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ON IMAGES OF REAL REPRESENTATIONS OF SPECIAL LINEAR GROUPS OVER COMPLETE DISCRETE VALUATION RINGS

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Abstract. In this paper, we investigate the abstract homomorphisms of the special linear group $\mathrm{SL}_n(\mathfrak{O})$ over complete discrete valuation rings with finite residue field into the general linear group $\mathrm{GL}_m(\mathbb{R})$ over the field of real numbers. We show that for $m < 2n$, every such homomorphism factors through a finite index subgroup of $\mathrm{SL}_n(\mathfrak{O})$. For $\mathfrak{O}$ with positive characteristic, this result holds for all $m \in \mathbb{N}$.

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1. Introduction. Borel and Tits showed in 1973 that in "most" cases, abstract homomorphisms between algebraic groups are in fact algebraic [**4**], i.e. *any* homomorphism $\varphi\colon G(k) \to G'(k')$ "almost" arises out from a field-morphism $k \to k'$.

In 1975 Margulis showed that higher rank lattices are superrigid. Employing the Borel–Harish Chandra theorems, this means that if R and k are a suitably chosen ring and field respectively then, any abstract homomorphism $G(R) \to G'(k)$ again *almost* arises out of a ring-morphism $R \to k$.

These results beg the following motivating question:

Question. Let R and R' be rings and G and G' be group schemes so that $G(R)$ and $G'(R')$ are well defined. When are the homomorphisms $G(R) \to G'(R')$ dictated by ring-morphisms $R \to R'$?

We purposefully leave G and G' vaguely defined. The reader may consider algebraic group schemes, or even the group generated by *elementary unipotent* matrices over R, which will be defined shortly. Answering questions along these lines, we have

- [**4**] Let k be an infinite field, G and G' be absolutely almost simple algebraic groups with G simply connected or G' adjoint, and G generated by k-unipotents. Modulo the finite centres of G and G', any abstract homomorphism $G(k) \to G'(k')$ with Zariski-dense image arises out of a field homomorphism $k \to k'$.

- [**2**, **3**, **8**] Let $\mathfrak{O}$ be the ring of integers of a number field k and G be higher rank and defined over k. Let $G'(\mathbb{C})$ be non compact. Then, any Zariski-dense homomorphism $G(\mathfrak{O}) \to G'(\mathbb{C})$ arises from a ring-morphism $\mathfrak{O} \to \mathbb{C}$.
- [**5**] Let $n \geqslant 3$. Every homomorphism $\mathrm{SL}_n(\mathbb{Z}[x]) \to \mathrm{GL}_D\overline{\mathbb{Q}}$ is not injective. This is a reflection of the fact that $\mathbb{Z}[x]$ does not admit a unital ring embedding into $\overline{\mathbb{Q}}$.
- [**11**] Let $n \geqslant 3$. Any semisimple representation $\mathrm{SL}_n(\mathbb{Z}[x_1, \ldots, x_m]) \to \mathrm{SL}_D\mathbb{C}$ is virtually the direct sum of tensor products of ring homomorphisms $\mathbb{Z}[x_1, \ldots, x_m] \to \mathbb{C}$.
- [**7**] Let $\mathbb{Z}\langle x, y\rangle$ be the free non-commutative ring on x and y. The group $\mathrm{EL}_3(\mathbb{Z}\langle x, y\rangle)$ generated by elementary unipotents over the ring $\mathbb{Z}\langle x, y\rangle$ does not have a faithful finite dimensional representation over any field.
- The most recent result is due to Igor Rapinchuk [**10**]. It applies to the very general context of higher rank universal Chevalley–Demazure group schemes, describing their abstract representations into $\mathrm{GL}_D(\mathbb{K})$, where $\mathbb{K}$ is an algebraically closed field. We state an example which we feel both captures the essence of the result and is relevant to our current work. Let $\mathfrak{O}$ be a local principal ideal ring and $n \geqslant 3$. Let $\varphi\colon \mathrm{SL}_n(\mathfrak{O}) \to \mathrm{GL}_D(\mathbb{C})$ be an abstract homomorphism. If the image is not finite then there exists a commutative $\mathbb{C}$-algebra B, an embedding $\iota{:}\mathrm{SL}_n(\mathfrak{O}) \to \mathrm{SL}_n\mathbb{C}$ (induced from a ring embedding $\mathfrak{O} \to B$) so that, up to finite index, φ factors through ι composed with a $\mathbb{C}$-algebraic map $\mathrm{SL}_n\mathbb{C} \to \mathrm{GL}_D(\mathbb{C})$. We remark that the general nature of this theorem makes us believe that, with some additional work, our result for $n \geqslant 3$ may be deduced from his. On the other hand, our inductive proof holds in the case of $n = 2$, and is therefore distinct from Rapinchuk's.

