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SOME SUFFICIENT CONDITIONS FOR GRAPHS TO HAVE (g, f)-FACTORS

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Suppose that G is a graph with vertex set $V(G)$ and edge set $E(G)$, and let g and f be two non-negative integer-valued functions defined on $V(G)$ such that $g(x) \leqslant f(x)$ for each $x \in V(G)$. A (g, f)-factor of G is a spanning subgraph F of G such that $g(x) \leqslant d_F(x) \leqslant f(x)$ for each $x \in V(F)$. In this paper, some sufficient conditions for a graph to have a (g, f)-factor are given.

1. INTRODUCTION

The graphs considered in this paper will be finite undirected simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, the degree of x in G is denoted by $d_G(x)$. Suppose g and f are two non-negative integer-valued functions defined on $V(G)$ such that $g(x) \leqslant f(x)$ for each $x \in V(G)$. Then a (g, f)-factor of graph G is defined as a spanning subgraph F of G such that $g(x) \leqslant d_F(x) \leqslant f(x)$ for each $x \in V(F)$. And if $g(x) = a$ and $f(x) = b$ for each $x \in V(F)$, then a (g, f)-factor is called an $[a, b]$-factor. In particular, G is called a (g, f)-graph if G itself is a (g, f)-factor. A graph G is called a (g, f, n)-critical graph if after deleting any n vertices of G the remaining graph of G has a (g, f)-factor. If G is a (g, f, n)-critical graph, then we also say that G is (g, f, n)-critical. If $g(x) = a$ and $f(x) = b$, then a (g, f, n)-critical graph is simply called an (a, b, n)-critical graph. If $g(x) = f(x)$ (respectively, $g(x) = f(x) = k$) for each $x \in V(G)$, then a (g, f, n)-critical graph is simply called an (f, n)-critical graph (a (k, n)-critical graph). If $k = 1$, then a (k, n)-critical graph is simply called an n-critical graph. A matching in a graph G is a set of edges of G with the property that no two edges are adjacent. A k-matching is a matching of size k. A matching M is said to be maximum if G has no a matching K with $|K| > |M|$.

A fractional (g, f)-factor is a function h that assigns to each edge of a graph G a number in [0,1], so that for each vertex x we have $g(x) \leqslant d_G^h(x) \leqslant f(x)$, where $d_G^h(x) = \sum\limits_{e \ni x} h(e)$ (the sum is taken over all edges incident to x) is a fractional degree of x in G. And if $g(x) = a$ and $f(x) = b$ for each $x \in V(G)$, then a fractional (g, f)-factor is

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called a fractional $[a, b]$-factor. The other terminologies and notations not given in this paper can be found in [**1**, **7**].

Many authors have investigated $[a, b]$-factors [**2**, **3**, **8**], (g, f)-factors [**4**, **9**, **10**], factorisations [**11**]. There is a sufficient condition for a graph G to have a (g, f)-factor which was given by Guizhen Liu.

Theorem 1. ([**4**]) *Let G be a graph, and let g and f are two non-negative integer-valued functions defined on $V(G)$ such that $g(x) < f(x)$ for each $x \in V(G)$. If $g(x) \leqslant d_G(x)$ and $\big(f(x) - 1\big)d_G(y) \geqslant \big(d_G(x) - 1\big)g(y)$ for each $x, y \in V(G)$, then G has a (g, f)-factor containing any edge e of G.*

In [**5**], Liu and Zhang gave a sufficient condition for the existence of a fractional (g, f)-factor in a graph G.

Theorem 2. ([**5**]) *Let G be a graph, and let g and f be two non-negative integer-valued functions defined on $V(G)$ such that $g(x) \leqslant f(x)$ for each $x \in V(G)$. If $g(x) \leqslant d_G(x)$ and $f(x)d_G(y) \geqslant d_G(x)g(y)$ for each $x, y \in V(G)$, then G has a fractional (g, f)-factor.*

In [**6**], Guizhen Liu and Lanju Zhang made the following theorem.

Theorem 3. ([**6**]) *Let G be a graph, and let g and f be two non-negative integer-valued functions defined on $V(G)$ such that $g(x) < f(x)$ for each $x \in V(G)$, then G has a fractional (g, f)-factor if and only if G has a (g, f)-factor.*

According to Theorems 2 and 3, we easily obtain the following result.

