorphisms.
(2) The compositions $G\circ F$ and $F\circ G$ are naturally isomorphic to the identity functors.
(3) F and G are exact functors between $\mathcal{C}$ and $\mathcal{D}$ which are regarded as fully exact subcategories of $\mathcal{A}$ and $\mathcal{B}$, respectively.

Precisely, the functor F satisfies the property: if $0\to X_1\to X_2\to X_3\to 0$ is a short exact sequence in $\mathcal{A}$ with $X_i\in\mathcal{C}$ for $1\leqslant i\leqslant 3$, then $0\to F(X_3)\to F(X_2)\to F(X_1)\to 0$ is a short exact sequence in $\mathcal{B}$. The functor G satisfies a similar property.

https://doi.org/10.4153/S0008414X24000580 Published online by Cambridge University Press

Note that F and G in Definition 1.1(3) were called *strictly exact* functors by Huisgen-Zimmermann in [24]. Given that exact structures of resolving subcategories (as fully exact subcategories, see Section 2.2) of module categories are highlighted in our discussions, we prefer to use the terminology of *exact* functors between exact categories in Definition 1.1(3).

Clearly, Morita duality and Miyachita's duality are resolving dualities. Now, we address the following basic questions on resolving dualities.

(1) How to characterize general dualities of resolving subcategories of finitely generated modules over algebras?

(2) Are there any more resolving dualities between resolving subcategories of module categories associated with special modules over algebras?

(3) Are there applications of resolving dualities to homological invariants?

This article is devoted to providing partial answers to these questions. To state our results, we first introduce some notation and definitions.

Throughout the article, all algebras considered are Artin algebras and all modules are finitely generated left modules. Let A be an algebra. We denote by A-mod the category of left A-modules and by A^{op} the opposite algebra of A. Let T be an A-module. We denote by ${}^{\perp}({}_AT)$ the full subcategory of A-mod consisting of modules X with $\mathrm{Ext}_A^n(X,T)=0$ for all $n\geqslant 1$, and by $\mathcal{W}({}_AT)$ the full subcategory of ${}^{\perp}({}_AT)$ consisting of modules X which has an $\mathrm{add}(T)$-coresolution such that it stays exact after applying the functor $\mathrm{Hom}_A(-,T)$. Then ${}^{\perp}({}_AA)$ and $\mathcal{W}({}_AA)$ are exactly the categories of *semi-Gorenstein-projective* and *Gorenstein-projective* A-modules (see Definitions 2.2 and 2.3), respectively. Given a natural number n, the full subcategories of A-mod consisting of modules with projective, Gorenstein-projective and semi-Gorenstein-projective dimension at most n are denoted by $\mathcal{P}^{\leqslant n}(A)$, $\mathcal{GP}^{\leqslant n}(A)$ and $\mathcal{SGP}^{\leqslant n}(A)$, respectively. For simplicity, we write $\mathcal{P}(A)$, $\mathcal{GP}(A)$, and $\mathcal{SGP}(A)$ for $\mathcal{P}^{\leqslant 0}(A)$, $\mathcal{GP}^{\leqslant 0}(A)$, and $\mathcal{SGP}^{\leqslant 0}(A)$, respectively. As usual, the category of A-modules with finite projective dimension is denoted by $\mathcal{P}^{<\infty}(A)$.

Following [31, 41], T is called a *Wakamatsu tilting* A-module if both A and T belong to $\mathcal{W}({}_AT)$. Further, if ${}_AT$ has finite projective dimension and ${}_AA$ has a finite $\mathrm{add}(T)$-coresolution, then T is called a *tilting* A-module. Let B be another algebra and M an A-B-bimodule. We say that M is *faithfully balanced* if the canonical algebra homomorphisms $B\to \mathrm{End}_A(M)^{op}$ and $A\to\mathrm{End}_{B^{op}}(M)$ are isomorphisms. If, in addition, the module ${}_AM$ is Wakamatsu tilting (respectively, tilting), then M is called a *Wakamatsu tilting (respectively, tilting) A-B-bimodule*. It is known that if T is a Wakamatsu tilting A-module, then it is automatically a Wakamatsu tilting A-B-bimodule with $B:=\mathrm{End}_A(T)^{op}$, the endomorphism algebra of T over A. Similarly, if T is a tilting A-module, it is a tilting A-B-bimodule.

Our main result conveys that any resolving dualities between full subcategories of module categories of algebras are always afforded by Wakamatsu tilting bimodules, which are tilting bimodules when the dualities can be restricted to smaller resolving subcategories.

Theorem 1.2 *Let A and B be Artin algebras, and let $\mathcal{C}\subseteq A$-mod and $\mathcal{D}\subseteq B^{op}$-mod be full subcategories. Suppose that $F:\mathcal{C}\to\mathcal{D}$ and $G:\mathcal{D}\to\mathcal{C}$ are inverse resolving dualities. Then:*

https://doi.org/10.4153/S0008414X24000580 Published online by Cambridge University Press

(1) *There exists a Wakamatsu tilting bimodule ${}_AT_B$ such that:*

(a) $T_B \cong F({}_AA)$ *and* ${}_AT \cong G(B_B)$;
(b) $F \cong Hom_A(-,T)|_{\mathcal{C}}$ *and* $G \cong Hom_{B^{op}}(-,T)|_{\mathcal{D}}$;
(c) $\mathcal{C} \subseteq \mathcal{W}({}_AT)$ *and* $\mathcal{D} \subseteq \mathcal{W}(T_B)$.

(2) *The bimodule ${}_AT_B$ is a tilting bimodule if and only if F and G can be restricted to any one of inverse dualities of the following types:*

$$\mathcal{C} \cap \mathcal{P}^{<\infty}(A) \simeq \mathcal{D} \cap \mathcal{P}^{<\infty}(B^{op}), \qquad \mathcal{C} \cap \mathcal{P}^{\leqslant n}(A) \simeq \mathcal{D} \cap \mathcal{P}^{\leqslant m}(B^{op}),$$

$$\mathcal{C} \cap \mathcal{GP}^{\leqslant n}(A) \simeq \mathcal{D} \cap \mathcal{GP}^{\leqslant m}(B^{op}), \qquad \mathcal{C} \cap \mathcal{SGP}^{\leqslant n}(A) \simeq \mathcal{D} \cap \mathcal{SGP}^{\leqslant m}(B^{op}),$$

where n and m are some natural numbers.

In Theorem 1.2(1), the existence of an (even faithfully balanced) A-B-bimodule T satisfying (a) and (b) is not new, which follows directly from [1, Theorem 23.5]. The really new thing in Theorem 1.2(1) is that T is a Wakamatsu tilting A-B-bimodule and satisfies (c). In order to illustrate the assertion (c) in Theorem 1.2(1), let us recall the resolving dualities established in [42, Theorem 4.2]: for any Wakamatsu tilting bimodule ${}_AT_B$, the contravariant Hom-functors $\mathrm{Hom}_A(-,T) : A\text{-mod} \to B^{op}\text{-mod}$ and $\mathrm{Hom}_{B^{op}}(-,T) : B^{op}\text{-mod} \to A\text{-mod}$ can be restricted to inverse resolving dualities

$$(\diamond) \qquad \mathcal{W}({}_AT) \simeq \mathcal{W}(T_B).$$

Thus (c) in Theorem 1.2(1) implies that $(\diamond)$ are "*maximal*" resolving dualities between subcategories of A-mod and B^{op}-mod which can be restricted to the resolving dualities between $\mathcal{C}$ and $\mathcal{D}$. Moreover, Theorem 1.2(2) provides sufficient and necessary conditions for Wakamatsu tilting bimodules to be tilting from the viewpoint of resolving dualities (see Corollary 3.7). This might be helpful for understanding the Wakamatsu tilting conjecture.

Theorem 1.2 generalizes Morita duality, Miyachita's duality, and Huisgen-Zimmermann's correspondence; see Corollary 3.7 and Remark 3.8 for explanation. Moreover, combining Theorem 1.2 with $(\diamond)$, we obtain a Gorenstein version of both Miyachita's duality (see Theorem 2.5) and Huisgen-Zimmermann's correspondence (see Theorem 2.6).

Corollary 1.3 *Let A and B be Artin algebras.*

(1) *If ${}_AT_B$ is a tilting bimodule, then the functors $Hom_A(-,T)$ and $Hom_{B^{op}}(-,T)$ can be restricted to inverse resolving dualities ${}^{\perp}({}_AT) \cap \mathcal{GP}^{\leqslant \ell}(A) \simeq {}^{\perp}(T_B) \cap \mathcal{GP}^{\leqslant \ell}(B^{op})$, where ℓ denotes the projective dimension of ${}_AT$.*

(2) *Let $\mathcal{C} \subseteq A$-mod and $\mathcal{D} \subseteq B^{op}$-mod be full subcategories. Suppose that there are inverse resolving dualities between $\mathcal{C}$ and $\mathcal{D}$. If $\mathcal{GP}(A) \subseteq \mathcal{C} \subseteq \mathcal{GP}^{\leqslant n}(A)$ and $\mathcal{GP}(B^{op}) \subseteq \mathcal{D} \subseteq \mathcal{GP}^{\leqslant m}(B^{op})$ for some $n, m \in \mathbb{N}$, then there is a tilting bimodule ${}_AT_B$ such that:*

$$\mathcal{C} = {}^{\perp}({}_AT) \cap \mathcal{GP}^{\leqslant \ell}(A) = {}^{\perp}({}_AT) \cap \mathcal{GP}^{<\infty}(A),$$

$$\mathcal{D} = {}^{\perp}(T_B) \cap \mathcal{GP}^{\leqslant \ell}(B^{op}) = {}^{\perp}(T_B) \cap \mathcal{GP}^{<\infty}(B^{op}),$$

where ℓ denotes the projective dimension of ${}_AT$.

It should be pointed out that the resolving dualities in Corollary 1.3(1) are the same as the dualities $(\diamond)$. But this new form in Corollary 1.3(1) is based on our explicit characterization of the categories $\mathcal{W}({}_AT)$ and $\mathcal{W}(T_B)$ in terms of Gorenstein-projective modules (see Corollary 3.4 for details), which seems to be new and will be used in applications. Similar to Theorem 2.6, we hope that $\mathcal{GP}^{\leqslant n}(A)$ and $\mathcal{GP}^{\leqslant m}(B^{op})$ in Corollary 1.3(2) can be replaced by $\mathcal{GP}^{<\infty}(A)$ and $\mathcal{GP}^{<\infty}(B^{op})$, respectively. At this moment, we have no idea how to reach this goal in general, but there are two interesting cases in which this does hold true: (a) both A and B are *CM-finite*, that is, up to isomorphism, there are only finitely many indecomposable Gorenstein-projective A-modules and B-modules; (b) both A and B^{op} have finite finitistic dimension (see Corollary 3.6).

Finally, we apply the resolving dualities in Corollary 1.3 to establish triangle equivalences and homological invariants of algebras linked by tilting bimodules. Similar applications of Miyachita's duality to homotopy categories and algebraic K-groups associated with projective modules are given in Corollary 4.3.

Let $\mathcal{E}$ be an exact category in the sense of Quillen. For $* \in \{\emptyset, +, -, b\}$, the $*$-*derived category* of $\mathcal{E}$, denoted by $\mathscr{D}^*(\mathcal{E})$, is defined to be the Verdier quotient of $*$-homotopy category $\mathscr{K}^*(\mathcal{E})$ of $\mathcal{E}$ by the full triangulated subcategory of strictly exact complexes (see Section 2.2). When $\mathcal{E}$ is a small category (that is, the class of isomorphism classes of objects of $\mathcal{E}$ is a set), we denote by $K_n(\mathcal{E})$ the *nth algebraic K-group* of $\mathcal{E}$ for each $n \in \mathbb{N}$ (see [37]). In particular, $K_n(\mathcal{GP}(A))$ is called the *nth Gorenstein algebraic K-group* of A. When $\mathcal{E}$ is a weakly 1-Gorenstein exact category with finite morphism spaces and finite extension spaces, the *semi-derived Ringel–Hall algebra* of $\mathcal{E}$, denoted by $\mathcal{SDH}(\mathcal{E})$, was defined in [29] (see also [11, 20, 28] for some cases) and applied to stimulate further interactions between communities on Hall algebras and on quantum symmetric pairs. Given any finite-dimensional algebra A over a finite field, $\mathcal{GP}(A)$ is a weakly 1-Gorenstein exact category. If A is 1-Gorenstein, then A-mod is also weakly 1-Gorenstein. In this case, $\mathcal{SDH}(A\text{-mod})$ is called the *semi-derived Ringel–Hall algebra* of A and simply denoted by $\mathcal{SDH}(A)$.

Now, our applications of resolving dualities to homological invariants are given in the following result. In particular, an unexpected application is that semi-derived Ringel–Hall algebras of Gorenstein-projective modules are preserved under tilting. This generalizes some results on the invariance of semi-derived Ringel–Hall algebras of Gorenstein algebras recently obtained by Lu and Wang (see [29]).

Corollary 1.4 *Let A and B be Artin algebras, and let ${}_AT_B$ be a tilting bimodule. Then:*

(1) *There is a triangle equivalence $\mathscr{D}(\mathcal{GP}(A)) \simeq \mathscr{D}(\mathcal{GP}(B))$ which can be restricted to an equivalence $\mathscr{D}^*(\mathcal{GP}(A)) \simeq \mathscr{D}^*(\mathcal{GP}(B))$ for any $* \in \{+, -, b\}$.*
(2) *$K_n(\mathcal{GP}(A)) \simeq K_n(\mathcal{GP}(B))$ for any $n \in \mathbb{N}$.*
(3) *Suppose that A is a finite-dimensional algebra over a finite field and ${}_AT$ has projective dimension at most 1. Then $\mathcal{SDH}(\mathcal{GP}(A)) \cong \mathcal{SDH}(\mathcal{GP}(B))$ as algebras.*

Corollary 1.4(3) improves [29, Corollary A23] in which both A and B are required to be 1-Gorenstein algebras. By Corollary 1.4(3) and Proposition 4.10, we can also provide a new proof of [29, Theorems C(1) and D]. Moreover, our proof is based on resolving dualities rather than covariant equivalences, and thus completely different from the proof given in [29]. This will be explained in Corollary 4.12 and Remark 4.13.

Another crucial idea in our strategy is that a new definition of semi-derived Ringel–Hall algebras of weakly 1-Gorenstein exact categories is introduced, which behaves better under resolving dualities (see Proposition 4.8).

2 Preliminaries

In this section, we give some definitions and collect facts which are used in the article.

2.1 Notation and definitions

Let A be an Artin R-algebra, that is, R is a commutative Artin ring and A is an R-algebra which is finitely generated as an R-module. As usual, A^{op} stands for the opposite algebra of A. Denote by A-mod the category of finitely generated left A-modules. The kernel, image, and cokernel of a homomorphism f in A-mod are denoted by $\mathrm{Ker}(f)$, $\mathrm{Im}(f)$, and $\mathrm{Coker}(f)$, respectively.

Let T be an A-module. We define

$$^{\perp}({}_AT) := \{M \in A\text{-mod} \mid \mathrm{Ext}_A^i(M,T) = 0 \ for\ all\ i \geqslant 1\}.$$

Similarly, $({}_AT)^{\perp}$ can be defined. Denote by $\mathrm{add}({}_AT)$ the full subcategory of A-mod consisting of direct summands of finite direct sums of copies of T. The nth syzygy of ${}_AT$ is denoted by $\Omega_A^n(T)$ for each $n \in \mathbb{N}$.

2.2 Derived categories of exact categories

An exact category $\mathcal{E}$ (in the sense of Quillen) is by definition an additive category endowed with a class of conflations closed under isomorphism and satisfying certain axioms (see [26, 37] for details). When the additive category is abelian, the class of conflations coincides with the class of short exact sequences. An additive functor $F : \mathcal{E} \to \mathcal{E}'$ between exact categories $\mathcal{E}$ and $\mathcal{E}'$ is said to be *exact* if it sends the conflations in $\mathcal{E}$ to the ones in $\mathcal{E}'$.