Let $\mathfrak{O}$ be a complete discrete valuation ring with finite residue field. The typical examples of such rings are $\mathbf{Z}_p$(the ring of p-adic integers) and $\mathbf{F}_q[[t]]$(the ring of formal power series with coefficients over a finite field). Our main result is the following:

THEOREM 1.1. *For every $n \in \mathbb{N}$ and $D < 2n$, the image of any abstract homomorphism $\varphi : \mathrm{SL}_n(\mathfrak{O}) \to \mathrm{GL}_D(\mathbb{R})$ is finite. Furthermore, if $\mathfrak{O}$ has positive characteristic then the image of φ is finite for all D.*

REMARK. The proof of Theorem 1.1 is completely elementary. In particular, it does not rely on Margulis super-rigidity.

The connection between this result and our motivating question is as follows: in the absence of unital ring-morphisms from $\mathfrak{O} \to \mathbb{R}$, the result means that these abstract homomorphisms are indeed, up to finite index, dictated by ring-morphisms $\mathfrak{O} \to \mathbb{R}$. Namely, up to restricting to a finite index subgroup, they arise from the zero map $\mathfrak{O} \to 0 \in \mathbb{R}$. This interpretation is clear in the context of our proof. Our objective is to show that a sufficient amount of the ring structure can be expressed in terms of the group structure of SL_n.

Fix $x \in \mathfrak{O}$ and $i \neq j$. We denote the elementary unipotent matrix with 1's on the diagonal, x in the $(i, j)^{th}$-entry, and 0's elsewhere by $E_{i,j}(x) \in \mathrm{SL}_n(\mathfrak{O})$. Consider the following two equations:

$$[E_{1,2}(x), E_{2,3}(y)] = E_{1,3}(xy),$$
$$E_{1,3}(x) \cdot E_{1,3}(y) = E_{1,3}(x + y).$$

This shows that if $n \geqslant 3$ both the additive and multiplicative structures of a ring are embedded in the group structure of SL_n. This is not possible for $n = 2$ but there is still a sufficient amount of information that is held about the ring inside the group structure of SL_2, provided the ring has many units. The task is then to pass this information, via the homomorphism from the source to the target, which is the essence of the proof.

A consequence of our result is that if $D < 2n$ then the D-dimensional real representations of $SL_n(\mathfrak{O})$, as an abstract group, are continuous in the local-topology.

2. Algebraic facts. In this section, we give a few algebraic facts that we shall need for the proof of Theorem 1.1. Recall $\mathfrak{O}$ is a complete discrete valuation ring and therefore is a principal ideal domain with a unique maximal ideal. Let π be a fixed generator of the maximal ideal of $\mathfrak{O}$. Being a discrete valuation ring, $\mathfrak{O}$ has a natural topology on it and we shall consider this topology on $\mathfrak{O}$ in the sequel.

LEMMA 2.1. *For any $\mathfrak{O}$ with zero characteristic, an additive subgroup is of finite index if and only if it contains a subgroup of the form $\pi^k\mathfrak{O}$.*

Proof. Let A be a finite index subgroup of $\mathfrak{O}$. Then A is both open and closed as a subgroup of $\mathfrak{O}$. The ring $\mathfrak{O}$ is a finite extension of $\mathbb{Z}_p$ and therefore there exist $x_1, x_2, \ldots, x_g \in \mathfrak{O}$ which generate $\mathfrak{O}$ over $\mathbb{Z}_p$. By hypothesis $\mathfrak{O}/A$ finite implies that there exists an integer m such that for all $1 \leqslant i \leqslant g$ the elements mx_i, and therefore $\mathbb{Z}_p[mx_1, mx_2, \ldots .mx_g]$, are contained in the kernel of the projection map $\mathfrak{O} \to \mathfrak{O}/A$. But then A closed implies that $m\mathbb{Z}_p[x_1, \ldots, x_g] = \pi^{\mathrm{val}(m)}\mathfrak{O}$ is contained in A. □

LEMMA 2.2 (Generalized Hensel's Lemma). *Let $f(x) \in \mathfrak{O}[x]$ be a polynomial. If there exists $a \in \mathfrak{O}$ such that*

$$f(a) \equiv 0(\mathrm{mod}\, f'(a)^2\pi\mathfrak{O}),$$

then there exists $a_0 \in \mathfrak{O}$ satisfying

$$f(a_0) = 0 \text{ and } a_0 \equiv a(\mathrm{mod}\, f'(a)\pi\mathfrak{O}).$$

If $f'(a)$ is a nonzero divisor in $\mathfrak{O}$, then a_0 is unique.

For a proof see [**9**, Theorem 2.24].