Theorem 4. *Let G be a graph, and let g and f be two non-negative integer-valued functions defined on $V(G)$ such that $g(x) < f(x)$ for each $x \in V(G)$. If $g(x) \leqslant d_G(x)$ and $f(x)d_G(y) \geqslant d_G(x)g(y)$ for each $x, y \in V(G)$, then G has a (g, f)-factor.*

2. The Proof of Main Theorems

In this paper, we generalise Theorems 1 and 4 and obtain the following theorems.

Theorem 5. *Let G be a graph, and let g and f be two non-negative integer-valued functions defined on $V(G)$ such that $g(x) < f(x)$ for each $x \in V(G)$, and M is an $(rk - r + 1)$-matching of G. If $g(x) \leqslant d_G(x)$ and $\big(f(x) - k\big)d_G(y) \geqslant \big(d_G(x) - k\big)g(y)$ for each $x, y \in V(G)$, then G has a (g, f)-factor containing M, where r and k are two positive integers.*

Proof: If $g(x) \leqslant d_G(x) \leqslant f(x)$ for each $x \in V(G)$, then G is a (g, f)-graph. By the definition of a (g, f)-graph, the theorem holds. In the following we assume that $g(x) < f(x) \leqslant d_G(x)$ for each $x \in V(G)$. We apply induction on k.

If $k = 1$, then we have

$$\big(f(x) - 1\big)d_G(y) \geqslant \big(d_G(x) - 1\big)g(y)$$

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for each $x, y \in V(G)$. According to Theorem 1, G has a (g, f)-factor containing M.

Suppose that the statement holds for $k = n$, that is, if

$$\big(f(x) - n\big)d_G(y) \geqslant \big(d_G(x) - n\big)g(y)$$

for each $x, y \in V(G)$, then G has a (g, f)-factor containing M. Let us proceed to the induction step.

If $k = n+1$, then $\big(f(x)-(n+1)\big)d_G(y) \geqslant \big(d_G(x)-(n+1)\big)g(y)$ for each $x, y \in V(G)$. In the following we prove that G has a (g, f)-factor containing M.

Let $H \subseteq M$ and $|H| = r$, and let $M' = M - H$, $G' = G - H$. We define $g'(x)$ and $f'(x)$ on $V(G)$ as follows,

$$g'(x) = \begin{cases} g(x) - 1, & x \in V(H) \\ g(x), & x \notin V(H) \end{cases}$$

$$f'(x) = \begin{cases} f(x) - 1, & x \in V(H) \\ f(x), & x \notin V(H) \end{cases}$$

Clearly, G has a (g, f)-factor containing M if and only if G' has a (g', f')-factor containing M'. In view of the induction hypothesis, we only need to prove

$$\big(f'(x) - n\big)d_{G'}(y) \geqslant \big(d_{G'}(x) - n\big)g'(y)$$

for each $x, y \in V(G')$. Now we consider four cases.

CASE 1. If $x \in V(H)$, $y \in V(H)$, then

$$d_{G'}(x) = d_G(x) - 1,\ f'(x) = f(x) - 1,\ d_{G'}(y) = d_G(y) - 1,\ g'(y) = g(y) - 1.$$

Thus, we have

$$\begin{aligned}
\big(f'(x) - n\big)d_{G'}(y) &= \big[f(x) - (n+1)\big]\big(d_G(y) - 1\big) \\
&= \big[f(x) - (n+1)\big]d_G(y) - f(x) + (n+1) \\
&\geqslant \big[d_G(x) - (n+1)\big]g(y) - f(x) + (n+1) \\
&= \big[d_G(x) - (n+1)\big]\big(g'(y) + 1\big) - f(x) + (n+1) \\
&= \big[d_G(x) - (n+1)\big]g'(y) + d_G(x) - (n+1) - f(x) + (n+1) \\
&= \big(d_{G'}(x) - n\big)g'(y) + d_G(x) - f(x) \\
&\geqslant (d_{G'}(x) - n)g'(y)
\end{aligned}$$

CASE 2. If $x \in V(H)$, $y \notin V(H)$, then

$$d_{G'}(x) = d_G(x) - 1,\ f'(x) = f(x) - 1,\ d_{G'}(y) = d_G(y),\ g'(y) = g(y).$$

In this case, we get that

$$\begin{aligned}(f'(x)-n)d_{G'}(y) &= (f(x)-(n+1))d_G(y)\\ &\geqslant [d_G(x)-(n+1)]g(y)\\ &= (d_{G'}(x)-n)g'(y)\end{aligned}$$