Let $\mathcal{E}$ be an exact category and $\mathcal{F}$ a full subcategory of $\mathcal{E}$. If $\mathcal{F}$ is closed under extensions in $\mathcal{E}$, then $\mathcal{F}$, endowed with the conflations in $\mathcal{E}$ having their terms in $\mathcal{F}$, is an exact category, and the inclusion $\mathcal{F} \subseteq \mathcal{E}$ is a fully faithful exact functor. In this case, $\mathcal{F}$ is called a *fully exact subcategory* of $\mathcal{E}$ (see [26, Section 4]). Clearly, resolving subcategories (see Definition 1.1(1)) of module categories are fully exact subcategories. *Throughout the article, we always regard resolving subcategories of module categories as exact categories.*

Let $\mathcal{A}$ be an abelian category, and let $\mathcal{E}$ be a fully exact subcategory of $\mathcal{A}$. Denote by $\mathscr{C}(\mathcal{E})$ the category of complexes over $\mathcal{E}$ and by $\mathscr{K}(\mathcal{E})$ the homotopy category of complexes over $\mathcal{E}$. A complex $X \in \mathscr{C}(\mathcal{E})$ is said to be *strictly exact* if it is an exact complex over $\mathcal{A}$ and all of its boundaries belong to $\mathcal{E}$. Let $\mathscr{K}_{ac}(\mathcal{E})$ be the full subcategory of $\mathscr{K}(\mathcal{E})$ consisting of those complexes which are isomorphic to strictly exact complexes. Then $\mathscr{K}_{ac}(\mathcal{E})$ is a full triangulated subcategory of $\mathscr{K}_{ac}(\mathcal{E})$ closed under direct summands. The *unbounded derived category* of $\mathcal{E}$, denoted by $\mathscr{D}(\mathcal{E})$, is defined to be the Verdier quotient of $\mathscr{K}(\mathcal{E})$ by $\mathscr{K}_{ac}(\mathcal{E})$. Similarly, the bounded-below, bounded-above, and bounded derived categories $\mathscr{D}^+(\mathcal{E})$, $\mathscr{D}^-(\mathcal{E})$, and $\mathscr{D}^b(\mathcal{E})$ can be defined.

https://doi.org/10.4153/S0008414X24000580 Published online by Cambridge University Press

In Section 4, we need the following result (see [26, Section 11] and [36, Proposition A.5.8]).

Lemma 2.1 *Let $\mathcal{F}$ and $\mathcal{E}$ be fully exact subcategories of $\mathcal{A}$ with $\mathcal{F} \subseteq \mathcal{E}$. Assume that $\mathcal{E}$ is closed under direct summands in $\mathcal{A}$ and the following two conditions hold:*

(a) *For an exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ with $X \in \mathcal{E}$, if $Y, Z \in \mathcal{F}$, then $X \in \mathcal{F}$.*

(b) *There is a natural number n such that, for each object $E \in \mathcal{E}$, there is an exact sequence in $\mathcal{A}$*

$$0 \to F_n \xrightarrow{f_n} \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} E \to 0$$

with $F_i \in \mathcal{F}$ and $\mathrm{Im}(f_i) \in \mathcal{E}$ for all $0 \leqslant i \leqslant n$.

Then the inclusion $\mathcal{F} \subseteq \mathcal{E}$ induces a triangle equivalence $\mathscr{D}(\mathcal{F}) \to \mathscr{D}(\mathcal{E})$ which can be restricted to an equivalence $\mathscr{D}^*(\mathcal{F}) \to \mathscr{D}^*(\mathcal{E})$ for any $* \in \{+, -, b\}$.

2.3 (Semi-)Gorenstein-projective modules and $\mathcal{X}$-(co)resolutions

A complete projective resolution over A is by definition an exact complex

$$P^\bullet : \cdots \to P^{-1} \xrightarrow{d_P^{-1}} P^0 \xrightarrow{d_P^0} P^1 \to \cdots \to P^n \xrightarrow{d_P^n} P^{n+1} \to \dots$$

of projective A-modules such that the complex $\mathrm{Hom}_A(P^\bullet, A)$, obtained by applying $\mathrm{Hom}_A(-, A)$ to $P^\bullet$, is again exact.

Definition 2.2 [17, 18] An A-module M is said to be *Gorenstein-projective* if there is a complete projective resolution $P^\bullet$ over A such that M is isomorphic to the image of $d_P^{-1} : P^{-1} \to P^0$.

Gorenstein-projective modules were called modules of G-dimension zero in [3] or totally reflexive modules in [6, Section 2]. A generalization of Gorenstein-projective modules is the following.

Definition 2.3 [40] An A-module M is said to be *semi-Gorenstein-projective* provided that $\mathrm{Ext}^i_A(M, A) = 0$ for all $i \geqslant 1$.

Denote by $\mathcal{GP}(A)$ and $\mathcal{SGP}(A)$ the categories of Gorenstein-projective and semi-Gorenstein-projective A-modules, respectively. Then $\mathcal{GP}(A)$ and $\mathcal{SGP}(A)$ are resolving subcategories of A-mod. Moreover, $\mathcal{GP}(A) \subseteq \mathcal{SGP}(A)$, but the converse of the inclusion is not true in general (see [40] for examples).

Let $\mathcal{X}$ be a full subcategory of A-mod. An $\mathcal{X}$*-resolution* of an A-module M is by definition an exact sequence of A-modules $\cdots \to X_2 \to X_1 \to X_0 \to M \to 0$ with $X_i \in \mathcal{X}$ for all i. The length of the resolution is defined to be the supremum of i such that $X_i \neq 0$. Now, the $\mathcal{X}$*-resolution dimension* of M, denote by $\mathrm{res.dim}_{\mathcal{X}}(M)$, is defined to be the minimal natural number n such that M has an $\mathcal{X}$-resolution of length n, or ∞ if no such n exists. The concept of $\mathcal{X}$-coresolutions and $\mathcal{X}$-coresolution dimension of M can be defined dually.

The following result is known in the literature.

Lemma 2.4 *Let $\mathcal{X}$ be a resolving subcategory of A-mod and M an A-module. The following are equivalent for a natural number n:*

https://doi.org/10.4153/S0008414X24000580 Published online by Cambridge University Press

(1) *res.dim*$_{\mathcal{X}}({}_AM) \leqslant n$.
(2) $\Omega_A^n(M) \in \mathcal{X}$.
(3) *For any* $\mathcal{X}$*-resolution* $\cdots \to X_2 \to X_1 \to X_0 \to M \to 0$ *of* M, $\mathrm{Ker}(X_{n-1} \to X_{n-2}) \in \mathcal{X}$.

Associated with $\mathcal{X}$, there are

$$\mathcal{X}^{\leqslant n}(A) := \{M \in A\text{-mod} \mid \mathrm{res.dim}_{\mathcal{X}}({}_AM) \leqslant n\},$$

$$\mathcal{X}^{<\infty}(A) := \{M \in A\text{-mod} \mid \mathrm{res.dim}_{\mathcal{X}}({}_AM) < \infty\}.$$

Clearly, if $\mathcal{X}$ is a resolving subcategory of A-mod, then so are $\mathcal{X}^{\leqslant n}(A)$ and $\mathcal{X}^{<\infty}(A)$. Denote by $\mathcal{P}(A)$ the category of projective A-modules. For simplicity, we set

$$\mathcal{P}^{\leqslant n}(A) := \mathcal{P}(A)^{\leqslant n}(A), \quad \mathcal{P}^{<\infty}(A) := \mathcal{P}(A)^{<\infty}(A),$$

$$\mathcal{GP}^{\leqslant n}(A) := \mathcal{GP}(A)^{\leqslant n}(A), \quad \mathcal{GP}^{<\infty}(A) := \mathcal{GP}(A)^{<\infty}(A),$$

$$\mathcal{SGP}^{\leqslant n}(A) := \mathcal{SGP}(A)^{\leqslant n}(A), \quad \mathcal{SGP}^{<\infty}(A) := \mathcal{SGP}(A)^{<\infty}(A).$$

2.4 Resolving dualities induced by tilting modules

An A-module T is called an *n-tilting* module (see [10, 22, 23, 33]) if the following conditions are satisfied:

(T1) $T \in \mathcal{P}^{\leqslant n}(A)$, that is, the projective dimension of ${}_AT$, denoted by $\mathrm{proj.dim}({}_AT)$, is at most n.
(T2) $\mathrm{Ext}_A^j(T, T) = 0$ for all $j \geqslant 1$.
(T3) There exists an exact sequence of A-modules $0 \to {}_AA \to T_0 \to \cdots \to T_n \to 0$ with $T_i \in \mathrm{add}({}_AT)$ for all $0 \leqslant i \leqslant n$.

A module ${}_AT$ is said to be *tilting* if it is n-tilting for some natural number n. If, in addition, $\mathcal{P}^{<\infty}(A) \subseteq {}^{\perp}({}_AT)$, then ${}_AT$ is said to be *strong tilting* (see [4]).

Given a tilting module ${}_AT$ with $B := \mathrm{End}_A(T)^{op}$, we see that T_B is also a tilting module and A is isomorphic to $\mathrm{End}_{B^{op}}(T)$ as algebras. In this case, we will say that ${}_AT_B$ is a tilting bimodule.

The following theorem is readily deduced from [33, Theorem 3.5].

Theorem 2.5 (Miyashita's duality) *For a tilting bimodule* ${}_AT_B$, *let*

$$\mathcal{C} := {}^{\perp}({}_AT) \cap \mathcal{P}^{<\infty}(A) \text{ and } \mathcal{D} := {}^{\perp}(T_B) \cap \mathcal{P}^{<\infty}(B^{op}).$$

Then the restricted Hom-functors $\mathrm{Hom}_A(-, T)|_{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ *and* $\mathrm{Hom}_{B^{op}}(-, T)|_{\mathcal{D}} : \mathcal{D} \to \mathcal{C}$ *are inverse resolving dualities. Further, if* ${}_AT_B$ *is strong tilting, then there are inverse resolving dualities* $\mathcal{P}^{<\infty}(A) \simeq \mathcal{P}^{<\infty}(B^{op})$.

Recently, Huisgen-Zimmermann has supplemented Miyashita's duality by showing that resolving dualities between subcategories of $\mathcal{P}^{<\infty}(A)$ and $\mathcal{P}^{<\infty}(B^{op})$ are always afforded by tilting bimodules (see [24, Theorem 1]).

Theorem 2.6 (Huisgen-Zimmermann's correspondence) *Let* $\mathcal{C} \subseteq \mathcal{P}^{<\infty}(A)$ *and* $\mathcal{D} \subseteq \mathcal{P}^{<\infty}(B^{op})$ *be resolving subcategories of* A-mod *and* B^{op}*-mod, respectively. If*

https://doi.org/10.4153/S0008414X24000580 Published online by Cambridge University Press

$F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are inverse resolving dualities, then there exists a tilting bimodule ${}_AT_B$ such that:

(a) *$F \cong Hom_A(-, T)|_{\mathcal{C}}$ and $G \cong Hom_{B^{op}}(-, T)|_{\mathcal{D}}$,*
(b) *$\mathcal{C} = {}^{\perp}({}_AT) \cap \mathcal{P}^{<\infty}(A)$ and $\mathcal{D} = {}^{\perp}(T_B) \cap \mathcal{P}^{<\infty}(B^{op})$, and*
(c) *$\mathcal{C}$ consists of those modules M which have finite $\mathrm{add}({}_AT)$-coresolutions and $\mathcal{D}$ consists of those modules N which have finite $\mathrm{add}(T_B)$-coresolutions.*

By Theorem 2.6(c), we have

$${}^{\perp}({}_AT) \cap \mathcal{P}^{<\infty}(A) = {}^{\perp}({}_AT) \cap \mathcal{P}^{\leqslant \ell}(A) \text{ and } {}^{\perp}(T_B) \cap \mathcal{P}^{<\infty}(B^{op}) = {}^{\perp}(T_B) \cap \mathcal{P}^{\leqslant \ell}(B^{op}),$$

where ℓ denotes the projective dimension of ${}_AT$.

3 Correspondence between Wakamatsu tilting modules and resolving dualities

In this section, we discuss resolving subcategories of module categories related to Wakamatsu tilting modules and establish relationships between resolving dualities and Wakamatsu tilting modules. In case, these dualities can be restricted to resolving subcategories of modules with finite projective, Gorenstein-projective or semi-Gorenstein-projective dimension, the associated Wakamatsu tilting modules are shown to be tilting. In particular, we show Theorem 1.2 and Corollary 1.3.

3.1 Basic facts on Wakamatsu tilting modules

Let T be an A-module. We denote by $\mathrm{cogen}^*({}_AT)$ the full subcategory of A-mod consisting of modules M which admits an exact sequence of A-modules $0 \to M \to T_0 \to T_1 \to T_2 \to \cdots$ with $T_i \in \mathrm{add}({}_AT)$ for all $i \geqslant 0$ such that applying the functor $\mathrm{Hom}_A(-, T)$ to the sequence still yields an exact sequence. Equivalent characterizations of $\mathrm{cogen}^*({}_AT)$ are given in the following result (for example, see [30, Lemmas 2.2 and 2.4]).

Lemma 3.1 *Let $B = End_A(T)^{op}$. For an A-module M, the following statements are equivalent.*

(1) *$M \in \mathrm{cogen}^*({}_AT)$.*
(2) *The map $\sigma_M : M \to Hom_{B^{op}}(Hom_A(M, T), T)$, $m \mapsto [f \mapsto f(m)]$ for $m \in M$ and $f \in Hom_A(M, T)$ is an isomorphism and $Hom_A(M, T) \in {}^{\perp}(T_B)$.*
(3) *The canonical maps*

$$Ext^i_A(N, M) \to Ext^i_{B^{op}}(Hom_A(M, T), Hom_A(N, T))$$

induced by the functor $Hom_A(-, T)$ are isomorphisms for all $N \in {}^{\perp}({}_AT)$ and $i \geqslant 0$.

In the introduction, we have defined

$$\mathcal{W}({}_AT) := {}^{\perp}({}_AT) \cap \mathrm{cogen}^*({}_AT).$$

Clearly, $T \in \mathcal{W}({}_AT)$ if and only if $\mathrm{Ext}^i_A(T, T) = 0$ for all $i \geqslant 1$. It follows from Lemma 3.1(3) that the functor $\mathrm{Hom}_A(-, T) : A\text{-mod} \to B^{op}\text{-mod}$ can be restricted to a fully faithful functor $\mathcal{W}({}_AT) \to B^{op}$-mod which preserves extension groups of

https://doi.org/10.4153/S0008414X24000580 Published online by Cambridge University Press

modules. However, in general, $\mathcal{W}({}_AT)$ is not a resolving subcategory of A-mod since it may not contain projective A-modules. Following [41], an A-module T is said to be *Wakamatsu tilting* if $T \in \mathcal{W}({}_AT)$ and ${}_AA \in \mathcal{W}({}_AT)$. This is also equivalent to the following two conditions:

(1) $\mathrm{End}_{B^{op}}(T) \cong A$, where $B = \mathrm{End}_A(T)^{op}$;

(2) $\mathrm{Ext}^i_A(T,T) = 0 = \mathrm{Ext}^i_{B^{op}}(T,T)$ for all $i \geqslant 1$.

By these conditions, if ${}_AT$ is Wakamatsu tilting, then T_B is also Wakamatsu tilting. So T is a Wakamatsu tilting A-B-bimodule.

Now, we collect some basic properties of Wakamatsu tilting modules.

Lemma 3.2 [31, Proposition 2.11] *Let T be a Wakamatsu tilting A-module. Then:*

(1) $\mathcal{W}({}_AT)$ is a resolving subcategory of A-mod.

(2) $\mathcal{W}({}_AT) \cap \mathcal{W}({}_AT)^{\perp} = \mathrm{add}({}_AT)$ and ${}^{\perp}(\mathcal{W}({}_AT)^{\perp}) = \mathcal{W}({}_AT)$.

(3) T is an injective cogenerator for $\mathcal{W}({}_AT)$, that is, for any $X \in \mathcal{W}({}_AT)$, there is an exact sequence $0 \to X \to I_0 \to X_1 \to 0$ in A-mod such that $I_0 \in add({}_AT)$ and $X_1 \in \mathcal{W}({}_AT)$.

For simplicity, sometimes we write (X,Y) for $\mathrm{Hom}_A(X,Y)$ in our proofs for A-modules X and Y.

Lemma 3.3 *Let ${}_AT$ be a Wakamatsu tilting module. Then:*

(1) ${}^{\perp}({}_AT) \cap \mathcal{P}^{\leqslant n}(A) = \mathcal{W}({}_AT) \cap \mathcal{P}^{\leqslant n}(A)$ for any $n \geqslant 0$.

(2) ${}_AT \in \mathcal{P}^{<\infty}(A)$ if and only if ${}_AT \in \mathcal{GP}^{<\infty}(A)$ and $\mathcal{GP}(A) \subseteq \mathcal{W}({}_AT)$.

(3) If ${}_AT \in \mathcal{P}^{<\infty}(A)$, then $\mathcal{W}({}_AT) \cap \mathcal{GP}^{\leqslant n}(A) = {}^{\perp}({}_AT) \cap \mathcal{GP}^{\leqslant n}(A)$ for any $n \geqslant 0$.

(4) If ${}_AT$ is a tilting module of projective dimension ℓ, then $\mathcal{W}({}_AT) \subseteq \mathcal{GP}^{\leqslant \ell}(A)$.