LEMMA 2.3. *For any $\mathfrak{O}$ with zero characteristic, there is a positive integer r and an element $q \in \mathfrak{O}^*$ so that $q^4 = -r$.*

Proof. It is enough to prove this result for $\mathbb{Z}_p$ as $\mathfrak{O}$ is a finite extension of $\mathbb{Z}_p$. For $\mathbb{Z}_p$, the proof follows by applying Lemma 2.2 to the following $f(x) \in \mathbb{Z}_p[x]$.

$$f(x) = \begin{cases} x^4 + 31, & \text{if } p = 2; \\ x^4 + (p-1), & \text{otherwise.} \end{cases}$$

□

Recall that, for a ring $\mathcal{R}$ (not necessarily unital), the elementary unipotent matrices $E_{ij}(x) \in M_n(\mathcal{R})$ for $x \in \mathcal{R}$ and $i \neq j$ are the matrices with 1's on the diagonal, x in the (i, j)th-entry, and 0's elsewhere. We denote by $EL_n(\mathcal{R})$ the group generated by the set of elementary unipotents $\{E_{i,j}(x) \in M_n(\mathcal{R}) : x \in \mathcal{R} \text{ and } i \neq j\}$.

LEMMA 2.4 ([**1**, Proposition 5.1]).

(1) *The group* $\mathrm{SL}_n(\mathfrak{O})$ *is generated by elementary unipotents for* $n \geqslant 2$.
(2) *The subgroup* $\mathrm{EL}_n(\pi^k\mathfrak{O})$ *is of finite index in* $\mathrm{SL}_n(\mathfrak{O})$, *for* $n \geqslant 2$.

COROLLARY 2.5. *If* $\rho : \mathrm{SL}_n(\mathfrak{O}) \to G$ *is a representation so that for some* $i \neq j$ *the image* $\rho(E_{i,j}(\mathfrak{O}))$ *is finite then* $\rho(\mathrm{SL}_n(\mathfrak{O}))$ *is finite.*

Proof. If the image $\rho(E_{i,j}(\mathfrak{O}))$ is finite, then there is some k so that $E_{i,j}(\pi^k\mathfrak{O}) \leqslant \ker(\rho)$. For any $r \neq s$ with $1 \leqslant r, s \leqslant n$, the groups $E_{i,j}(\pi^k\mathfrak{O})$ and $E_{r,s}(\pi^k\mathfrak{O})$ are conjugate in $\mathrm{SL}_n(\mathfrak{O})$ therefore the group $E_{r,s}(\pi^k\mathfrak{O})$ is also contained in the kernel of ρ. This means that $\mathrm{EL}_n(\pi^k\mathfrak{O}) \leqslant \ker(\rho)$ and hence by Lemma 2.4 the kernel has finite index in $\mathrm{SL}_n(\mathfrak{O})$. □

PROPOSITION 2.6. *Every finite index subgroup of* $\mathrm{SL}_n(\mathfrak{O})$ *has finite abelianization, i.e. it is strongly almost perfect. Furthermore, if either* $|\mathfrak{O}/\pi\mathfrak{O}| > 3$ *or* $n > 2$ *then* $\mathrm{SL}_n(\mathfrak{O})$ *is perfect.*

Proof. Let $G \leqslant \mathrm{SL}_n(\mathfrak{O})$ be a finite index subgroup. Then, for each i, j with $i \neq j$ the subgroup $G \cap E_{i,j}(\mathfrak{O})$ must be of finite index in $E_{i,j}(\mathfrak{O})$ and hence $G \geqslant \mathrm{EL}_n(\pi^k\mathfrak{O})$ for some k. Therefore, it is sufficient to show that $\mathrm{EL}_n(\pi^k\mathfrak{O})$ has finite abelianization.

For $n \geqslant 3$ this follows from the Steinberg relations which in fact shows that both $\mathrm{EL}_n(\pi^k\mathfrak{O})$ and $\mathrm{SL}_n(\mathfrak{O})$ are perfect.

For the case of $n = 2$ we further subdivide to consider two cases according to whether $|\mathfrak{O}/\pi\mathfrak{O}| > 3$ or $|\mathfrak{O}/\pi\mathfrak{O}| \leqslant 3$.

Assume $|\mathfrak{O}/\pi\mathfrak{O}| > 3$. Then, there is an $\xi \in \mathfrak{O}^*$ such that $\xi^2 - 1$ is invertible. Indeed, $(\mathfrak{O}/\pi\mathfrak{O})^*$ is a cyclic group of order greater than 2, which means that there is an element of order greater than 2. Let ξ be a lift of this element under the natural map $\mathfrak{O} \to (\mathfrak{O}/\pi\mathfrak{O})$. Then $\xi^2 - 1$ is not in the kernel $\pi\mathfrak{O}$ and hence $\xi^2 - 1$ is invertible. Then by Lemma 1.6 [**1**] which states that if there is $\xi \in \mathfrak{O}^*$ such that $\xi^2 - 1$ is invertible then $\mathrm{SL}_n(\mathfrak{O})$ is perfect, we obtain our result.