CASE 3. If $x \notin V(H)$, $y \in V(H)$,then

$$d_{G'}(x)=d_G(x),\ f'(x)=f(x),\ d_{G'}(y)=d_G(y)-1,\ g'(y)=g(y)-1.$$

Thus,we have

$$\begin{aligned}(f'(x)-n)d_{G'}(y) &= (f(x)-n)(d_G(y)-1)\\ &= [f(x)-(n+1)](d_G(y)-1)+d_G(y)-1\\ &= [f(x)-(n+1)]d_G(y)+d_G(y)-1-f(x)+(n+1)\\ &\geqslant [d_G(x)-(n+1)]g(y)+d_G(y)-f(x)+n\\ &= (d_G(x)-n)g(y)-g(y)+d_G(y)-f(x)+n\\ &\geqslant (d_G(x)-n)(g'(y)+1)-f(x)+n\\ &= (d_G(x)-n)g'(y)+d_G(x)-n-f(x)+n\\ &= (d_{G'}(x)-n)g'(y)+d_G(x)-f(x)\\ &\geqslant (d_{G'}(x)-n)g'(y)\end{aligned}$$

CASE 4. If $x \notin V(H)$, $y \notin V(H)$, then

$$d_G(x)=d_{G'}(x),\ f(x)=f'(x),\ d_G(y)=d_{G'}(y),\ g(y)=g'(y).$$

In this case, we have

$$\begin{aligned}(f'(x)-n)d_{G'}(y) &= (f(x)-n)d_G(y)\\ &= [f(x)-(n+1)]d_G(y)+d_G(y)\\ &\geqslant [d_G(x)-(n+1)]g(y)+d_G(y)\\ &= (d_G(x)-n)g(y)+d_G(y)-g(y)\\ &\geqslant (d_G(x)-n)g(y)\\ &= (d_{G'}(x)-n)g'(y)\end{aligned}$$

Thus, the induction hypothesis guarantees the existence of a (g',f')-factor containing M' in G'. Hence, G has a (g,f)-factor containing M.

This completes the proof. □

In view of the proof of Theorem 5, we justify similarly the following Theorem 6.

Theorem 6. *Let G be a graph, and let g and f be two non-negative integer-valued functions defined on $V(G)$ such that $g(x) < f(x)$ for each $x \in V(G)$. If $g(x) \leqslant d_G(x)$ and $\big(f(x) - k\big)d_G(y) \geqslant \big(d_G(x) - k\big)g(y)$ for each $x, y \in V(G)$, then G has a (g, f)-factor containing any k edges of G, where k is one non-negative integer.*

In Theorems 5 and 6, if $k = 1$, then we obtain Theorem 1. Furthermore, we have the following results.

Theorem 7. *Let G be a graph, and let g and f be two non-negative integer-valued functions defined on $V(G)$ such that $g(x) < f(x)$ for each $x \in V(G)$. If $g(x) \leqslant d_G(x)$ and $f(x)\big(d_G(y) - n\big) \geqslant d_G(x)g(y)$ for each $x, y \in V(G)$, then G is (g, f, n)-critical. Here n is a non-negative integer.*

Proof: Let $U \subseteq V(G)$, and $|U| = n$, and let $G' = G - U$. By assumption, we have

$$d_G(x) \geqslant d_{G'}(x) \geqslant d_G(x) - n$$

for each $x \in V(G')$. Thus, we get

$$f(x)d_{G'}(y) \geqslant f(x)\big(d_G(y) - n\big) \geqslant d_G(x)g(y) \geqslant d_{G'}(x)g(y)$$

for each $x, y \in V(G')$.

By Theorem 4, G' has a (g, f)-factor. From the definition of a (g, f, n)-critical graph, G is (g, f, n)-critical.

The proof is complete. ∎

Theorem 8. *Let G be a graph, and let g and f be two non-negative integer-valued functions defined on $V(G)$ such that $g(x) < f(x)$ for each $x \in V(G)$, and M is a maximum matching of G. If $g(x) \leqslant d_G(x)$ and $f(x)\big(d_G(y) - 1\big) \geqslant d_G(x)g(y)$ for each $x, y \in V(G)$, then G has a (g, f)-factor excluding M.*

Proof: Let $G' = G - M$. In this case, we have

$$d_G(x) \geqslant d_{G'}(x) \geqslant d_G(x) - 1$$

for each $x \in V(G')$. Thus, we get

$$f(x)d_{G'}(y) \geqslant f(x)\big(d_G(y) - 1\big) \geqslant d_G(x)g(y) \geqslant d_{G'}(x)g(y)$$

for each $x, y \in V(G')$.

In view of Theorem 4, G' has a (g, f)-factor. That is to say, G has a (g, f)-factor excluding M.

This completes the proof. ∎

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