Proof (1) It suffices to prove ${}^{\perp}({}_AT) \cap \mathcal{P}^{\leqslant n}(A) \subseteq \mathrm{cogen}^*({}_AT)$. Let $M \in {}^{\perp}({}_AT) \cap \mathcal{P}^{\leqslant n}(A)$. Then there exists an exact sequence in A-mod

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0,$$

where P_i is projective for all $0 \leqslant i \leqslant n$. This yields the following exact sequence in B^{op}-mod:

$$0 \to \mathrm{Hom}_A(M,T) \to \mathrm{Hom}_A(P_0,T) \to \mathrm{Hom}_A(P_1,T) \to \cdots \to \mathrm{Hom}_A(P_n,T) \to 0.$$

Since $\mathrm{Hom}_A(P_i,T) \in \mathrm{add}(T_B) \subseteq {}^{\perp}(T_B)$ for all $0 \leqslant i \leqslant n$ and since ${}^{\perp}(T_B)$ is a resolving subcategory of B^{op}-mod, we have $\mathrm{Hom}_A(M,T) \in {}^{\perp}(T_B)$. It follows that there is a commutative diagram with exact rows in A-mod:

$$\begin{tikzcd}
0 \arrow[r] & P_n \arrow[r] \arrow[d, "\sigma_{P_n}"] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, "\sigma_{P_0}"] & M \arrow[r] \arrow[d, "\sigma_M"] & 0 \\
0 \arrow[r] & ((P_n,T),T) \arrow[r] & \cdots \arrow[r] & ((P_0,T),T) \arrow[r] & ((M,T),T) \arrow[r] & 0.
\end{tikzcd}$$

Since σ_{P_i} is an isomorphism for all $0 \leqslant i \leqslant n$, so is σ_M. Thus $M \in \mathrm{cogen}^*({}_AT)$ by Lemma 3.1.

(2) Assume that ${}_AT \in \mathcal{GP}^{<\infty}(A)$ and $\mathcal{GP}(A) \subseteq \mathcal{W}({}_AT)$. Since ${}_AT \in \mathcal{GP}^{<\infty}(A)$, we see from [15, Lemma 2.17] that there is an exact sequence $0 \to T \to H \to G \to 0$ in A-mod, where $H \in \mathcal{P}^{<\infty}(A)$ and $G \in \mathcal{GP}(A)$. As $\mathcal{GP}(A) \subseteq \mathcal{W}({}_AT)$, the exact sequence

https://doi.org/10.4153/S0008414X24000580 Published online by Cambridge University Press

$0 \to T \to H \to G \to 0$ splits. Thus ${}_AT$ is isomorphic to a direct summand of H, which forces $T \in \mathcal{P}^{<\infty}(A)$.

Conversely, assume ${}_AT \in \mathcal{P}^{<\infty}(A)$. Then $\mathcal{GP}(A) \subseteq {}^{\perp}({}_AT)$. Let M be an A-module in $\mathcal{GP}(A)$. It is enough to prove that M is in $\mathcal{W}({}_AT)$. There exists a complete projective resolution over A

$$P^\bullet : \cdots \to P^{-1} \overset{d^{-1}}{\to} P^0 \overset{d^0}{\to} P^1 \overset{d^1}{\to} \dots$$

such that $M \cong \mathrm{Im}(d^{-1})$. Set $K^i := \mathrm{Im}(d^i)$ for all $i \in \mathbb{Z}$. Then $K^i \in \mathcal{GP}(A)$. As $\mathcal{GP}(A) \subseteq {}^{\perp}({}_AT)$, the complex $\mathrm{Hom}_A(P^\bullet, T)$ is again exact. Hence we have the following commutative diagrams:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\sigma_M"] & P^0 \arrow[r] \arrow[d, "\cong"] & K^1 \arrow[r] \arrow[d, "\sigma_{K^1}"] & 0 \\
0 \arrow[r] & ((M,T),T) \arrow[r] & ((P^0,T),T) \arrow[r] & (((K^1,T),T) &
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & K^1 \arrow[r] \arrow[d, "\sigma_{K^1}"] & P^1 \arrow[r] \arrow[d, "\cong"] & K^2 \arrow[r] \arrow[d, "\sigma_{K^2}"] & 0 \\
0 \arrow[r] & ((K^1,T),T) \arrow[r] & ((P^1,T),T) \arrow[r] & ((K^2,T),T). &
\end{tikzcd}$$

This implies that both σ_{K^1} and σ_M are injective, and therefore σ_M is an isomorphism by the snake lemma. Similarly, we can show that $\sigma_{K^i} : K^i \to ((K^i, T), T)$ is an isomorphism for any $i \in \mathbb{Z}$. Thus $\mathrm{Ext}^i_B(\mathrm{Hom}_A(M,T),T) = 0$ for all $i \geqslant 1$. By Lemma 3.1, $M \in \mathrm{cogen}^*({}_AT)$. Since $M \in {}^{\perp}({}_AT)$, we have $M \in \mathcal{W}({}_AT)$.

(3) It suffices to show ${}^{\perp}({}_AT) \cap \mathcal{GP}^{\leqslant n}(A) \subseteq \mathcal{W}({}_AT)$. Let $M \in {}^{\perp}({}_AT) \cap \mathcal{GP}^{\leqslant n}(A)$. By [15, Lemma 2.17], there exists an exact sequence $0 \to M \to H \to G \to 0$ in A-mod with $\mathrm{proj.dim}({}_AH) \leqslant n$ and $G \in \mathcal{GP}(A)$. As ${}_AT \in \mathcal{P}^{<\infty}(A)$, we have $G \in {}^{\perp}({}_AT)$. This implies $H \in {}^{\perp}({}_AT)$. Hence $H \in \mathcal{W}({}_AT)$ by (1). Since $G \in \mathcal{W}({}_AT)$ by (2), $M \in \mathcal{W}({}_AT)$ by Lemma 3.2.

(4) Let $M \in \mathcal{W}({}_AT)$. There exists an exact sequence $0 \to M \to X_0 \to X_1 \to \dots$ in A-mod with $X_i \in \mathrm{add}({}_AT)$ for all $i \geqslant 0$ such that it stays exact after applying $\mathrm{Hom}_A(-,T)$. Since $\mathrm{proj.dim}({}_AT) = \ell$, we see from the horseshoe lemma that there exists an exact sequence in A-mod

$$0 \to \Omega^{\ell}_A(M) \to Q_0 \to Q_1 \to \dots,$$

where Q_i is projective for each $i \geqslant 0$. Recall that there exists an exact sequence of A-modules $0 \to {}_AA \to T_0 \to \cdots \to T_\ell \to 0$ with $T_i \in \mathrm{add}({}_AT)$ for all $0 \leqslant i \leqslant \ell$. Since $M \in {}^{\perp}({}_AT)$, there are isomorphisms $\mathrm{Ext}^i_A(\Omega^{\ell}_A(M), A) \cong \mathrm{Ext}^{\ell+i}_A(M,A) \cong \mathrm{Ext}^i_A(M, T_\ell) = 0$ for all $i \geqslant 1$. This implies $M \in \mathcal{GP}^{\leqslant \ell}(A)$. ■

The following corollary is an immediate consequence of Lemma 3.3.

https://doi.org/10.4153/S0008414X24000580 Published online by Cambridge University Press

Corollary 3.4 *Let ${}_AT$ be a Wakamatsu tilting module. Then:*

(1) ${}^{\perp}({}_AT) \cap \mathcal{P}^{<\infty}(A) = \mathcal{W}({}_AT) \cap \mathcal{P}^{<\infty}(A)$.

(2) *If ${}_AT \in \mathcal{P}^{<\infty}(A)$, then $\mathcal{W}({}_AT) \cap \mathcal{GP}^{<\infty}(A) = {}^{\perp}({}_AT) \cap \mathcal{GP}^{<\infty}(A)$.*

(3) *If ${}_AT$ is a tilting module of projective dimension ℓ, then*

$$\mathcal{W}({}_AT) = {}^{\perp}({}_AT) \cap \mathcal{GP}^{\leqslant \ell}(A) = {}^{\perp}({}_AT) \cap \mathcal{GP}^{<\infty}(A),$$

$$\mathcal{W}(T_B) = {}^{\perp}(T_B) \cap \mathcal{GP}^{\leqslant \ell}(B^{op}) = {}^{\perp}(T_B) \cap \mathcal{GP}^{<\infty}(B^{op}).$$

3.2 Proofs of Theorem 1.2 and Corollary 1.3

We begin this subsection with a proof of Theorem 1.2(1).

Proof of Theorem 1.2(1) Recall from [1, Theorem 23.5] that, given inverse (not necessarily resolving) dualities $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ with ${}_AA \in \mathcal{C}$ and $B_B \in \mathcal{D}$, there exists a faithfully balanced bimodule ${}_AT_B$ with $T_B \cong F({}_AA) \in \mathcal{D}$ and ${}_AT \cong G(B_B) \in \mathcal{C}$ such that $F \cong \mathrm{Hom}_A(-, T)|_{\mathcal{C}}$ and $G \cong \mathrm{Hom}_{B^{op}}(-, T)|_{\mathcal{D}}$. Now, assume that F and G are resolving dualities. They are exact functors between exact categories $\mathcal{C}$ and $\mathcal{D}$. Let $\delta : 0 \to T \to X \to M \to 0$ be an exact sequence in A-mod with $M \in \mathcal{C}$. Since $\mathcal{C}$ is closed under extensions in A-mod, X belongs to $\mathcal{C}$. As F is exact, the sequence $F(\delta)$: $0 \to F(M) \to F(X) \to F(T) \to 0$ is exact. It follows from $F(T) \cong FG(B_B) \cong B_B$ that $F(\delta)$ splits in B^{op}-mod. Note that GF is naturally isomorphic to the identity functor of $\mathcal{C}$. Thus δ splits in A-mod. This implies that $\mathrm{Ext}^1_A(M, T) = 0$ for all $M \in \mathcal{C}$. Since $\mathcal{C} \subseteq A$-mod is a resolving subcategory, $\Omega^i_A(M) \in \mathcal{C}$ for all $M \in \mathcal{C}$ and $i \geqslant 1$. Consequently, $\mathrm{Ext}^i_A(M, T) \cong \mathrm{Ext}^1_A(\Omega^{i-1}_A(M), T) = 0$. This yields $\mathcal{C} \subseteq {}^{\perp}({}_AT)$. In particular, ${}_AT \in {}^{\perp}({}_AT)$. Similarly, one can prove $\mathcal{D} \subseteq {}^{\perp}(T_B)$, which gives rise to $T_B \in {}^{\perp}(T_B)$. Since ${}_AT_B$ is faithfully balanced, it is Wakamatsu tilting.

To prove (c), it suffices to show the inclusion $\mathcal{C} \subseteq \mathcal{W}({}_AT)$, while the inclusion $\mathcal{D} \subseteq \mathcal{W}(T_B)$ can be shown dually.

In fact, as $\mathcal{C} \subseteq {}^{\perp}({}_AT)$, we only need to show $\mathcal{C} \subseteq \mathrm{cogen}^*({}_AT)$. For each $X \in \mathcal{C}$, let

$$\cdots \to Q_1 \xrightarrow{g_1} Q_0 \xrightarrow{g_0} F(X) \to 0$$

be a projective resolution of $F(X)$ in B^{op}-mod. Clearly, $F(X) \in \mathcal{D}$ and $Q_i \in \mathcal{D}$ for all $i \geqslant 0$. Since $\mathcal{D}$ is a resolving subcategory of B^{op}-mod, $\mathrm{Ker}(g_i) \in \mathcal{D}$ for all $i \geqslant 0$. By the exactness of G, we obtain an exact sequence of A-modules $0 \to X \to G(Q_0) \to G(Q_1) \to \dots$ in which $G(Q_i) \in \mathrm{add}(G(B_B)) = \mathrm{add}({}_AT)$ for all $i \geqslant 0$. Thus the exactness of F implies $X \in \mathrm{cogen}^*({}_AT)$. This finishes the proof of Theorem 1.2(1). ■

To show Theorem 1.2(2), we first show the following result.

Lemma 3.5 *Let ${}_AT_B$ be the Wakamatsu tilting bimodule associated with inverse resolving dualities $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ in Theorem 1.2(1). Suppose that $\mathcal{U} \subseteq A$-mod and $\mathcal{V} \subseteq B^{op}$-mod are resolving subcategories. Let m and n be natural numbers. Then:*

(1) *F and G restrict to inverse dualities $\mathcal{C} \cap \mathcal{U}^{\leqslant n}(A) \simeq \mathcal{D} \cap \mathcal{V}^{\leqslant m}(B^{op})$ if and only if the following condition holds:*

$$(*) \quad F(\mathcal{C} \cap \mathcal{U}) \subseteq \mathcal{D} \cap \mathcal{V}^{\leqslant m}(B^{op}) \quad \textit{and} \quad G(\mathcal{D} \cap \mathcal{V}) \subseteq \mathcal{C} \cap \mathcal{U}^{\leqslant n}(A).$$

https://doi.org/10.4153/S0008414X24000580 Published online by Cambridge University Press

(2) *Assume that the condition* $(*)$ *holds. The following statements are true.*

(a) $\mathcal{C} \cap \mathcal{U}^{<\infty}(A) = \mathcal{C} \cap \mathcal{U}^{\leqslant n}(A)$. *If* $\mathcal{U} \subseteq \mathcal{C}$, *then* $\mathcal{C} \cap \mathcal{U}^{<\infty}(A) = {}^{\perp}({}_AT) \cap \mathcal{U}^{<\infty}(A)$.

(b) $\mathcal{D} \cap \mathcal{V}^{<\infty}(B^{op}) = \mathcal{D} \cap \mathcal{V}^{\leqslant m}(B^{op})$. *If* $\mathcal{V} \subseteq \mathcal{D}$, *then* $\mathcal{D} \cap \mathcal{V}^{<\infty}(B^{op}) = {}^{\perp}(T_B) \cap \mathcal{V}^{<\infty}(B^{op})$.

(c) *If* $\mathcal{U} \subseteq \mathcal{SGP}(A)$ *and* $\mathcal{V} \subseteq \mathcal{SGP}(B^{op})$, *then* ${}_AT_B$ *is a tilting bimodule.*

Proof (1) The necessity of (1) is clear since $\mathcal{U} \subseteq \mathcal{U}^{\leqslant n}(A)$ and $\mathcal{V} \subseteq \mathcal{V}^{\leqslant m}(B^{op})$. It suffices to show the sufficiency of (1). Assume that the condition $(*)$ holds. Let $M \in \mathcal{C} \cap \mathcal{U}^{\leqslant n}(A)$. We need to show $F(M) \in \mathcal{D} \cap \mathcal{V}^{\leqslant m}(B^{op})$.

Let $\cdots \to P_1 \to P_0 \to M \to 0$ be a projective resolution of ${}_AM$ and let $s := \max\{m, n\}$. Since $\mathcal{C}$ and $\mathcal{U}$ are resolving subcategories of A-mod, $\Omega_A^s(M) \in \mathcal{C} \cap \mathcal{U}$ by Lemma 2.4. It follows from $F(\mathcal{C} \cap \mathcal{U}) \subseteq \mathcal{D} \cap \mathcal{V}^{\leqslant m}(B^{op})$ that $F(\Omega_A^s(M)) \in \mathcal{D} \cap \mathcal{V}^{\leqslant m}(B^{op})$. Since $M \in \mathcal{C} \subseteq {}^{\perp}({}_AT)$ by Theorem 1.2(1), there is an exact sequence in B^{op}-mod:

$$0 \to F(M) \to F(P_0) \to F(P_1) \to \cdots \to F(P_{s-1}) \to F(\Omega_A^s(M)) \to 0.$$

Observe that both $\mathcal{D}$ and $\mathcal{V}^{\leqslant m}(B^{op})$ are resolving subcategories of B^{op}-mod, and so is their intersection $\mathcal{D} \cap \mathcal{V}^{\leqslant m}(B^{op})$. Since $F(P_i) \in F(\mathrm{add}({}_AA)) \subseteq F(\mathcal{C} \cap \mathcal{U})$ for all $0 \leqslant i \leqslant s-1$, we have $F(M) \in \mathcal{D} \cap \mathcal{V}^{\leqslant m}(B^{op})$.

(2) It is enough to prove (a) and (c) since (b) is dual to (a).

(a) To prove $\mathcal{C} \cap \mathcal{U}^{<\infty}(A) = \mathcal{C} \cap \mathcal{U}^{\leqslant n}(A)$, it suffices to prove $\mathcal{C} \cap \mathcal{U}^{<\infty}(A) \subseteq \mathcal{C} \cap \mathcal{U}^{\leqslant n}(A)$. Let $X \in \mathcal{C} \cap \mathcal{U}^{<\infty}(A)$. Then there is a natural number t such that $\Omega_A^t(X) \in \mathcal{C} \cap \mathcal{U}$ by Lemma 2.4. It follows from the proof of the sufficiency of (1) that $F(X) \in \mathcal{D} \cap \mathcal{V}^{\leqslant m}(B^{op})$. Since $X \cong GF(X)$, we have $X \in \mathcal{C} \cap \mathcal{U}^{\leqslant n}(A)$.