For general $|\mathfrak{O}/\pi\mathfrak{O}|$ and $n = 2$, Observe that

$$\begin{pmatrix} 1 & \pi^k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pi^k & 1 \end{pmatrix} = \begin{pmatrix} 1 + \pi^{2k} & \pi^k \\ \pi^k & 1 \end{pmatrix}.$$

Therefore, after multiplying by suitable elements of $\mathrm{EL}_2(\pi^k\mathfrak{O})$ we see that for some $x \in \mathfrak{O}$ the following element belongs to $\mathrm{EL}_2(\pi^k\mathfrak{O})$:

$$\begin{pmatrix} 1 + \pi^{2k}x & 0 \\ 0 & (1 + \pi^{2k}x)^{-1} \end{pmatrix}.$$

Let $q = 1 + \pi^{2k}x$. Then, $q^2 - 1 = \pi^{k_0}x'$ for some $k_0 \geqslant 2k$ and $x' \in \mathfrak{O}^*$. Therefore, the commutator subgroup of $\mathrm{EL}_2(\pi^k\mathfrak{O})$ contains

$$\left(\begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & (q^2 - 1)t \\ 0 & 1 \end{pmatrix}, \text{ for every } t \in \pi^k\mathfrak{O}, \tag{1}$$

and in particular, contains the subgroup

$$\begin{pmatrix} 1 & \pi^{k_0+k}\mathfrak{O} \\ 0 & 1 \end{pmatrix}.$$

Considering the transpose analogue of the commutator relation (1) we see that the commutator subgroup of $\mathrm{EL}_2(\pi^k\mathfrak{O})$ contains the finite index subgroup $\mathrm{EL}_2(\pi^{k_1}\mathfrak{O})$ for $k_1 = k_0 + k$. Hence $\mathrm{EL}_2(\pi^k\mathfrak{O})$ has finite abelianization.

□

LEMMA 2.7. *If $S \leqslant \mathrm{GL}_D(\mathbb{R})$ is a solvable subgroup then there exists a finite index subgroup $S_0 \trianglelefteq S$ such that $[S_0, S_0]$ is unipotent and is conjugate to an upper triangular unipotent group via an element of $\mathrm{GL}_D(\mathbb{R})$.*

Proof. Let S_0 be the finite index subgroup so that the Zariski closure $\overline{S}_0^Z(\mathbb{C})$ is Zariski-connected. By the Lie–Kolchin theorem **[6]** $\overline{S}_0^Z(\mathbb{C})$ is conjugate into the upper triangular group and the commutator subgroup $[\overline{S}_0^Z(\mathbb{C}), \overline{S}_0^Z(\mathbb{C})]$ is unipotent. This means that $[S_0, S_0] \leqslant [\overline{S}_0^Z(\mathbb{C}), \overline{S}_0^Z(\mathbb{C})]$ is unipotent. Since the entries of S are in $\mathbb{R}$, there is an $\mathbb{R}$-basis which upper-triangulates the unipotent group $[S_0, S_0]$. □

For a ring $\mathcal{R}$ we will denote by $N_n(\mathcal{R}), U_n(\mathcal{R}), D_n(\mathcal{R}) \leqslant \mathrm{EL}_n(\mathcal{R})$ the maximal upper triangular group, maximal upper triangular unipotent group, and the maximal diagonal group respectively.

LEMMA 2.8. *If N_0 is a finite index subgroup of $N_n(\mathfrak{O})$ then $U_n(\mathfrak{O}) \cap [N_0, N_0]$ has finite index in $U_n(\mathfrak{O})$.*

Proof. The proof is by induction on n.

For $n = 2$, let $N_0 \leqslant N_2(\mathfrak{O})$ be the finite index subgroup of interest. Observe that, since $N_0 \cap D_2(\mathfrak{O})$ is finite index in $D_2(\mathfrak{O})$, there is an integer $k \geqslant 0$ such that

$$\begin{pmatrix} 1+\pi^k & 0 \\ 0 & (1+\pi^k)^{-1} \end{pmatrix} \in N_0.$$

Similarly, $N_0 \cap E_{12}(\mathfrak{O})$ has finite index in $E_{12}(\mathfrak{O})$ and by Lemma 2.1 contains $E_{12}(\pi^r\mathfrak{O})$ for some positive integer r.

Apply the commutation relation (1) with $q = 1 + \pi^k$ and $t \in \pi^r\mathfrak{O}$ and we see that $[N_0, N_0] \cap U_n(\mathfrak{O})$ contains the finite index subgroup

$$\begin{pmatrix} 1 & \pi^{k+r}\mathfrak{O} \\ 0 & 1 \end{pmatrix}.$$

Now, assume it is true for n and let us show it for $n+1$. Consider $N_n(\mathfrak{O}) \hookrightarrow N_{n+1}(\mathfrak{O})$ by taking the last column of $N_{n+1}(\mathfrak{O})$ to be trivial. Similarly, we have $U_n(\mathfrak{O}) \hookrightarrow U_{n+1}(\mathfrak{O})$. Let $N_0 \leqslant N_{n+1}(\mathfrak{O})$ be the finite index subgroup in question. And let $N_0' = N_0 \cap N_n(\mathfrak{O})$.

Consider, $[N_0', N_0'] \cap U_n(\mathfrak{O})$. Then, by induction $[N_0', N_0'] \cap U_n(\mathfrak{O}) \geqslant U_n(\pi^k\mathfrak{O})$ for some $k \geqslant 0$. Observing that $[N_0, N_0] \cap U_{n+1}(\mathfrak{O})$ is normal in $U_{n+1}(\mathfrak{O})$ the following commutator relation gives the desired result:

$$[E_{i,n}(\pi^k\mathfrak{O}), E_{n,n+1}(\mathfrak{O})] = E_{i,n+1}(\pi^k\mathfrak{O}).$$

□

Combining Lemmas 2.7 and 2.8, we obtain the following:

LEMMA 2.9. *Let* $\varphi : N_n(\mathfrak{O}) \to \mathrm{GL}_D(\mathbb{R})$ *be a homomorphism. Then, there exists a normal finite index subgroup* N_0 *of* $N_n(\mathfrak{O})$ *such that* $U_0 = [N_0, N_0] \cap U_n(\mathfrak{O})$ *is of finite index in* $U_n(\mathfrak{O})$ *and so that the image* $\varphi(U_0)$ *is unipotent.*

3. Proof of Theorem 1.1.

3.1. Proof in positive characteristic. Let $\mathfrak{O}$ be a complete discrete valuation ring of positive characteristic. Let φ: $\mathrm{SL}_n(\mathfrak{O}) \to \mathrm{GL}_D(\mathbb{R})$ be a homomorphism. We shall show that the image of φ is finite.

With Lemma 2.9 we find a finite index subgroup $U_0 \leqslant U$ so that $\varphi(U_0)$ is unipotent. The ring $\mathfrak{O}$ has positive characteristic implies all the elements in U_0, and therefore of $\varphi(U_0)$, have finite order. Being a unipotent subgroup of $\mathrm{GL}_D(\mathbb{R})$ we obtain $\varphi(U_0)$ is finite. By Corollary 2.5 we conclude that the image $\varphi(\mathrm{SL}_n(\mathfrak{O}))$ is finite.

3.2. Proof in characteristic zero. Now onwards we assume that $\mathfrak{O}$ is a complete discrete valuation ring of zero characteristic. We use induction for this case. We prove this for $n = 2$ first.

Step 1: $\mathrm{SL}_2(\mathfrak{O}) \to \mathrm{GL}_2(\mathbb{R})$

Proof in this case follows from the following proposition combined with Corollary 2.5.

PROPOSITION 3.1. *For any representation* $\varphi : N_2(\mathfrak{O}) \to \mathrm{GL}_2(\mathbb{R})$ *the image* $\varphi(U_2(\mathfrak{O}))$ *is finite.*

Proof. With the representation fixed, let $U_0 \leqslant U_2(\mathfrak{O})$ be the finite index subgroup guaranteed by Lemma 2.9 so that $\varphi(U_0)$ is unipotent.

If the 1-eigen space of $\varphi(U_0)$ is 2-dimensional then the map φ factors through U_0 and the result follows. Therefore, assume by contradiction that it is 1 dimensional. Since the image $\varphi(U_0)$ has $\mathbb{R}$-entries, the 1-eigen space is defined over $\mathbb{R}$ and so, up to post composing with an inner automorphism of $\mathrm{GL}_2\mathbb{R}$ we may assume that the image $\varphi(U_0)$ is upper triangular unipotent.

Since the image of the centralizer (respectively normalizer) of U_0 must centralize (respectively normalize) the image of U_0 we see that the image $\varphi(U_2(\mathfrak{O}))$ is upper triangular with ± 1 on the diagonal (and respectively the image $\varphi(N_2(\mathfrak{O}))$ is upper triangular).