Suppose $\mathcal{U} \subseteq \mathcal{C}$. By Theorem 1.2(1), we only need to check ${}^{\perp}({}_AT) \cap \mathcal{U}^{<\infty}(A) \subseteq \mathcal{C} \cap \mathcal{U}^{<\infty}(A)$. Let $Y \in {}^{\perp}({}_AT) \cap \mathcal{U}^{<\infty}(A)$. Then there exists an exact sequence in A-mod:

$$0 \to U_t \to U_{t-1} \to \cdots \to U_1 \to U_0 \to Y \to 0,$$

where $U_i \in \mathcal{U}$ for all $0 \leqslant i \leqslant t \in \mathbb{N}$. Note that $\mathcal{C} \subseteq \mathcal{W}({}_AT) \subseteq {}^{\perp}({}_AT)$ by Theorem 1.2(1). Since $Y \in {}^{\perp}({}_AT)$ and $\mathcal{U} \subseteq \mathcal{C}$, applying $F := \mathrm{Hom}_A(-, T)$ to the above exact sequence yields an exact sequence in B^{op}-mod:

$$0 \to F(Y) \to F(U_0) \to F(U_1) \to \cdots \to F(U_{t-1}) \to F(U_t) \to 0.$$

As $\mathcal{U} \subseteq \mathcal{C}$, we see from the condition $(*)$ that $F(U_i) \in F(\mathcal{U}) \subseteq \mathcal{D} \cap \mathcal{V}^{\leqslant m}(B^{op})$ for all $0 \leqslant i \leqslant t$. Note that $\mathcal{D} \cap \mathcal{V}^{\leqslant m}(B^{op})$ is a resolving subcategory of B^{op}-mod. Thus $F(Y) \in \mathcal{D} \cap \mathcal{V}^{\leqslant m}(B^{op})$. By the exactness of G, the following sequence

$$0 \to GF(U_t) \to GF(U_{t-1}) \to \cdots \to GF(U_1) \to GF(U_0) \to GF(Y) \to 0$$

is exact in A-mod. Recall that GF is naturally isomorphic to the identity functor of $\mathcal{C}$. Since $U_i \in \mathcal{U} \subseteq \mathcal{C}$ for all $0 \leqslant i \leqslant t$, we have $Y \cong GF(Y) \in G(\mathcal{D}) \subseteq \mathcal{C}$.

(c) Since ${}_AT \cong G(B_B)$ by Theorem 1.2(1), we obtain ${}_AT \in \mathcal{C} \cap \mathcal{U}^{\leqslant s}(A)$. As $\mathcal{C}$ and $\mathcal{U}$ are resolving subcategories of A-mod, it follows from Lemma 2.4 that $\Omega_A^i(T) \in \mathcal{C} \cap \mathcal{U}$ for all $i \geqslant s$. In particular, $\Omega_A^{s+1}(T) \in \mathcal{C} \cap \mathcal{U}$. By $(*)$, $F(\Omega_A^{s+1}(T)) \in \mathcal{D} \cap \mathcal{V}^{\leqslant s}(B^{op})$.

https://doi.org/10.4153/S0008414X24000580 Published online by Cambridge University Press

Moreover, from the minimal projective resolution $\cdots \to P_2 \to P_1 \to P_0 \to {}_AT \to 0$ of the module ${}_AT$, we obtain an exact sequence in B^{op}-mod

$$0 \to B \to F(P_0) \to F(P_1) \to F(P_2) \to \dots,$$

where $F(P_i) \cong \mathrm{Hom}_A(P_i, T) \in \mathrm{add}(T_B)$ for all $i \geqslant 0$. Since $\mathcal{D} \subseteq {}^{\perp}(T_B)$ by Theorem 1.2(1)(c) and $F(\Omega_A^{s+1}(T)) \in \mathcal{D}$, it is clear that $\mathrm{Ext}^j_{B^{op}}(F(\Omega_A^{s+1}(T)), F(P_i)) = 0$ for all $j \geqslant 1$ and $i \geqslant 0$. Consequently, there are isomorphisms

$$\begin{aligned}\mathrm{Ext}^1_{B^{op}}\big(F(\Omega_A^{s+1}(T)), F(\Omega_A^{s}(T))\big) &\cong \mathrm{Ext}^{s+1}_{B^{op}}\big(F(\Omega_A^{s+1}(T)), B\big)\\ &\cong \mathrm{Ext}^1_{B^{op}}\big(\Omega^s_{B^{op}}(F(\Omega_A^{s+1}(T))), B\big).\end{aligned}$$

By Lemma 2.4, $\Omega^s_{B^{op}}(F(\Omega_A^{s+1}(T))) \in \mathcal{V}$. Since $\mathcal{V} \subseteq \mathcal{SGP}(B^{op}) = {}^{\perp}(B_B)$ by assumption, we have $\mathrm{Ext}^1_{B^{op}}\big(\Omega^s_{B^{op}}(F(\Omega_A^{s+1}(T))), B\big) = 0$, and hence $\mathrm{Ext}^1_{B^{op}}\big(F(\Omega_A^{s+1}(T)), F(\Omega_A^{s}(T))\big) = 0$. This implies that the sequence

$$0 \to F(\Omega_A^{s}(T)) \to F(P_s) \to F(\Omega_A^{s+1}(T)) \to 0$$

splits. Thus $F(\Omega_A^{s}(T))$ is a direct summand of $F(P_s)$, and belongs to $\mathrm{add}(T_B)$. Since $\Omega_A^s(T) \in \mathcal{C}$, we have $\Omega_A^s(T) \cong GF(\Omega_A^s(T)) \in \mathrm{add}({}_AA)$. This forces $\mathrm{proj.dim}({}_AT) \leqslant s$. Similarly, we can show $\mathrm{proj.dim}(T_B) \leqslant s$. Thus ${}_AT_B$ is tilting. ∎

Proof of Theorem 1.2(2) The sufficiency of (2) is a direct consequence of Lemma 3.5. Now, assume that ${}_AT_B$ is a tilting bimodule with $\ell = \mathrm{proj.dim}({}_AT)$. By Theorem 1.2(1) and Corollary 3.4(3), $\mathcal{C} \subseteq \mathcal{W}({}_AT) \subseteq \mathcal{GP}^{\leqslant \ell}(A)$ and $\mathcal{D} \subseteq \mathcal{W}(T_B) \subseteq \mathcal{GP}^{\leqslant \ell}(B^{op})$. It follows that $\mathcal{C} \cap \mathcal{GP}^{\leqslant \ell}(A) = \mathcal{C}$ and $\mathcal{D} \cap \mathcal{GP}^{\leqslant \ell}(B^{op}) = \mathcal{D}$. Since $\mathcal{GP}^{\leqslant \ell}(A) \subseteq \mathcal{SGP}^{\leqslant \ell}(A)$, we also have $\mathcal{C} \cap \mathcal{SGP}^{\leqslant \ell}(A) = \mathcal{C}$ and $\mathcal{D} \cap \mathcal{SGP}^{\leqslant \ell}(B^{op}) = \mathcal{D}$. Thus the assertions on (semi-)Gorenstein-projective modules in the necessity of (2) automatically hold.

Now, we apply Lemma 3.5 to show the assertion on projective modules in the necessity of (2).

Let $\mathcal{U} := \mathrm{add}({}_AA)$ and $\mathcal{V} := \mathrm{add}(B_B)$. Since $\mathcal{C} \subseteq A$-mod and $\mathcal{D} \subseteq B^{op}$-mod are resolving subcategories, $\mathcal{U} = \mathcal{U} \cap \mathcal{C}$ and $\mathcal{V} = \mathcal{V} \cap \mathcal{D}$. Clearly, we have $F(M) = \mathrm{Hom}_A(M, T) \in \mathrm{add}(T_B)$ for any $M \in \mathcal{U}$. Since $T_B = F(A) \in \mathcal{D} \cap \mathcal{V}^{\leqslant \ell}(B^{op})$, it is clear that $F(M) \in \mathcal{D} \cap \mathcal{V}^{\leqslant \ell}(B^{op})$. Dually, $G(N) \in \mathcal{C} \cap \mathcal{U}^{\leqslant \ell}(A)$ for any $N \in \mathcal{V}$. By Lemma 3.5(1), F and G can be restricted to inverse resolving dualities $\mathcal{C} \cap \mathcal{P}^{\leqslant \ell}(A) \simeq \mathcal{D} \cap \mathcal{P}^{\leqslant \ell}(B^{op})$. Further, by Lemma 3.5(2), we have $\mathcal{C} \cap \mathcal{P}^{<\infty}(A) = \mathcal{C} \cap \mathcal{P}^{\leqslant \ell}(A)$ and $\mathcal{D} \cap \mathcal{P}^{\leqslant \ell}(B^{op}) = \mathcal{D} \cap \mathcal{P}^{<\infty}(B^{op})$. This finishes the proof of (2). ∎

Proof of Corollary 1.3 (1) follows from Corollary 3.4(3) and the resolving dualities $\mathcal{W}({}_AT) \simeq \mathcal{W}(T_B)$ by [42, Theorem 4.2].

(2) Suppose $\mathcal{C} \subseteq \mathcal{GP}^{\leqslant n}(A)$ and $\mathcal{D} \subseteq \mathcal{GP}^{\leqslant m}(B^{op})$ for some $n, m \in \mathbb{N}$. Then we have

$$\mathcal{C} \cap \mathcal{GP}^{\leqslant n}(A) = \mathcal{C} \text{ and } \mathcal{D} \cap \mathcal{GP}^{\leqslant m}(B^{op}) = \mathcal{D}.$$

It follows from Theorem 1.2(2) that ${}_AT_B$ is a tilting bimodule. In Lemma 3.5, we set $\mathcal{U} := \mathcal{GP}(A)$ and $\mathcal{V} := \mathcal{GP}(B^{op})$, and then the condition $(*)$ is satisfied. Further, assume $\mathcal{GP}(A) \subseteq \mathcal{C}$ and $\mathcal{GP}(B^{op}) \subseteq \mathcal{D}$. Then we have $\mathcal{C} = {}^{\perp}({}_AT) \cap \mathcal{GP}^{<\infty}(A)$ and $\mathcal{D} = {}^{\perp}({}_AT) \cap \mathcal{GP}^{<\infty}(B^{op})$ by Lemma 3.5(2). Since T is tilting, we have $\mathcal{C} = {}^{\perp}({}_AT) \cap \mathcal{GP}^{\leqslant \ell}(A)$ and $\mathcal{D} = {}^{\perp}(T_B) \cap \mathcal{GP}^{\leqslant \ell}(B^{op})$ by Corollary 3.4(3). ∎

https://doi.org/10.4153/S0008414X24000580 Published online by Cambridge University Press

Inspired by Theorem 2.6, we are going to provide two sufficient conditions for replacing $\mathcal{GP}^{\leqslant n}(A)$ and $\mathcal{GP}^{\leqslant m}(B^{op})$ in Corollary 1.3(2) with $\mathcal{GP}^{<\infty}(A)$ and $\mathcal{GP}^{<\infty}(B^{op})$, respectively.

Recall from [8] that an algebra A is said to be *of finite Cohen–Macaulay type*, or simply, *CM-finite*, if there are only finitely many isomorphism classes of indecomposable finitely generated Gorenstein-projective modules. Clearly, A is CM-finite if A is of finite representation type. More examples of CM-finite algebras can be found in [39].

The *(little) finitistic dimension* of A, denoted by $\mathrm{findim}(A)$, is defined to be the supremum of projective dimensions of all finitely generated A-modules of finite projective dimension. Related to finitistic dimension, there is a famous conjecture in the representation theory of algebras, called the *finitistic dimension conjecture*, which says that $\mathrm{findim}(A) < \infty$ for any Artin algebra A (for example, see Bass [7] and [5, Conjectures]).

Corollary 3.6 *Let A and B be algebras, and let $\mathcal{C} \subseteq A$-mod and $\mathcal{D} \subseteq B^{op}$-mod be full subcategories. Suppose that there are inverse resolving dualities between $\mathcal{C}$ and $\mathcal{D}$ such that*

$$\mathcal{GP}(A) \subseteq \mathcal{C} \subseteq \mathcal{GP}^{<\infty}(A) \text{ and } \mathcal{GP}(B^{op}) \subseteq \mathcal{D} \subseteq \mathcal{GP}^{<\infty}(B^{op}).$$

Then there is a tilting bimodule ${}_AT_B$ such that

$$\mathcal{C} = {}^{\perp}({}_AT) \cap \mathcal{GP}^{<\infty}(A) \text{ and } \mathcal{D} = {}^{\perp}(T_B) \cap \mathcal{GP}^{<\infty}(B^{op})$$

under either of the following conditions:

(a) *both A and B are CM-finite;*

(b) $\mathrm{findim}(A) < \infty$ *and* $\mathrm{findim}(B^{op}) < \infty$.

Proof Assume that (a) holds. Let X be the direct sum of all non-isomorphic, indecomposable, finitely generated Gorenstein-projective A-modules. Since A is CM-finite, X is a finite direct sum and $\mathrm{add}({}_AX) = \mathcal{GP}(A)$. As $\mathcal{GP}(A) \subseteq \mathcal{C}$ and $\mathcal{D} \subseteq \mathcal{GP}^{<\infty}(B^{op})$, we have $F(X) \in \mathcal{GP}^{<\infty}(B^{op})$. This implies that there exists a natural number m such that $F(X) \in \mathcal{GP}^{\leqslant m}(B^{op})$, and thus $F(\mathcal{C} \cap \mathcal{GP}(A)) = F(\mathcal{GP}(A)) \subseteq \mathcal{D} \cap \mathcal{GP}^{\leqslant m}(B^{op})$. Since $\mathrm{Hom}_B(-, B) : \mathcal{GP}(B) \to \mathcal{GP}(B^{op})$ is a duality of exact categories and B is CM-finite, B^{op} is also CM-finite. In a similar way, we can show that $G(\mathcal{D} \cap \mathcal{GP}(B^{op})) = G(\mathcal{GP}(B^{op})) \subseteq \mathcal{C} \cap \mathcal{GP}^{\leqslant n}(A)$ for some natural number n. Now, Corollary 3.6 follows from Lemma 3.5.

Assume that (b) holds. Thanks to [43, Lemma 4.4], we have

$$\mathcal{GP}^{<\infty}(A) = \mathcal{GP}^{\leqslant s}(A) \text{ and } \mathcal{GP}^{<\infty}(B^{op}) = \mathcal{GP}^{\leqslant t}(B^{op}),$$

where $s = \mathrm{findim}(A)$ and $t = \mathrm{findim}(B^{op})$. Thus Corollary 3.6 follows from Corollary 1.3(2). ■

Finally, we provide several equivalent characterizations for Wakamatsu tilting bimodules to be tilting. This may be helpful for understanding the Wakamatsu tilting conjecture from the viewpoint of resolving dualities.

Corollary 3.7 *A Wakamatsu tilting bimodule ${}_AT_B$ is tilting if and only if the functors $\mathrm{Hom}_A(-, T)$ and $\mathrm{Hom}_{B^{op}}(-, T)$ can be restricted to any one of inverse resolving dualities of the following types:*

https://doi.org/10.4153/S0008414X24000580 Published online by Cambridge University Press

(1) $\mathcal{W}({}_AT) \cap \mathcal{P}^{<\infty}(A) \simeq \mathcal{W}(T_B) \cap \mathcal{P}^{<\infty}(B^{op})$;
(2) ${}^{\perp}({}_AT) \cap \mathcal{P}^{<\infty}(A) \simeq {}^{\perp}(T_B) \cap \mathcal{P}^{<\infty}(B^{op})$;
(3) $\mathcal{W}({}_AT) \cap \mathcal{GP}^{\leqslant n}(A) \simeq \mathcal{W}(T_B) \cap \mathcal{GP}^{\leqslant m}(B^{op})$ *for some* $n, m \in \mathbb{N}$;
(4) ${}^{\perp}({}_AT) \cap \mathcal{GP}^{\leqslant n}(A) \simeq {}^{\perp}(T_B) \cap \mathcal{GP}^{\leqslant m}(B^{op})$ *for some* $n, m \in \mathbb{N}$;
(5) $\mathcal{W}({}_AT) \cap \mathcal{SGP}^{\leqslant n}(A) \simeq \mathcal{W}(T_B) \cap \mathcal{SGP}^{\leqslant m}(B^{op})$ *for some* $n, m \in \mathbb{N}$.