This gives rise to an additive map $\psi_A : \mathfrak{O} \to \mathbb{R}$ and multiplicative maps $\psi_i : \mathfrak{O}^* \to \mathbb{R}^*$ as follows:

$$\varphi\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\pm 1 & \psi_A(x)\\ 0 & \pm 1\end{pmatrix},$$

and

$$\varphi\begin{pmatrix}q & 0\\ 0 & q^{-1}\end{pmatrix} = \begin{pmatrix}\psi_1(q) & *\\ 0 & \psi_2(q)\end{pmatrix} = \begin{pmatrix}\psi_1(q) & 0\\ 0 & \psi_2(q)\end{pmatrix}\begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix}.$$

Consider the following relation for $q^2 \in \mathfrak{O}^*$, $x \in \mathfrak{O}$, $r \in \mathbb{Z}$:

$$\begin{pmatrix}q^2 & 0\\ 0 & q^{-2}\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}q^{-2} & 0\\ 0 & q^2\end{pmatrix} = \begin{pmatrix}1 & q^4x\\ 0 & 1\end{pmatrix}.$$

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Using our definitions of ψ_i and ψ_A, after applying φ to both sides of the equation above and observing that $\left(\begin{smallmatrix}1 & * \\ 0 & 1\end{smallmatrix}\right)$ centralizes the image of U_0 we get the following:

$$\begin{pmatrix}\psi_1(q)^2 & * \\ 0 & \psi_2(q)^{-2}\end{pmatrix}\begin{pmatrix}\pm 1 & \psi_A(x) \\ 0 & \pm 1\end{pmatrix}\begin{pmatrix}\psi_1(q)^2 & * \\ 0 & \psi_2(q)^{-2}\end{pmatrix} = \begin{pmatrix}\pm 1 & \psi_A(q^4 x) \\ 0 & \pm 1\end{pmatrix}.$$

Performing the matrix multiplication, we obtain the following equation, which holds for every $x \in \mathfrak{O}$ and $q \in \mathfrak{O}^*$:

$$\psi_1(q)^2\psi_2(q)^2\psi_A(x) = \psi_A(q^4 x). \tag{2}$$

By Lemma 2.3, we can find $q \in \mathfrak{O}^*$ so that q^4 is a negative integer, say $-r$. Using the fact that additive maps between abelian groups are $\mathbb{Z}$-equivariant, equation (2) becomes

$$(\psi_1(q)^2\psi_2(q)^2 + r)\psi_A(x) = 0.$$

But the above expression is in $\mathbb{R}$ so that $\psi_1(q)^2\psi_2(q)^2 + r$ must be positive. Therefore we must have $\psi_A(x) = 0$ for all $x \in \mathfrak{O}$. This contradicts our assumption that the 1-eigen space of $\varphi(U_0)$ is 1 dimensional. □

Step 2: $\mathrm{SL}_2(\mathfrak{O}) \to \mathrm{GL}_3(\mathbb{R})$

Proof. We begin by giving the proof in case $\varphi\colon \mathrm{SL}_2(\mathfrak{O}) \to \mathrm{GL}_3(\mathbb{R})$ is reducible. We then show that any representation into $\mathrm{GL}_3(\mathbb{R})$ must either be reducible or have finite image.

If φ is reducible, then there is an invariant subspace V of dimension one or two. By extending a basis for V to a basis of $\mathbb{R}$, we may conjugate with an element of $\mathrm{GL}_3(\mathbb{R})$ so that $\varphi(\mathrm{SL}_2(\mathfrak{O}))$ is an upper block triangular subgroup of $\mathrm{GL}_3(\mathbb{R})$. This gives rise to a map from the image $\varphi(\mathrm{SL}_2(\mathfrak{O})) \to \mathrm{GL}_1(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$ with abelian kernel. Applying the previously established fact that any representation $\mathrm{SL}_2(\mathfrak{O}) \to \mathrm{GL}_2(\mathbb{R})$ has finite image, we see that $\varphi(\mathrm{SL}_2(\mathfrak{O}))$ contains a finite index abelian subgroup. But, as $\mathrm{SL}_2(\mathfrak{O})$ is strongly almost perfect (Lemma 2.6), we deduce that $\varphi(\mathrm{SL}_2(\mathfrak{O}))$ is finite.

We now show that either φ is reducible or has finite image. As before, we apply Lemma 2.9 to find U_0 of finite index in $U_2(\mathfrak{O})$ so that $\varphi(U_0)$ is unipotent.

Let $V_1 \subset \mathbb{R}^3$ be the 1-eigen space of $\varphi(U_0)$. Recall that it is $N_2(\mathfrak{O})$ invariant since $U_0 \trianglelefteq N$. If V_1 is a 3-dimensional space then the image of U_0 is trivial and hence by Corollary 2.5, we get that the image of $SL_2(\mathfrak{O})$ is finite. If V_1 is not 3-dimensional, then either V_1 or $\mathbb{R}^3/V_1$ is two dimensional.

Again, since V_1 is $N_2(\mathfrak{O})$-invariant, we get two homomorphisms $N_2(\mathfrak{O}) \to \mathrm{GL}(V_1)$ and $N_2(\mathfrak{O}) \to \mathrm{GL}(\mathbb{R}^3/V_1)$. By Proposition 3.1, we must have that the image of $U_2(\mathfrak{O})$ in each is finite. In particular, by choice of V_1 the image of U_0 in both $\mathrm{GL}(V_1)$ and $\mathrm{GL}(\mathbb{R}^3/V_1)$ is trivial.