Proof Recall that the functors $F := \mathrm{Hom}_A(-, T)$ and $G := \mathrm{Hom}_{B^{op}}(-, T)$ between A-mod and B^{op}-mod can be restricted to inverse resolving dualities $\mathcal{W}({}_AT) \simeq \mathcal{W}(T_B)$ by [42, Theorem 4.2]. So, the necessity of Corollary 3.7 follows from Theorem 1.2(2) and Lemma 3.3(1)(3). As to the sufficiency of Corollary 3.7, all the cases except (4) are direct consequences of Theorem 1.2(2) and Lemma 3.3(1). It remains to show the case (4). Suppose that F and G are restricted to the inverse resolving dualities ${}^{\perp}({}_AT) \cap \mathcal{GP}^{\leqslant n}(A) \simeq {}^{\perp}(T_B) \cap \mathcal{GP}^{\leqslant m}(B^{op})$ for some $n, m \in \mathbb{N}$. By Theorem 1.2(1), ${}^{\perp}({}_AT) \cap \mathcal{GP}^{\leqslant n}(A) \subseteq \mathcal{W}({}_AT)$ and ${}^{\perp}(T_B) \cap \mathcal{GP}^{\leqslant m}(B^{op}) \subseteq \mathcal{W}(T_B)$. This implies that ${}^{\perp}({}_AT) \cap \mathcal{GP}^{\leqslant n}(A) = \mathcal{W}({}_AT) \cap \mathcal{GP}^{\leqslant n}(A)$ and ${}^{\perp}(T_B) \cap \mathcal{GP}^{\leqslant n}(B^{op}) = \mathcal{W}(T_B) \cap \mathcal{GP}^{\leqslant n}(B^{op})$, and then we return to the case (3). ■

Remark 3.8 In Theorem 1.2, if $\mathcal{C} = A$-mod and $\mathcal{D} = B^{op}$-mod, then ${}_AT$ and T_B are injective cogenerators since A-mod $= \mathcal{W}({}_AT)$ and B^{op}-mod $= \mathcal{W}(T_B)$. By Corollary 3.7(2), we obtain Miyachita's duality (see Theorem 2.5). Further, if $F : \mathcal{C} \to \mathcal{D}$ is a resolving duality with $\mathcal{C} \subseteq \mathcal{P}^{<\infty}(A)$ and $\mathcal{D} \subseteq \mathcal{P}^{<\infty}(B^{op})$, then the conditions (a) and (b) in Huisgen-Zimmermann's correspondence (see Theorem 2.6) also follow from Theorem 1.2 and Lemma 3.5.

4 Applications to establish homological invariances under tilting

In this section, we apply the resolving dualities established in the former section to homological invariants of algebras. On the one hand, we employ Corollary 1.3 to construct triangle equivalences of derived categories of Gorenstein-projective modules and to show that higher algebraic K-groups of Gorenstein-projective modules are invariant under tilting. On the other hand, we show that semi-derived Ringel–Hall algebras of Gorenstein-projective modules over algebras are preserved under tilting (see Theorem 4.9), which generalizes some results in [29, Proposition A5].

4.1 Derived equivalences and algebraic K-groups of Gorenstein-projective modules

Throughout this subsection, for a small exact category $\mathcal{E}$, we denote by $K(\mathcal{E})$ the K-theory space of $\mathcal{E}$ in the sense of Quillen (see [37]), and by $K_n(\mathcal{E})$ the nth homotopy group of $K(\mathcal{E})$ (called the nth algebraic K-group of $\mathcal{E}$).

Let us recall two classical results on algebraic K-theory of exact categories. One is usually called the "resolution theorem" (see, for example, [37, Section 4]); the other conveys that algebraic K-groups of exact categories are invariant under dualities.

Lemma 4.1 *Let $\mathcal{E}'$ be a full subcategory of a small exact category $\mathcal{E}$. Assume that the following two conditions hold:*

https://doi.org/10.4153/S0008414X24000580 Published online by Cambridge University Press

(a) *If $X \rightarrowtail Y \twoheadrightarrow Z$ is a conflation in $\mathcal{E}$ with $Z \in \mathcal{E}'$, then $Y \in \mathcal{E}'$ if and only if $X \in \mathcal{E}'$.*
(b) *For any object $M \in \mathcal{E}$, there is an exact sequence in $\mathcal{E}$:*

$$0 \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 \to M \to 0$$

such that $M_i \in \mathcal{E}'$ for all $0 \leqslant i \leqslant n$.

Then the inclusion $\mathcal{E}' \subseteq \mathcal{E}$ of exact categories induces a homotopy equivalence of K-theory space $K(\mathcal{E}') \xrightarrow{\simeq} K(\mathcal{E})$. In particular, $K_n(\mathcal{E}') \simeq K_n(\mathcal{E})$ for all $n \in \mathbb{N}$.

Lemma 4.2 *If $F : \mathcal{A}_1 \to \mathcal{A}_2$ is a duality of small exact categories, then $K_n(\mathcal{A}_1) \simeq K_n(\mathcal{A}_2)$ for all $n \in \mathbb{N}$.*

Proof Since F induces an equivalence $\mathcal{A}_1 \xrightarrow{\simeq} \mathcal{A}_2{}^{op}$ of small exact categories, it follows that $K_n(\mathcal{A}_1) \simeq K_n(\mathcal{A}_2{}^{op})$ for each $n \in \mathbb{N}$. Lemma 4.2 follows from $K_n(\mathcal{A}_2) \simeq K_n(\mathcal{A}_2{}^{op})$ (see [37, Section 2]). ∎

Proof of Corollary 1.4(1)(2) (1) Let $\ell := \text{proj.dim}({}_A T)$. Set $\mathcal{A}_1 := {}^{\perp}({}_A T) \cap \mathcal{GP}^{\leqslant \ell}(A)$ and $\mathcal{A}_2 := {}^{\perp}(T_B) \cap \mathcal{GP}^{\leqslant \ell}(B^{op})$. By Corollary 1.3(1), $F := \text{Hom}_A(-, T) : \mathcal{A}_1 \to \mathcal{A}_2$ is a duality of small exact categories. Let $G := \text{Hom}_{B^{op}}(-, B) : \mathcal{GP}(B^{op}) \to \mathcal{GP}(B)$. Then G is also a duality of small exact categories. Since $\mathcal{GP}(A) \subseteq \mathcal{A}_1$ and $\mathcal{GP}(B^{op}) \subseteq \mathcal{A}_2$, there is a diagram of exact categories:

$$\begin{tikzcd}
& \mathcal{GP}(A) \arrow[r, "\subseteq"] & \mathcal{A}_1 \arrow[d, "F"] \\
\mathcal{GP}(B) & \mathcal{GP}(B^{op}) \arrow[l, "G"', "\simeq"] \arrow[r, "\subseteq"] & \mathcal{A}_2.
\end{tikzcd} \tag{4.1}$$

Clearly, $\mathcal{A}_1 \subseteq A$-mod and $\mathcal{A}_2 \subseteq B^{op}$-mod are resolving subcategories. Now, we apply Lemma 2.1 to the inclusions of exact categories in (4.1), and then obtain the following diagram of triangle equivalences of unbounded derived categories of exact categories:

$$\begin{tikzcd}
& \mathscr{D}(\mathcal{GP}(A)) \arrow[r, "\simeq"] & \mathscr{D}(\mathcal{A}_1) \arrow[d, "\simeq"] \\
\mathscr{D}(\mathcal{GP}(B)) & \mathscr{D}(\mathcal{GP}(B^{op}))^{op} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \mathscr{D}(\mathcal{A}_2)^{op}.
\end{tikzcd}$$

This implies a triangle equivalence $\mathscr{D}(\mathcal{GP}(A)) \simeq \mathscr{D}(\mathcal{GP}(B))$, which can be restricted to an equivalence $\mathscr{D}^*(\mathcal{GP}(A)) \simeq \mathscr{D}^*(\mathcal{GP}(B))$ for any $* \in \{+, -, b\}$ by Lemma 2.1.

(2) By Lemma 4.1, $K_n(\mathcal{GP}(A)) \simeq K_n(\mathcal{A}_1)$ and $K_n(\mathcal{GP}(B^{op})) \simeq K_n(\mathcal{A}_2)$. By Lemma 4.2, $K_n(\mathcal{A}_1) \simeq K_n(\mathcal{A}_2)$ and $K_n(\mathcal{GP}(B)) \simeq K_n(\mathcal{GP}(B^{op}))$. Thus $K_n(\mathcal{GP}(A)) \simeq K_n(\mathcal{GP}(B))$. ∎

By use of resolving dualities, we can show the following result, of which all the assertions except the equivalence $\mathscr{K}(\mathcal{P}(A)) \simeq \mathscr{K}(\mathcal{P}(B))$ are known (for example, see [16, 22, 38]).

https://doi.org/10.4153/S0008414X24000580 Published online by Cambridge University Press

Corollary 4.3 *Let A and B be algebras and ${}_AT_B$ a tilting bimodule. Then:*

(1) *There is a triangle equivalence $\mathscr{K}(\mathcal{P}(A)) \simeq \mathscr{K}(\mathcal{P}(B))$ which can be restricted to an equivalence $\mathscr{K}^*(\mathcal{P}(A)) \simeq \mathscr{K}^*(\mathcal{P}(B))$ for any $* \in \{+,-,b\}$.*
(2) *$K_n(\mathcal{P}(A)) \simeq K_n(\mathcal{P}(B))$ for any $n \in \mathbb{N}$.*

Proof Recall that a strictly exact complex $X \in \mathscr{C}(\mathcal{P}(A))$ is exact in $\mathscr{C}(A\text{-mod})$ with all of its boundaries in $\mathcal{P}(A)$. Then X is contractible and thus zero in $\mathscr{K}(\mathcal{P}(A))$. By the construction of derived categories in Section 2.2, $\mathscr{K}^*(\mathcal{P}(A)) = \mathscr{D}^*(\mathcal{P}(A))$ for any $* \in \{\varnothing,+,-,b\}$. Set

$$\ell := \mathrm{proj.dim}({}_AT),\ \ \mathcal{B}_1 := {}^{\perp}({}_AT) \cap \mathcal{P}^{\leqslant \ell}(A),\ \ \mathcal{B}_2 := {}^{\perp}(T_B) \cap \mathcal{P}^{\leqslant \ell}(B^{op}).$$

Then $\mathcal{B}_1 = {}^{\perp}({}_AT) \cap \mathcal{P}^{<\infty}(A)$ and $\mathcal{B}_2 = {}^{\perp}(T_B) \cap \mathcal{P}^{<\infty}(B^{op})$ by Theorem 2.6(c), and $F : \mathcal{B}_1 \to \mathcal{B}_2$ is a duality of small exact categories by Theorem 2.6 (see also Corollary 3.7(2)). Now, in the proof of Corollary 1.4(1)(2), we replace $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{GP}(A)$ and $\mathcal{GP}(B^{op})$ with $\mathcal{B}_1$ and $\mathcal{B}_2$, $\mathcal{P}(A)$ and $\mathcal{P}(B^{op})$, respectively, and then show Corollary 4.3 similarly. ∎

4.2 Semi-derived Ringel–Hall algebras of weakly 1-Gorenstein exact categories

In this subsection, we first recall the definition of semi-derived Ringel–Hall algebras of weakly 1-Gorenstein exact categories from [29], and then introduce a new definition for these algebras (up to isomorphism) which behaves better under resolving dualities (see Proposition 4.8).

Let $k := \mathbb{F}_q$ be a finite field and $\mathcal{A}$ a small exact category linear over k. For each $X \in \mathcal{A}$, we define

$$\begin{aligned}\text{Ext-proj.dim}X &:= \min\{i \in \mathbb{N} \mid \mathrm{Hom}_{\mathscr{D}(\mathcal{A})}(X, Y[j]) = 0 \text{ for all } Y \in \mathcal{A} \text{ and all } j > i\},\\ \text{Ext-inj.dim}X &:= \min\{i \in \mathbb{N} \mid \mathrm{Hom}_{\mathscr{D}(\mathcal{A})}(Y, X[j]) = 0 \text{ for all } Y \in \mathcal{A} \text{ and all } j > i\}.\end{aligned}$$

The following four subcategories of $\mathcal{A}$ are defined in the appendix of [29].

$$\begin{aligned}\mathcal{P}^{\leqslant i}(\mathcal{A}) &= \{X \in \mathcal{A} \mid \text{Ext-proj.dim}X \leqslant i\},\\ \mathcal{I}^{\leqslant i}(\mathcal{A}) &= \{X \in \mathcal{A} \mid \text{Ext-inj.dim}X \leqslant i\},\\ \mathcal{P}^{<\infty}(\mathcal{A}) &= \{X \in \mathcal{A} \mid \text{Ext-proj.dim}X < \infty\},\\ \mathcal{I}^{<\infty}(\mathcal{A}) &= \{X \in \mathcal{A} \mid \text{Ext-inj.dim}X < \infty\}.\end{aligned}$$

The category $\mathcal{A}$ is said to be *weakly Gorenstein* if $\mathcal{P}^{<\infty}(\mathcal{A}) = \mathcal{I}^{<\infty}(\mathcal{A})$; *weakly d-Gorenstein* if it is weakly Gorenstein and $\mathcal{P}^{<\infty}(\mathcal{A}) = \mathcal{P}^{\leqslant d}(\mathcal{A}) = \mathcal{I}^{\leqslant d}(\mathcal{A})$.

We consider an exact category $\mathcal{A}$ satisfying the following conditions:

(E-a) $\mathcal{A}$ is a small exact category with finite morphism spaces and finite extension spaces, i.e.,

$$|\mathrm{Hom}_{\mathcal{A}}(M,N)| < \infty, \quad |\mathrm{Ext}^1_{\mathcal{A}}(M,N)| < \infty;$$

(E-b) $\mathcal{A}$ is linear over $k = \mathbb{F}_q$;
(E-c) $\mathcal{A}$ is weakly 1-Gorenstein;
(E-d) for any object $X \in \mathcal{A}$, there exists a deflation $P_X \to X$ with $P_X \in \mathcal{P}^{<\infty}(\mathcal{A})$.

https://doi.org/10.4153/S0008414X24000580 Published online by Cambridge University Press

Clearly, if A is a finite-dimensional algebra over k, then the Frobenius category $\mathcal{GP}(A)$ satisfies (E-a)–(E-d). If, in addition, A is 1-Gorenstein (that is, both ${}_AA$ and A_A have injective dimension at most 1), then the abelian category A-mod also satisfies (E-a)–(E-d). The following result supplies a class of weakly 1-Gorenstein exact categories which may be neither Frobenius nor abelian categories in general.