Therefore, up to post-composing φ with the transpose inverse automorphism of $\mathrm{GL}_3(\mathbb{R})$ if necessary, we may assume that the 1-eigen space of $\varphi(U_0)$ has dimension two.

Now, since U_0 and U_0^t (the group consisting of transpose matrices of U_0) are conjugate inside $\mathrm{SL}_2(\mathfrak{O})$, the 1-eigen space of the image $\varphi(U_0^t)$ has dimension two as well. Therefore, the intersection of these two 2-dimensional spaces must be non-trivial in $\mathbb{R}^3$ which means that the image of the group $\langle U_0, U_0^t\rangle$ has a non-trivial 1-eigen space.

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The group $\langle U_0, U_0^t\rangle$ is of finite index in $\mathrm{SL}_2(\mathfrak{O})$. Up to passing to a further finite index subgroup if necessary, we may assume that it is normal in $\mathrm{SL}_2(\mathfrak{O})$ and hence the non-trivial 1-eigen space of this finite index normal subgroup is invariant under $\mathrm{SL}_2(\mathfrak{O})$. This means that φ is reducible.

□

Step 3: The general case

Proof. Now onwards, whenever we speak of $\mathrm{SL}_{n-1}(\mathfrak{O}) \leqslant \mathrm{SL}_n(\mathfrak{O})$ we mean that we view $\mathrm{SL}_{n-1}(\mathfrak{O})$ as a subgroup of $\mathrm{SL}_n(\mathfrak{O})$ embedded in the upper left-hand corner of $\mathrm{SL}_n(\mathfrak{O})$.

To proceed by induction, we assume that the image of any homomorphism $\mathrm{SL}_{n-1}(\mathfrak{O}) \to \mathrm{GL}_{2n-3}(\mathbb{R})$ is finite. By considering $\mathrm{SL}_{n-1}(\mathfrak{O}) \leqslant \mathrm{SL}_n(\mathfrak{O})$ and using Corollary 2.5 we get that $\mathrm{SL}_n\mathfrak{O} \to \mathrm{GL}_D(\mathbb{R})$ has finite image for all $D < 2n-3$.

We are left to prove that if $2n-3 < D \leqslant 2n-1$ then the image of $\varphi : \mathrm{SL}_n(\mathfrak{O}) \to \mathrm{GL}_D(\mathbb{R})$ is finite. The following argument works for both $D = 2n-2$ and $D = 2n-1$. The argument for $D = 2n-1$ follows by the induction hypothesis. After proving for $D = 2n-2$, we apply the same argument for $D = 2n-1$ and use the result for $D = 2n-2$ in this.

As before, let U_0 be determined by Lemma 2.9. Let

$$L = \{(l_{ij}) \in \mathrm{SL}_n(\mathfrak{O}) \mid l_{ii} = 1, l_{ij} = 0\ \forall\, i \neq j \,\text{and}\, j \neq n\},$$

be the abelian subgroup of $\mathrm{SL}_n(\mathfrak{O})$ consisting of matrices having 1's on the diagonal and non-trivial entries only in the last column. It is easily verified that L is normalized by $\mathrm{SL}_{n-1}(\mathfrak{O}) \leqslant \mathrm{SL}_n(\mathfrak{O})$. By intersecting L with U_0, we obtain a finite index subgroup L_0 of L whose image is unipotent. By Lemma 2.1, we can pass to a further finite index subgroup and assume that there exists an integer m such that

$$L_0 = \{(l_{ij}) \in L \mid l_{ij} \in \pi^m\mathfrak{O}\ \forall\ i \neq j\},$$

is contained in U_0 and is also normalized by $\mathrm{SL}_{n-1}(\mathfrak{O})$.

The image $\varphi(L_0)$ is unipotent, therefore there exists a flag

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = \mathbb{R}^D, \tag{3}$$

of susbpaces of $\mathbb{R}^D$ with the property that V_1 is the maximal 1-eigen space for $\varphi(L_0)$ and V_j is the maximal 1-eigen space for the quotient action on $\mathbb{R}^D/V_{j-1}$. Since $\varphi(L_0)$ is normalized by $\varphi(\mathrm{SL}_{n-1}(\mathfrak{O}))$, the flag in (3) is preserved by $\varphi(\mathrm{SL}_{n-1}(\mathfrak{O}))$.

If $k = 1$ then $\mathbb{R}^D$ is the 1-eigenspace of $\varphi(L_0)$, that is to say the image of L_0 is trivial. Therefore the image of $E_{1,n}(\mathfrak{O}) \leqslant L$ is finite and again, Corollary 2.5 shows that the image of $\mathrm{SL}_n(\mathfrak{O})$ is finite.