Lemma 4.4 *Let A be a finite-dimensional algebra over the field k and ${}_AT_B$ a tilting bimodule with* $\mathrm{proj.dim}({}_AT) \leqslant 1$. *Define $\mathcal{A} := {}^{\perp}({}_AT) \cap \mathcal{GP}^{\leqslant 1}(A)$. Then $\mathcal{A}$ is weakly 1-Gorenstein satisfying (E-a)–(E-d) and $\mathcal{P}^{<\infty}(\mathcal{A}) = {}^{\perp}({}_AT) \cap \mathcal{P}^{\leqslant 1}(A)$.*

Proof By Corollary 3.4(3), $\mathcal{A} = \mathcal{W}({}_AT)$. Since $\mathcal{A}$ is a resolving subcategory of A-mod by Lemma 3.2(1), it is a small exact category of which the projective objects are exactly projective A-modules. By Lemma 3.2(3), $\mathrm{add}({}_AT)$ equals the full subcategory of $\mathcal{A}$ consisting of injective objects. So, $\mathcal{A}$ as an exact category has enough projective objects and injective objects. Clearly, $\mathcal{P}^{<\infty}(\mathcal{A}) = \mathcal{A} \cap \mathcal{P}^{<\infty}(A) = {}^{\perp}({}_AT) \cap \mathcal{P}^{\leqslant 1}(A)$. Further, since $\mathrm{proj.dim}({}_AT) \leqslant 1$, it follows from Theorem 2.6(c) that ${}^{\perp}({}_AT) \cap \mathcal{P}^{\leqslant 1}(A) = {}^{\perp}({}_AT) \cap \mathcal{P}^{<\infty}(A)$ which consists of those A-modules having finite $\mathrm{add}({}_AT)$-coresolutions. Thus $\mathcal{P}^{<\infty}(\mathcal{A}) = \mathcal{I}^{<\infty}(\mathcal{A})$, that is, $\mathcal{A}$ is weakly Gorenstein. Since $\mathcal{P}^{<\infty}(\mathcal{A}) \subseteq \mathcal{P}^{\leqslant 1}(\mathcal{A})$, we have $\mathcal{P}^{<\infty}(\mathcal{A}) = \mathcal{P}^{\leqslant 1}(\mathcal{A})$. Moreover, by the resolving duality $\mathrm{Hom}_A(-,T) : {}^{\perp}({}_AT) \cap \mathcal{P}^{\leqslant 1}(A) \to {}^{\perp}(T_B) \cap \mathcal{P}^{\leqslant 1}(B^{op})$ in Theorem 2.5, each object of ${}^{\perp}({}_AT) \cap \mathcal{P}^{\leqslant 1}(A)$ has an $\mathrm{add}({}_AT)$-coresolution of length at most 1. This implies $\mathcal{I}^{<\infty}(\mathcal{A}) = \mathcal{I}^{\leqslant 1}(\mathcal{A})$, and therefore $\mathcal{A}$ is weakly 1-Gorenstein. Since $\mathcal{A} \subseteq A$-mod is a resolving subcategory and k is a finite field, it can be checked that $\mathcal{A}$ satisfies (E-a)–(E-d). ■

Now, let $\mathcal{A}$ be an exact category which satisfies (E-a)–(E-d). Denote by $\mathrm{Iso}(\mathcal{A})$ the set of isomorphism classes of objects in $\mathcal{A}$ and by $K_0(\mathcal{A})$ the Grothendieck group of $\mathcal{A}$. Let $\mathcal{H}(\mathcal{A})$ be the *Ringel–Hall algebra* of $\mathcal{A}$, that is, $\mathcal{H}(\mathcal{A}) = \bigoplus_{[M]\in \mathrm{Iso}(\mathcal{A})} \mathbb{Q}[M]$ as $\mathbb{Q}$-vector spaces with the multiplication given by

$$[M] \diamond [N] := \sum_{[L]\in \mathrm{Iso}(\mathcal{A})} \frac{|\mathrm{Ext}^1_{\mathcal{A}}(M,N)_L|}{|\mathrm{Hom}_{\mathcal{A}}(M,N)|}[L],$$

where $\mathrm{Ext}^1_{\mathcal{A}}(M,N)_L$ stands for the subset of $\mathrm{Ext}^1_{\mathcal{A}}(M,N)$ parameterizing all extensions in which the middle term is isomorphic to L. Then $\mathcal{H}(\mathcal{A})$ is a $K_0(\mathcal{A})$-graded algebra. For $M \in \mathcal{A}$ and $K \in \mathcal{P}^{\leqslant 1}(\mathcal{A})$, define

$$\langle K, M\rangle = \dim_k \mathrm{Hom}_{\mathcal{A}}(K,M) - \dim_k \mathrm{Ext}^1_{\mathcal{A}}(K,M),$$

$$\langle M, K\rangle = \dim_k \mathrm{Hom}_{\mathcal{A}}(M,K) - \dim_k \mathrm{Ext}^1_{\mathcal{A}}(M,K).$$

These formulas descend bilinear forms (called *Euler forms*), again denoted by $\langle \cdot, \cdot \rangle$, on the Grothendieck groups $K_0(\mathcal{P}^{\leqslant 1}(\mathcal{A}))$ and $K_0(\mathcal{A})$.

To introduce semi-derived Ringel–Hall algebras of weakly 1-Gorenstein exact categories, we first recall the definition of (left or right) denominator subsets of rings and their relations with Ore localizations. For more details, we refer to [27, Chapter 4] and [32, Chapter 2].

https://doi.org/10.4153/S0008414X24000580 Published online by Cambridge University Press

Let R be a ring with identity, and let S be a subset of R closed under multiplications with $1 \in S$. Following [27, Definition 10.5], S is called a *left denominator subset* of R if the following conditions hold:

(i) For any $a \in R$ and $s \in S$, the intersection $Sa \cap Rs$ is not empty.

(ii) For any $r \in R$, if $rt = 0$ for some $t \in S$, there exists some $t' \in S$ such that $t'r = 0$.

If S satisfies only the condition (i), then S is called a *left Ore subset* of R. Similarly, we can define right denominator sets and right Ore sets. Now, Ore's localization theorem (for example, see [27, Theorem 10.6]) states that:

(1) the left Ore localization $[S^{-1}]R$ exists if and only if S is a left denominator subset of R;

(2) the right Ore localization $R[S^{-1}]$ exists if and only if S is a right denominator subset of R.

If S is a left and right denominator subset of R, then $[S^{-1}]R$ is called the *Ore localization* of R at S. In this case, up to isomorphism of rings, $[S^{-1}]R$, $R[S^{-1}]$ and the universal localization R_S of R at S are the same (for example, see [27, Corollary 10.14] or [13, Section 2.2]).

Let $I(\mathcal{A})$ be the two-sided ideal of $\mathcal{H}(\mathcal{A})$ generated by

$$\{[L] - [K \oplus M] \mid \exists \text{ an exact sequence } 0 \to K \to L \to M \to 0 \text{ with } K \in \mathcal{P}^{\leqslant 1}(\mathcal{A})\}.$$

We consider the following multiplicatively closed subset of the quotient $\mathcal{H}(\mathcal{A})/I(\mathcal{A})$ of $\mathcal{H}(\mathcal{A})$ by $I(\mathcal{A})$:

$$\mathcal{S}_{\mathcal{A}} := \{a[K] \in \mathcal{H}(\mathcal{A})/I(\mathcal{A}) \mid a \in \mathbb{Q}^{\times},\ K \in \mathcal{P}^{\leqslant 1}(\mathcal{A})\}.$$

Lemma 4.5 [29, Proposition A5] *$\mathcal{S}_{\mathcal{A}}$ is a right denominator subset of $\mathcal{H}(\mathcal{A})/I(\mathcal{A})$. Equivalently, the right Ore localization $(\mathcal{H}(\mathcal{A})/I(\mathcal{A}))[\mathcal{S}_{\mathcal{A}}^{-1}]$ of $\mathcal{H}(\mathcal{A})/I(\mathcal{A})$ with respect to $\mathcal{S}_{\mathcal{A}}$ exists.*

Following [29] (also cf. [11, 20, 28]), the algebra $(\mathcal{H}(\mathcal{A})/I(\mathcal{A}))[\mathcal{S}_{\mathcal{A}}^{-1}]$ is called the *semi-derived Ringel–Hall algebra* of $\mathcal{A}$ and denoted by $\mathcal{SDH}(\mathcal{A})$. Since the opposite category $\mathcal{A}^{op}$ of $\mathcal{A}$ is also a weakly 1-Gorenstein exact category, the algebra $\mathcal{SDH}(\mathcal{A}^{op})$ is well defined. However, at the present time, it is not clear whether $\mathcal{SDH}(\mathcal{A}) \cong (\mathcal{SDH}(\mathcal{A}^{op}))^{op}$ as algebras because the definition of $\mathcal{SDH}(\mathcal{A})$ seems not to be left–right symmetric. To solve this problem, we will introduce a new definition of $\mathcal{SDH}(\mathcal{A})$ up to isomorphism of algebras.

Let $J(\mathcal{A})$ be the two-sided ideal of $\mathcal{H}(\mathcal{A})$ generated by

$$\{[L] - [K \oplus M] \mid \exists \text{ exact sequence } 0 \to M \to L \to K \to 0 \text{ with } K \in \mathcal{I}^{\leqslant 1}(\mathcal{A})\}.$$

Then $I(\mathcal{A}^{op}) \cong J(\mathcal{A})$ and there is an isomorphism of algebras: $\mathcal{H}(\mathcal{A})/J(\mathcal{A}) \cong (\mathcal{H}(\mathcal{A}^{op})/I(\mathcal{A}^{op}))^{op}$. Similarly, we consider the multiplicatively closed subset $\mathcal{R}_{\mathcal{A}}$ of $\mathcal{H}(\mathcal{A})/J(\mathcal{A})$:

$$\mathcal{R}_{\mathcal{A}} := \{a[K] \in \mathcal{H}(\mathcal{A})/J(\mathcal{A}) \mid a \in \mathbb{Q}^{\times},\ K \in \mathcal{I}^{\leqslant 1}(\mathcal{A})\}.$$

The following result is the dual of Lemma 4.5.

Lemma 4.6 *$\mathcal{R}_{\mathcal{A}}$ is a left denominator subset of $\mathcal{H}(\mathcal{A})/J(\mathcal{A})$. Equivalently, the left Ore localization $[\mathcal{R}_{\mathcal{A}}^{-1}](\mathcal{H}(\mathcal{A})/J(\mathcal{A}))$ of $\mathcal{H}(\mathcal{A})/J(\mathcal{A})$ with respect to $\mathcal{R}_{\mathcal{A}}$ exists. Moreover, there is an isomorphism $[\mathcal{R}_{\mathcal{A}}^{-1}](\mathcal{H}(\mathcal{A})/J(\mathcal{A})) \cong (\mathcal{SDH}(\mathcal{A}^{op}))^{op}$ of algebras.*

Now, we consider the quotient $\mathcal{H}(\mathcal{A})/(I(\mathcal{A})+J(\mathcal{A}))$ of $\mathcal{H}(\mathcal{A})$ by the ideal $I(\mathcal{A})+J(\mathcal{A})$ and its multiplicatively closed subset

$$\Phi_{\mathcal{A}} := \{a[K] \in \mathcal{H}(\mathcal{A})/(I(\mathcal{A})+J(\mathcal{A})) \mid a \in \mathbb{Q}^{\times},\ K \in \mathcal{P}^{\leqslant 1}(\mathcal{A})\}.$$

Lemma 4.7 (1) *$\Phi_{\mathcal{A}}$ is a left and right denominator subset of the algebra $\mathcal{H}(\mathcal{A})/(I(\mathcal{A})+J(\mathcal{A}))$.*

(2) *There are isomorphisms of algebras:*

$$\begin{aligned}\mathcal{SDH}(\mathcal{A}) &\cong (\mathcal{H}(\mathcal{A})/(I(\mathcal{A})+J(\mathcal{A})))[\Phi_{\mathcal{A}}^{-1}]\\ &\cong [\Phi_{\mathcal{A}}^{-1}](\mathcal{H}(\mathcal{A})/(I(\mathcal{A})+J(\mathcal{A})))\\ &\cong [\mathcal{R}_{\mathcal{A}}^{-1}](\mathcal{H}(\mathcal{A})/J(\mathcal{A}))\\ &\cong (\mathcal{SDH}(\mathcal{A}^{op}))^{op}.\end{aligned}$$

Proof (1) Let $H := \mathcal{H}(\mathcal{A})$, $I := I(\mathcal{A})$ and $J := J(\mathcal{A})$. For any $K \in \mathcal{P}^{\leqslant 1}(\mathcal{A})$ and $M \in \mathcal{A}$, it follows from [29, Lemma A4] that $[M] \diamond [K] = q^{-\langle M,K\rangle}[M \oplus K]$ in H/I. Dually, $[K] \diamond [M] = q^{-\langle K,M\rangle}[M \oplus K]$ in H/J. Thus $q^{\langle M,K\rangle}[M] \diamond [K] = q^{\langle K,M\rangle}[K] \diamond [M]$ in $H/(I+J)$. This implies that $\Phi_{\mathcal{A}}$ is a left and right Ore subset of the algebra $H/(I+J)$. By a similar argument as in the proof of [29, Proposition A5], one can further show that $\Phi_{\mathcal{A}}$ is a left and right denominator subset of $H/(I+J)$.

(2) By (1), there is an isomorphism of algebras $(H/(I+J))[\Phi_{\mathcal{A}}^{-1}] \cong [\Phi_{\mathcal{A}}^{-1}](H/(I+J))$. By Lemma 4.6, it is enough to show the algebra isomorphism

$$\mathcal{SDH}(\mathcal{A}) \cong (H/(I+J))[\Phi_{\mathcal{A}}^{-1}].$$

The algebra isomorphism $[\Phi_{\mathcal{A}}^{-1}](H/(I+J)) \cong [\mathcal{R}_{\mathcal{A}}^{-1}](H/J)$ can be proved dually.

Let $\lambda_1 : H/I \to \mathcal{SDH}(\mathcal{A})$ and $\lambda : H/(I+J) \to (H/(I+J))[\Phi_{\mathcal{A}}^{-1}]$ be the localizations and let $\pi_1 : H/I \to H/(I+J)$ be the canonical surjection. Since $\pi_1(\mathcal{S}_{\mathcal{A}}) = \Phi_{\mathcal{A}}$, there is a unique homomorphism of algebras

$$\sigma : \mathcal{SDH}(\mathcal{A}) \to (H/(I+J))[\Phi_{\mathcal{A}}^{-1}]$$

such that $\sigma\lambda_1 = \lambda\pi_1$. By the statements following [29, Lemma A8], we have $\lambda_1(J) = 0$ in $\mathcal{SDH}(\mathcal{A})$. Then there exists a unique homomorphism of algebras

$$\widetilde{\lambda_1} : (H/(I+J))[\Phi_{\mathcal{A}}^{-1}] \to \mathcal{SDH}(\mathcal{A})$$

such that $\lambda_1 = \widetilde{\lambda_1}\pi_1$, and hence $\widetilde{\lambda_1}([K])$ is invertible in $\mathcal{SDH}(\mathcal{A})$ for any $[K] \in \Phi_{\mathcal{A}}$. This implies that λ induces a unique homomorphism of algebras

$$\rho : (H/(I+J))[\Phi_{\mathcal{A}}^{-1}] \to \mathcal{SDH}(\mathcal{A})$$

such that $\rho\lambda = \widetilde{\lambda_1}$. So we have following commutative diagram:

$$\begin{tikzcd}
H/I \arrow[r, "\lambda_1"] \arrow[d, "\pi_1"'] & \mathcal{SDH}(\mathcal{A}) \arrow[d, dotted, shift left, "\sigma"] \\
H/(I+J) \arrow[r, "\lambda"'] \arrow[ur, dotted, "\widetilde{\lambda_1}"] & (H/(I+J))[\Phi_{\mathcal{A}}^{-1}]. \arrow[u, dotted, shift left, "\rho"]
\end{tikzcd}$$

https://doi.org/10.4153/S0008414X24000580 Published online by Cambridge University Press

Since $\rho\sigma\lambda_1 = \rho\lambda\pi_1 = \widetilde{\lambda}_1\pi_1 = \lambda_1$, it follows from the universal property of λ_1 that $\rho\sigma = \mathrm{Id}$. On the other hand, $\sigma\rho\lambda\pi_1 = \sigma\widetilde{\lambda}_1\pi_1 = \sigma\lambda_1 = \lambda\pi_1$. Since π_1 is surjective, $\sigma\rho\lambda = \lambda$. By the universal property of λ, we have $\sigma\rho = \mathrm{Id}$. Thus σ and ρ are isomorphisms of algebras. ∎

Thanks to Lemma 4.7, up to isomorphism of algebras, we can define the *semi-derived Ringel–Hall algebra* of $\mathcal{A}$ to be the algebra $(\mathcal{H}(\mathcal{A})/(I(\mathcal{A}) + J(\mathcal{A})))[\Phi_{\mathcal{A}}^{-1}]$. This definition is left–right symmetric and applied to show the following result.

Proposition 4.8 *Let $F : \mathcal{A}_1 \to \mathcal{A}_2$ be a resolving duality of weakly 1-Gorenstein exact categories. Then there exists an isomorphism of algebras*

$$\Upsilon_F : \mathcal{SDH}(\mathcal{A}_1) \xrightarrow{\simeq} (\mathcal{SDH}(\mathcal{A}_2))^{op}$$
$$[M] \mapsto [F(M)].$$

Proof Note that F induces an equivalence $\mathcal{A}_1 \xrightarrow{\simeq} {\mathcal{A}_2}^{op}$ of weakly 1-Gorenstein exact categories. Then F induces an isomorphism of algebras: $\mathcal{SDH}(\mathcal{A}_1) \xrightarrow{\simeq} \mathcal{SDH}({\mathcal{A}_2}^{op})$. By Lemma 4.7(2), $\mathcal{SDH}({\mathcal{A}_2}^{op}) \simeq (\mathcal{SDH}(\mathcal{A}_2))^{op}$. Thus Proposition 4.8 holds. ∎

4.3 Invariance of semi-derived Ringel–Hall algebras under tilting

In this subsection, our main result is the following theorem which contains Corollary 1.4(3) in the introduction.

Theorem 4.9 *Let A be a finite-dimensional algebra over k and ${}_AT_B$ a tilting bimodule with* $\mathrm{proj.dim}({}_AT) \leqslant 1$*. Then there exists an isomorphism of algebras:*

$$\Xi : \ \mathcal{SDH}(\mathcal{GP}(A)) \xrightarrow{\simeq} \mathcal{SDH}(\mathcal{GP}(B))$$
$$[G] \mapsto q^{-\langle L,G\rangle}[Hom_A(T,L)]^{-1} \diamond [Hom_A(T,Z)],$$

where $f : G \to Z$ is a minimal left $({}_AT)^{\perp}$-approximation of G and $L =$ $\mathrm{Coker}(f)$.

When both A and B are 1-Gorenstein algebras, Theorem 4.9 is exactly [29, Corollary A23]. To show Theorem 4.9 for general algebras, we establish a crucial result as follows.

Proposition 4.10 *Let A be a finite-dimensional algebra over k, and let ${}_AT_B$ be a tilting bimodule with* $\mathrm{proj.dim}({}_AT) \leqslant 1$*. Set $\mathcal{A} := {}^{\perp}({}_AT) \cap \mathcal{GP}^{\leqslant 1}(A)$ and $\mathcal{B} := \mathcal{GP}(A)$. Then the embedding $\phi : \mathcal{H}(\mathcal{B}) \to \mathcal{H}(\mathcal{A})$ induces an algebra isomorphism $\widetilde{\phi} : \mathcal{SDH}(\mathcal{B}) \to \mathcal{SDH}(\mathcal{A})$. Furthermore, the inverse of $\widetilde{\phi}$ is given by $\widetilde{\psi} : [M] \mapsto q^{-\langle M,H_M\rangle}[G_M] \diamond [H_M]^{-1}$, where $M \in \mathcal{A}$, $G_M \in \mathcal{B}$ and $H_M \in add({}_AA)$ such that they fit into an exact sequence $0 \to H_M \to G_M \to M \to 0$ of A-modules.*

Proof Clearly, $\mathcal{B}$ is weakly 1-Gorenstein satisfying (E-a)–(E-d). By Lemma 4.4 and its proof, $\mathcal{A}$ is also weakly 1-Gorenstein satisfying (E-a)–(E-d) and $\mathcal{P}^{\leqslant 1}(\mathcal{A}) = {}^{\perp}({}_AT) \cap \mathcal{P}^{\leqslant 1}(A)$. This means that $\mathcal{SDH}(\mathcal{B})$ and $\mathcal{SDH}(\mathcal{A})$ are well defined. Moreover, since each object $M \in \mathcal{A}$ has Gorenstein dimension at most 1, the exact sequence in Proposition 4.10 always exists. By a similar proof of [29, Theorem C(1)], one can check that $\widetilde{\phi}$ is a surjective homomorphism of algebras and the map

https://doi.org/10.4153/S0008414X24000580 Published online by Cambridge University Press

$$\psi : \mathcal{H}(\mathcal{A}) \to \mathcal{SDH}(\mathcal{B}), \quad [M] \mapsto q^{-\langle M, H_M \rangle}[G_M] \diamond [H_M]^{-1}$$

is well defined. Next, we claim that ψ is a homomorphism of algebras. It suffices to show

$$\psi([M] \diamond [N]) = \psi([M]) \diamond \psi([N]) \tag{4.2}$$

for all $M, N \in \mathcal{A}$.

For this aim, we fix two exact sequences in A-mod:

$$0 \to H_1 \to G_1 \to M \to 0, \tag{4.3}$$

$$0 \to H_2 \to G_2 \to N \to 0, \tag{4.4}$$

where $H_1, H_2 \in \mathrm{add}({}_AA)$ and $G_1, G_2 \in \mathcal{B}$. Applying $\mathrm{Hom}_A(-, N)$ and $\mathrm{Hom}_A(G_1, -)$ to the sequences (4.3) and (4.4), respectively, we obtain the following diagram:

$$\begin{tikzcd} \mathrm{Ext}^1_A(M,N) \arrow[r, "\Theta"] \arrow[dr] & \mathrm{Ext}^1_A(G_1,G_2) \arrow[r] \arrow[d, "\cong"] & 0 \\ & \mathrm{Ext}^1_A(G_1,N). & \end{tikzcd} \tag{4.5}$$

Now, let $0 \to N \to L \to M \to 0$ be an exact sequence in $\mathcal{A}$. By (4.5), there is a commutative diagram with exact rows and columns:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & H_2 \arrow[r] \arrow[d] & H_L \arrow[r] \arrow[d] & H_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & G_2 \arrow[r] \arrow[d] & G_L \arrow[r] \arrow[d] & G_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & N \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0. & \end{tikzcd} \tag{4.6}$$

Note that $\langle L, H_L \rangle = \langle M + N, H_1 + H_2 \rangle = \langle M, H_1 \rangle + \langle M, H_2 \rangle + \langle N, H_1 \rangle + \langle N, H_2 \rangle$. Since $\mathcal{A}$ and $\mathcal{B}$ are fully exact subcategories of A-mod, there are equalities

$$|\mathrm{Ext}^1_{\mathcal{A}}(X_1, X_2)| = |\mathrm{Ext}^1_A(X_1, X_2)|, \quad |\mathrm{Hom}_{\mathcal{A}}(X_1, X_2)| = |\mathrm{Hom}_A(X_1, X_2)|,$$
$$|\mathrm{Ext}^1_{\mathcal{B}}(Y_1, Y_2)| = |\mathrm{Ext}^1_A(Y_1, Y_2)|, \quad |\mathrm{Hom}_{\mathcal{B}}(Y_1, Y_2)| = |\mathrm{Hom}_A(Y_1, Y_2)|,$$

for all $X_1, X_2 \in \mathcal{A}$ and $Y_1, Y_2 \in \mathcal{B}$. Thus

$$\begin{aligned}\psi([M]\diamond[N]) &= \psi\Big(\sum_{[L]\in\mathrm{Iso}(\mathcal{A})}\frac{|\mathrm{Ext}^1_A(M,N)_L|}{|\mathrm{Hom}_A(M,N)|}[L]\Big)\\ &= \sum_{[L]\in\mathrm{Iso}(\mathcal{A})}\frac{|\mathrm{Ext}^1_A(M,N)_L|}{|\mathrm{Hom}_A(M,N)|}q^{-\langle L,H_L\rangle}[G_L]\diamond[H_L]^{-1}\\ &= \Big(\sum_{[L]\in\mathrm{Iso}(\mathcal{A})}\frac{|\mathrm{Ext}^1_A(M,N)_L|}{|\mathrm{Hom}_A(M,N)|}q^{-\langle L,H_L\rangle}[G_L]\Big)\diamond[H_1\oplus H_2]^{-1}\\ &= \Big(\sum_{[L]\in\mathrm{Iso}(\mathcal{A})}\frac{|\mathrm{Ext}^1_A(M,N)_L|}{|\mathrm{Hom}_A(M,N)|}q^{-\langle M,H_1\rangle-\langle M,H_2\rangle-\langle N,H_1\rangle-\langle N,H_2\rangle}[G_L]\Big)\\ &\qquad\diamond[H_1\oplus H_2]^{-1}.\end{aligned}$$

Since $q^{\langle H_1,G_2\rangle}[H_1]\diamond[G_2] = [H_1\oplus G_2] = q^{\langle G_2,H_1\rangle}[G_2]\diamond[H_1]$, it follows that

$$\begin{aligned}\psi([M])\diamond\psi([N]) &= q^{-\langle M,H_1\rangle-\langle N,H_2\rangle}[G_1]\diamond[H_1]^{-1}\diamond[G_2]\diamond[H_2]^{-1}\\ &= q^{-\langle M,H_1\rangle-\langle N,H_2\rangle+\langle H_1,G_2\rangle-\langle G_2,H_1\rangle}[G_1]\diamond[G_2]\diamond[H_1]^{-1}\diamond[H_2]^{-1}\\ &= q^{-\langle M,H_1\rangle-\langle N,H_2\rangle+\langle H_1,G_2\rangle-\langle G_2,H_1\rangle}[G_1]\diamond[G_2]\diamond([H_2]\diamond[H_1])^{-1}\\ &= q^{-\langle M,H_1\rangle-\langle N,H_2\rangle+\langle H_1,G_2\rangle-\langle G_2,H_1\rangle+\langle H_2,H_1\rangle}[G_1]\diamond[G_2]\diamond[H_2\oplus H_1]^{-1}.\end{aligned}$$

Consequently, to prove (4.2), we only need to check

(4.7)
$$q^{-\langle M,H_2\rangle-\langle N,H_1\rangle}\sum_{[L]\in\mathrm{Iso}(\mathcal{A})}\frac{|\mathrm{Ext}^1_A(M,N)_L|}{|\mathrm{Hom}_A(M,N)|}[G_L] = q^{\langle H_1,G_2\rangle-\langle G_2,H_1\rangle+\langle H_2,H_1\rangle}[G_1]\diamond[G_2].$$

Now, we set $K := \mathrm{Ker}(\Theta)$ in the diagram (4.5). Then there exists an exact sequence

$$0\to\mathrm{Hom}_A(M,N)\to\mathrm{Hom}_A(G_1,N)\to\mathrm{Hom}_A(H_1,N)\to K\to 0.$$

Thus

$$|K| = q^{\dim_k\mathrm{Hom}_A(H_1,N)-\dim_k\mathrm{Hom}_A(G_1,N)+\dim_k\mathrm{Hom}_A(M,N)}. \tag{4.8}$$

For any $\delta' : 0\to G_2\to W\to G_1\to 0$ in the set $\mathrm{Ext}^1_A(G_1,G_2)_W$, one can show

$$|\{\delta\in\mathrm{Ext}^1_A(M,N)\mid\Theta(\delta)=\delta'\}| = |K|.$$

This leads to

$$|\{\delta\in\mathrm{Ext}^1_A(M,N)\mid\Theta(\delta)\in\mathrm{Ext}^1_A(G_1,G_2)_W\}| = |K||\mathrm{Ext}^1_A(G_1,G_2)_W|$$

and therefore

$$\sum_{[L]\in\mathrm{Iso}(\mathcal{A})}|\mathrm{Ext}^1_A(M,N)_L|[G_L] = |K|\sum_{[W]\in\mathrm{Iso}(\mathcal{A})}|\mathrm{Ext}^1_A(G_1,G_2)_W|[W]. \tag{4.9}$$

Since $G_1\in\mathcal{B}$ and $H_2\in\mathrm{add}({}_AA)$, the equality $\mathrm{Ext}^1_A(G_1,H_2)=0$ holds. This implies that the sequence $0\to\mathrm{Hom}_A(G_1,H_2)\to\mathrm{Hom}_A(G_1,G_2)\to\mathrm{Hom}_A(G_1,N)\to 0$ is exact. Consequently,

https://doi.org/10.4153/S0008414X24000580 Published online by Cambridge University Press

$$\dim_k \mathrm{Hom}_A(H_1, N) - \dim_k \mathrm{Hom}_A(G_1, N) = \langle H_1, N\rangle + \dim_k \mathrm{Hom}_A(G_1, H_2) - \dim_k \mathrm{Hom}_A(G_1, G_2). \tag{4.10}$$

So we have

$$\begin{aligned}
&q^{-\langle M,H_2\rangle-\langle N,H_1\rangle} \sum_{[L]\in \mathrm{Iso}(\mathcal{A})} \frac{|\mathrm{Ext}^1_A(M,N)_L|}{|\mathrm{Hom}_A(M,N)|}[G_L] \\
&= q^{-\langle M,H_2\rangle-\langle N,H_1\rangle+\dim_k \mathrm{Hom}_A(H_1,N)-\dim_k \mathrm{Hom}_A(G_1,N)} \\
&\quad\times \sum_{[W]\in \mathrm{Iso}(\mathcal{B})} |\mathrm{Ext}^1_A(G_1,G_2)_W|[W] \quad (\text{by } (4.8),\ (4.9)) \\
&= q^{-\langle M,H_2\rangle-\langle N,H_1\rangle+\langle H_1,N\rangle+\dim_k \mathrm{Hom}(G_1,H_2)} \sum_{[W]\in \mathrm{Iso}(\mathcal{B})} \frac{|\mathrm{Ext}^1_A(G_1,G_2)_W|}{\mathrm{Hom}_A(G_1,G_2)}[W] \quad (\text{by } (4.10)) \\
&= q^{-\langle M,H_2\rangle-\langle N,H_1\rangle+\langle H_1,G_2\rangle-\langle H_1,H_2\rangle+\dim_k \mathrm{Hom}_A(G_1,H_2)}[G_1]\diamond[G_2] \quad (\text{since } \langle H_1,G_2\rangle \\
&\qquad\qquad = \langle H_1, H_2+N\rangle) \\
&= q^{\langle H_1,G_2\rangle-\langle N,H_1\rangle} q^{-(\langle M,H_2\rangle+\langle H_1,H_2\rangle-\dim_k \mathrm{Hom}_A(G_1,H_2))}[G_1]\diamond[G_2] \\
&= q^{\langle H_1,G_2\rangle-\langle N,H_1\rangle}[G_1]\diamond[G_2] \quad (\text{since } \langle G_1,H_2\rangle = \langle M+H_1,H_2\rangle) \\
&= q^{\langle H_1,G_2\rangle-\langle G_2,H_1\rangle+\langle H_2,H_1\rangle}[G_1]\diamond[G_2] \quad (\text{since } \langle G_2,H_1\rangle = \langle H_2+N,H_1\rangle).
\end{aligned}$$

This shows that (4.7) is true, and thus (4.2) is also true. So, ψ is a homomorphism of algebras.

Finally, we show that ψ factorizes through the canonical surjection $\mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{A})/I(\mathcal{A})$.

Suppose $N \in \mathcal{P}^{\leqslant 1}(\mathcal{A})$. Since $H_2 \in \mathrm{add}({}_AA)$, we have $G_2 \in \mathcal{P}^{\leqslant 1}(A)$ (see the first column in the diagram (4.6)). As G_2 lies in $\mathcal{B}$, it is projective. This implies $G_L \cong G_1 \oplus G_2$, and therefore

$$\psi([L]) = q^{-\langle L,H_L\rangle}[G_L]\diamond[H_L]^{-1} = q^{-\langle M\oplus N, H_1\oplus H_2\rangle}[G_1\oplus G_2]\diamond[H_1\oplus H_2]^{-1} = \psi([N\oplus M]).$$

Consequently, ψ induces a homomorphism of algebras $\psi' : \mathcal{H}(\mathcal{A})/I(\mathcal{A}) \to \mathcal{SDH}(\mathcal{B})$. Since $\psi([K])$ is invertible in $\mathcal{SDH}(\mathcal{B})$ for any $K \in \mathcal{P}^{\leqslant 1}(\mathcal{A})$, ψ' induces a unique homomorphism of algebras $\widetilde{\psi} : \mathcal{SDH}(\mathcal{A}) \to \mathcal{SDH}(\mathcal{B})$. Clearly, $\widetilde{\psi}\widetilde{\phi} = \mathrm{Id}$, which means that $\widetilde{\phi}$ is injective. Since $\widetilde{\phi}$ is surjective, it is an isomorphism of algebras. ∎

A consequence of Proposition 4.10 is the following.

Corollary 4.11 *Let A be a finite-dimensional algebra over k, and let ${}_AT$ be a strong tilting module with* $\mathrm{proj.dim}({}_AT) \leqslant 1$. *Then there is an algebra isomorphism* $\mathcal{SDH}(\mathcal{GP}(A)) \cong \mathcal{SDH}(\mathcal{GP}^{\leqslant 1}(A))$.

Proof By Proposition 4.10, it suffices to show ${}^{\perp}({}_AT) \cap \mathcal{GP}^{\leqslant 1}(A) = \mathcal{GP}^{\leqslant 1}(A)$. Let $M \in \mathcal{GP}^{\leqslant 1}(A)$. By [15, Lemma 2.17], there exists an exact sequence $0 \to M \to H \to G \to 0$ in A-mod with $\mathrm{proj.dim}({}_AH) \leqslant 1$ and $G \in \mathcal{GP}(A)$. It follows from Lemma 3.3(2) that $G \in \mathcal{W}({}_AT)$. Since ${}_AT$ is a strong tiling module with $\mathrm{proj.dim}({}_AT) \leqslant 1$, we have $H \in {}^{\perp}({}_AT)$. Hence $M \in {}^{\perp}({}_AT)$. This implies $\mathcal{GP}^{\leqslant 1}(A) \subseteq {}^{\perp}({}_AT)$. ∎

Proof of Theorem 4.9 Let $\mathcal{A}_1 := {}^{\perp}({}_AT) \cap \mathcal{GP}^{\leqslant 1}(A)$ and $\mathcal{A}_2 := {}^{\perp}(T_B) \cap \mathcal{GP}^{\leqslant 1}(B^{op})$. It follows from Corollary 1.3(1) that $F := \mathrm{Hom}_A(-, T) : \mathcal{A}_1 \to \mathcal{A}_2$ is a duality of weakly

1-Gorenstein exact categories. Moreover, $G := \mathrm{Hom}_{B^{op}}(-, B) : \mathcal{GP}(B^{op}) \to \mathcal{GP}(B)$ is also a duality of weakly 1-Gorenstein exact categories. By Propositions 4.10 and 4.8, we obtain the following diagram of isomorphisms of algebras:

$$\begin{tikzcd}
\mathcal{SDH}(\mathcal{GP}(A)) \arrow[rr, "\widetilde{\phi}_A"] & & \mathcal{SDH}(\mathcal{A}_1) \arrow[d, "\Upsilon_F"] \\
& & (\mathcal{SDH}(\mathcal{A}_2))^{op} \arrow[d, "(\widetilde{\psi}_{B^{op}})^{op}"] \\
\mathcal{SDH}(\mathcal{GP}(B)) & & (\mathcal{SDH}(\mathcal{GP}(B^{op}))^{op} \arrow[ll, "(\Upsilon_G)^{op}"']
\end{tikzcd}$$

Define

$$\Xi := (\Upsilon_G)^{op}(\widetilde{\psi}_{B^{op}})^{op}\Upsilon_F\widetilde{\phi}_A : \mathcal{SDH}(\mathcal{GP}(A)) \to \mathcal{SDH}(\mathcal{GP}(B)).$$

Then Ξ is an isomorphism of algebras. It remains to show that, for any $G \in \mathcal{GP}(A)$,

$$\Xi([G]) = q^{-\langle L,G\rangle}[F(L)]^{-1} \diamond [F(Z)],$$

where L and Z are given in Theorem 4.9.