We now assume that $k > 1$. The argument proceeds in two cases depending on whether $2 \leqslant \dim(V_j) \leqslant D-2$ for some j or not. Assume that $2 \leqslant \dim(V_{j_0}) \leqslant D-2$ for some j_0. By assumption on D this means that the dimension, and co-dimension of V_{j_0} both satisfy the inequality

$$2 \leqslant \dim(V_{j_0}), D - \dim(V_{j_0}) < 2(n-1).$$

This now allows us to apply the induction hypothesis to the action of $\varphi(\mathrm{SL}_{n-1}(\mathfrak{O}))$ on both V_{j_0} and $\mathbb{R}^D/V_{j_0}$ and we get that the image of the map from $\varphi(\mathrm{SL}_{n-1}(\mathfrak{O}))$

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to $GL(V_{j_0}) \times GL(\mathbb{R}^D/V_{j_0})$ is finite. Let $\Gamma \leqslant SL_{n-1}(\mathfrak{O})$ be the finite index subgroup with trivial image in $GL(V_{j_0}) \times GL(\mathbb{R}^D/V_{j_0})$. Since the kernel of the map $stab(V_{j_0}) \to GL(V_{j_0}) \times GL(\mathbb{R}^D/V_{j_0})$ is abelian, we see that $\varphi(\Gamma)$ is abelian, and hence finite since $SL_{n-1}(\mathfrak{O})$ is strongly almost perfect (Lemma 2.6). In particular, this implies that the image of $E_{12}(\mathfrak{O})$ is finite, which concludes the proof in this case.

Now we are left with the case for which $\dim(V_j) = 1$ or $D-1$ for every $j = 1, \ldots, k-1$. This means that the flag (3) for $\varphi(L_0)$ is $\{0\} \subset V_1 \subset \mathbb{R}^D$, with V_1 being either of dimension or co-dimension one. Again, by postcomposing φ with the transpose inverse automorphism of $GL_D(\mathbb{R})$ if necessary, we can assume that the 1-eigen space of L_0 is $D-1$ dimensional.

Consider the n distinct conjugates of L that correspond to the distinct columns of $SL_n(\mathfrak{O})$. By taking these conjugates of L_0, we generate $EL_n(\pi^m\mathfrak{O})$. Each of these column spaces has a $D-1$ dimensional 1-eigenspace, let us call these $W_1, \ldots, W_n$. Then, $\cap_{i=1}^n W_i$ is a 1-eigenspace for $\varphi(EL_n(\pi^m\mathfrak{O}))$. The following shows that since $D \geqslant n+1$, the intersection is not trivial:

LEMMA 3.2. *Let $W_1, W_2, \ldots, W_n$ be co-dimension one subspaces in a D dimensional space. Then* $\dim(\cap_{i=1}^n W_i) \geqslant D-n$.

Proof. This result follows by $\dim(W_1 \cup W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$. □

Let us pass to a finite index subgroup of $EL_n(\pi^m\mathfrak{O})$ which is normal in $SL_n(\mathfrak{O})$. Then V, the 1-eigenspace for the image of this subgroup is at least $(D-n)$-dimensional, at most $(D-1)$-dimensional and $SL_n(\mathfrak{O})$-invariant.

This gives a map $\varphi(SL_n(\mathfrak{O})) \to GL(V) \times GL(\mathbb{R}^D/V)$. The dimension and co-dimension of V are both less than D. We have already established that this means that the image of $SL_n(\mathfrak{O})$ in $GL(V) \times GL(\mathbb{R}^D/V)$ is finite (notice that for $D = 2n-2$, we have $\dim(V) \leqslant 2n-3$ and result follows by induction and for $D = 2n-1$, $\dim(V) \leqslant 2n-2$ and result follows from $D = 2n-2$). We see that $\varphi(SL_n(\mathfrak{O}))$ has to contain a finite index abelian subgroup. But, as $SL_n(\mathfrak{O})$ is strongly almost perfect, we deduce that $\varphi(SL_n(\mathfrak{O}))$ is finite.

□

COROLLARY 3.3. *Assume that* $|\mathfrak{O}/\pi\mathfrak{O}| > 3$. *The image of any representation* $SL_2\mathfrak{O} \to GL_2\mathbb{R}$ *is trivial.*

Proof. Theorem 1.1 shows that the image of any representation $SL_2(\mathfrak{O}) \to GL_2(\mathbb{R})$ is finite, therefore compact, and hence contained in a conjugate of the maximal compact subgroup $SO_2(\mathbb{R})$. Since $SO_2(\mathbb{R})$ is abelian and $SL_2\mathfrak{O}$ is perfect whenever $|\mathfrak{O}/\pi\mathfrak{O}| > 3$, we conclude that the image is trivial.

□

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