In fact, since $G \in \mathcal{GP}(A)$, there is an exact sequence $0 \to K \to P \to G \to 0$ in A-mod such that $P \in \mathrm{add}({}_AA)$ and $K \in \mathcal{GP}(A)$. As ${}_AT$ is 1-tilting, there is an exact sequence $0 \to P \to T_0 \to T_1 \to 0$ in A-mod with $T_0, T_1 \in \mathrm{add}({}_AT)$. Consider the following pushout diagram ($\sharp$):

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & T_0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
& & T_1 \arrow[r, equal] \arrow[d] & T_1 \arrow[d] & \\
& & 0 & 0. &
\end{tikzcd}$$

Since T is a 1-tilting module, $({}_AT)^{\perp} = \mathrm{Gen}({}_AT)$, the smallest full subcategory of A-mod containing ${}_AT$ and being closed under direct sums and quotients. This forces $X \in ({}_AT)^{\perp}$. Since $T_1 \in \mathrm{add}({}_AT) \subseteq ({}_AT)^{\perp}$, the third column in the diagram ($\sharp$) implies that the map $G \to X$ is a left $({}_AT)^{\perp}$-approximation of G. Now, let $f : G \to Z$ be a minimal left $({}_AT)^{\perp}$-approximation with $L := \mathrm{Coker}(f)$. Then f is injective; Z and L are isomorphic to direct summands of X and T_1, respectively. In particular, $Z \in \mathrm{Gen}({}_AT)$ and $L \in \mathrm{add}({}_AT)$. Since $\mathcal{GP}(A) \cup \mathrm{add}({}_AT) \subseteq \mathcal{A}_1$ and $\mathcal{A}_1$ is closed under extensions in A-mod, we have $Z \in \mathcal{A}_1$. As $Z \in \mathrm{Gen}({}_AT)$, there exists an exact sequence $0 \to N \to H \to Z \to 0$ of A-modules with $H \in \mathrm{add}({}_AT)$. Since $\mathcal{A}_1 \subseteq A$-mod is a resolving subcategory, $N \in \mathcal{A}_1$. So, the sequence is exact in $\mathcal{A}_1$. Applying F to the sequence yields an exact sequence $0 \to F(Z) \to F(H) \to F(N) \to 0$ in $\mathcal{A}_2$. Since $F(H) \in \mathrm{add}(B_B)$ and $F(N) \in \mathcal{GP}^{\leqslant 1}(B^{op})$, it follows that $F(Z) \in \mathcal{GP}(B^{op})$. Hence, we have an exact sequence in $\mathcal{A}_2$

$$0 \longrightarrow F(L) \longrightarrow F(Z) \longrightarrow F(G) \longrightarrow 0 \tag{4.11}$$

such that $F(Z)$ is Gorenstein projective and $F(L)$ is projective. Moreover, there are equalities

$$\begin{aligned} \langle F(G), F(L)\rangle &= \dim_k \mathrm{Hom}_{B^{op}}(F(G), F(L)) - \dim_k \mathrm{Ext}^1_B(F(G), F(L)) \\ &= \dim_k \mathrm{Hom}_A(L, G) - \dim_k \mathrm{Ext}^1_B(L, G) \quad \text{(by Lemma 3.1)} \\ &= \langle L, G\rangle. \end{aligned} \tag{4.12}$$

Thus

$$\begin{aligned} \Xi([G]) &= ((\Upsilon_G)^{op}(\widetilde{\psi}_{B^{op}})^{op}\Upsilon_F\widetilde{\phi}_A)([G]) \\ &= (\Upsilon_G)^{op}(\widetilde{\psi}_{B^{op}})^{op}([F(G)]) \\ &= (\Upsilon_G)^{op}(q^{-\langle F(G),F(L)\rangle}[F(L)]^{-1} \diamond [F(Z)]) \quad \text{(by (4.11))} \\ &= q^{-\langle F(G),F(L)\rangle}[GF(L)]^{-1} \diamond [GF(Z)] \\ &= q^{-\langle F(G),F(L)\rangle}[\mathrm{Hom}_A(T,L)]^{-1} \diamond [\mathrm{Hom}_A(T,Z)] \quad \text{(by Lemma 3.1)} \\ &= q^{-\langle L,G\rangle}[\mathrm{Hom}_A(T,L)]^{-1} \diamond [\mathrm{Hom}_A(T,Z)] \quad \text{(by (4.12))}. \end{aligned}$$

This finishes the proof of Theorem 4.9. ■

Corollary 4.12 *Let A be a finite-dimensional algebra over k.*

(1) *[29, Theorem C(1)] If A is 1-Gorenstein, then there exists an isomorphism of algebras: $\mathcal{SDH}(A) \cong \mathcal{SDH}(\mathcal{GP}(A))$.*
(2) *[29, Theorem D] If ${}_AT_B$ is a tilting bimodule over finite-dimensional 1-Gorenstein algebras A and B, then there exists an isomorphism of algebras: $\mathcal{SDH}(A) \cong \mathcal{SDH}(B)$.*

Proof Let A be a finite-dimensional 1-Gorenstein k-algebra. Then A-mod is a weakly 1-Gorenstein algebra satisfying (E-a)–(E-d). Recall that $\mathcal{SDH}(A) := \mathcal{SDH}(A\text{-mod})$ is the semi-derived Ringel–Hall algebra of A. Set ${}_AT := \mathrm{Hom}_k(A, k)$, the ordinary injective cogenerator for A-mod. Then ${}_AT$ is strong tilting and $\mathrm{proj.dim}({}_AT) \leqslant 1$. Moreover, $A\text{-mod} = \mathcal{GP}^{\leqslant 1}(A) = {}^{\perp}({}_AT) \cap \mathcal{GP}^{\leqslant 1}(A)$. Now, (1) is a direct consequence of Corollary 4.11 and (2) follows from (1) and Theorem 4.9. ■

Remark 4.13 We point out that our proof of Corollary 4.12 is different from the proof given in [29].

(a) In the proof of [29, Theorem C(1)], under the assumption that A is 1-Gorenstein, the map $\psi : \mathcal{H}(\mathcal{A}) \to \mathcal{SDH}(\mathcal{GP}(A))$ (see the proof of Proposition 4.10 for $T := \mathrm{Hom}_k(A, k)$) was shown to induce a unique morphism of $\mathcal{T}(A)$-bimodules $\widetilde{\psi} : \mathcal{SDH}(\mathcal{A}) \to \mathcal{SDH}(\mathcal{GP}(A))$ by using an explicit description of $\mathcal{SDH}(\mathcal{A})$ as

https://doi.org/10.4153/S0008414X24000580 Published online by Cambridge University Press

a $\mathcal{T}(A)$-bimodule (see [29, Proposition A13]), where $\mathcal{A} = A$-mod and $\mathcal{T}(A) := \mathcal{SDH}(\mathcal{P}^{\leq 1}(A))$ is a subalgebra of $\mathcal{SDH}(A)$. In our proof, for a general 1-tilting module ${}_AT$, we prove that $\psi : \mathcal{H}(\mathcal{A}) \to \mathcal{SDH}(\mathcal{GP}(A))$ is an algebra homomorphism and automatically induces an algebra homomorphism $\widetilde{\psi} : \mathcal{SDH}(\mathcal{A}) \to \mathcal{SDH}(\mathcal{GP}(A))$ which is the inverse of $\widetilde{\phi}$ (see the proof of Proposition 4.10).

(b) In the proof of [29, Theorem D], when both A and B are 1-Gorenstein, the following algebra isomorphisms were established:

$$\mathcal{SDH}(A) \cong \mathcal{SDH}(\mathcal{X}) \cong \mathcal{SDH}(\mathcal{Y}) \cong \mathcal{SDH}(B),$$

where $\mathcal{X} := \{X \in A\text{-mod} \mid \mathrm{Ext}^1_A(T, X) = 0\}$ and $\mathcal{Y} := \{Y \in B\text{-mod} \mid \mathrm{Tor}^B_1(T, Y) = 0\}$ are weakly 1-Gorenstein exact categories, and the second isomorphism follows from the additive equivalence $\mathcal{X} \simeq \mathcal{Y}$ by the Brenner–Butler tilting theorem. However, for a general algebra A, the category $\mathcal{X}$ may not be weakly 1-Gorenstein, and therefore $\mathcal{SDH}(\mathcal{X})$ is not well defined. In our proof of Corollary 4.12(2), we use the resolving dualities in Corollary 1.3(1) and establish a series of algebra isomorphisms:

$$\mathcal{SDH}(A) \cong \mathcal{SDH}(\mathcal{GP}(A)) \cong \mathcal{SDH}(\mathcal{GP}(B)) \cong \mathcal{SDH}(B).$$

Acknowledgements Both authors are very grateful to the anonymous referee for suggestions on the language and exposition of the article. The authors would also like to thank Professor Xiao-Wu Chen for helpful discussions.

References

[1] F. W. Anderson and K. R. Fuller, *Rings and categories of modules*, Springer, New York, 1992.
[2] L. Angeleri-Hügel, D. Happel, and H. Krause (eds.), *Handbook of tilting theory*, London Mathematical Society Lecture Note Series, 332, Cambridge University Press, Cambridge, 2007.
[3] M. Auslander and M. Bridger, *Stable module category*. Mem. Amer. Math. Soc. **94**(1969), 1–146.
[4] M. Auslander and I. Reiten, *Applications of contravariantly finite subcategories*. Adv. Math. **86**(1991), 111–152.
[5] M. Auslander, I. Reiten, and S. O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, 36, Cambridge University Press, Cambridge, 1995.
[6] L. L. Avramov and A. Martsinkovsky, *Absolute, relative and Tate cohomology of modules of finite Gorenstein dimension*. Proc. Lond. Math. Soc. (3) **85**(2002), 393–440.
[7] H. Bass, *Finitistic dimension and a homological generalization of semiprimary rings*. Trans. Amer. Math. Soc. **95**(1960), 466–488.
[8] A. Beligiannis, *Cohen–Macaulay modules, (co)torsion pairs and virtually Gorenstein algebras*. J. Algebra **288**(2005), 137–211.
[9] A. Beligiannis and I. Reiten, *Homological and homotopical aspects of torsion theories*. Mem. Amer. Math. Soc. **188**(2007), 1–207.
[10] S. Brenner and M. C. R. Butler, *Generalizations of the Bernstein–Gelfand–Ponomarev reflection functors*. In: A. Dold and B.Eckmann (ed.), Representation theory II, Lecture Notes in Mathematics, 832, Springer, Berlin, 1980, pp. 103–169.
[11] T. Bridgeland, *Quantum groups via Hall algebras of complexes*. Ann. of Math. **177**(2013) 739–759.
[12] M. Broué, *Equivalences of blocks of group algebras*. In: V. Dlab and L.L. Scott (ed.), Finite dimensional algebras and related topics, Dordrecht, Kluwer, 1994, pp. 1–26.
[13] H. X. Chen and C. C. Xi, *Recollements induced from tilting modules over tame hereditary algebras*. Forum Math. **27**(2015), 1849–1901.
[14] H. X. Chen and C. C. Xi, *Symmetric subcategories, tilting modules, and derived recollements*. Rev. Mat. Iberoam. **39**(2023), 1771–1812.
[15] L. W. Christensen, A. Frankild, and H. Holm, *On Gorenstein projective, injective and flat dimensions–a functorial description with applications*. J. Algebra **302**(2006), 231–279.

[16] D. Daniel and B. Shipley, *K-theory and derived equivalences*. Duke Math. J. **124**(2004), no. 3, 587–617.

[17] E. E. Enochs and O. M. G. Jenda, *Gorenstein injective and projective modules*. Math. Z. **220**(1995), 611–633.

[18] E. E. Enochs and O. M. G. Jenda, *Relative homological algebra*, Walter de Gruyter, Berlin and New York, 2000.

[19] H. Enomoto, *Maximal self-orthogonal modules and a new generalization of tilting modules*. Preprint, 2023. arXiv:2301.13498.

[20] M. Gorsky, *Semi-derived and derived Hall algebras for stable categories*. Int. Math. Res. Not. **2018**(2018), 138–159.

[21] E. L. Green, I. Reiten, and Ø. Solberg, *Dualities on generalized Koszul algebras*. Mem. Amer. Math. Soc. **159**(2002), 1–67.

[22] D. Happel, *Triangulated categories in representation theory of finite dimensional algebras*, London Mathematical Society Lecture Note Series, 119, Cambridge University Press, Cambridge, 1988.

[23] D. Happel and C. M. Ringel, *Tilted algebras*. Trans. Amer. Math. Soc. **274**(1982), 399–443.

[24] B. Huisgen-Zimmermann, *Dualities from iterated tilting*. Isr. J. Math. **243**(2021), 315–353.

[25] B. Huisgen-Zimmermann and M. Saorín, *Dualities for modules of finite projective dimension*, Contemporary Mathematics, 761, American Mathematical Society, Providence, RI, 2021.

[26] B. Keller, *Derived categories and their uses*. In: M. Hazewinkel (ed.), Handbook of algebra, Amsterdam, vol. **1**, 1996, pp. 671–701.

[27] T. Y. Lam, *Lectures on modules and rings*, Graduate Texts in Mathematics, 189, Springer, New York, 1999.

[28] M. Lu and L. G. Peng, *Semi-derived Ringel–Hall algebras and Drinfeld double*. Adv. Math. **383**(2021), 107668.

[29] M. Lu and W. Q. Wang, *Hall algebras and quantum symmetric pairs I: Foundations*. Proc. Lond. Math. Soc. **124**(2022), 1–82.

[30] B. Ma and J. Sauter, *On faithfully balanced modules, F-cotilting and F-Auslander algebras*. J. Algebra **556**(2020), 1115–1164.

[31] F. Mantese and I. Reiten, *Wakamatsu tilting modules*. J. Algebra **278**(2004), 532–552.

[32] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, John Wiley and Sons, Chichester, 1987.

[33] Y. Miyashita, *Tilting modules of finite projective dimension*. Math. Z. **193**(1986), 113–146.

[34] K. Morita, *Duality for modules and its applications to the theory of rings with minimum condition*. Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **6**(1958), 85–142.

[35] S. Y. Pan and C. C. Xi, *Finiteness of finitistic dimension is invariant under derived equivalences*. J. Algebra **322**(2009), 21–24.

[36] L. Positselski, *Contraherent cosheaves*. Preprint, 2024, arXiv:1209.2995v12.

[37] D. Quillen, *Higher algebraic K-theory, I*. In: A Dold and B. Eckmann (ed.), Algebraic K-theory, I: Higher K-theories (Seattle, 1972), Lecture Notes in Mathematics, 341, Springer, Berlin, 1973, pp. 85–147.

[38] J. Rickard, *Morita theory for derived categories*. J. Lond. Math. Soc. **39**(1989), 436–456.

[39] C. M. Ringel, *The Gorenstein-projective modules for the Nakayama algebras I*. J. Algebra **385**(2013), 241–261.

[40] C. M. Ringel and P. Zhang, *Gorenstein-projective and semi-Gorenstein-projective modules*. Algebra Number Theory **14**(2020), 1–36.

[41] T. Wakamatsu, *On modules with trivial self-extensions*. J. Algebra **114**(1988), 106–114.

[42] T. Wakamatsu, *Tilting modules and Auslander's Gorenstein property*. J. Algebra **275**(2004), 3–39.

[43] C. C. Xi, *On the finitistic dimension conjecture, III: Related to the pair $eAe \supset A$*. J. Algebra **319**(2008), 3666–3688.

School of Mathematical Sciences & Academy for Multidisciplinary Studies, Capital Normal University, Beijing 100048, P. R. China
e-mail: chenhx@cnu.edu.cn

School of Mathematics, Hangzhou Normal University, Hangzhou 311121, P. R. China
e-mail: hujs@hznu.edu.cn