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DEFORMING A CONVEX DOMAIN INTO A DISK BY KLAIN'S CYCLIC REARRANGEMENT

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Abstract

For a convex domain, we use Klain's cyclic rearrangement to obtain a sequence of convex domains with increasing area and the same perimeter which converges to a disk. As a byproduct, we give a proof of the classical isoperimetric inequality in the plane.

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1. Introduction

It is an interesting problem to deform a compact region in the n-dimensional Euclidean space into another one. There are various schemes to solve the simplest case of this problem, that is to deform a compact region into a ball. Some employ curvature flows: the curve shortening flow can turn a planar compact region with C^2 boundary into a disk in a finite time (see [5–8]), the Gauss curvature flow can deform a convex body into a ball in 3-dimensional Euclidean space (see [1]), and the mean curvature flow can turn a convex body into a ball in n-dimensional Euclidean space (see [9]). Another approach uses symmetrisation for compact regions, such as the Steiner symmetrisation (see [2–4, 11, 12, 15]). In the 2-dimensional Euclidean plane, the Steiner symmetrisation preserves the area and decreases the perimeter of a compact domain. Klain [10] introduced another geometric method, the cyclic rearrangement, which preserves the perimeter of a star body while increasing its area (see Section 2).

The purpose of this note is to prove the following theorem by Klain's cyclic rearrangement.

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THEOREM 1.1. *Let K be a planar convex domain. Then there exists a sequence of convex domains with increasing area and the same perimeter which converges to a disk with the same perimeter as K.*

In order to prove this theorem, we have to solve two important problems. The first is to guarantee that Klain's cyclic rearrangement can be carried out. The second is to ensure that the cyclic rearrangement does not terminate until the convex domain turns into a disk. As an application of this theorem, we derive a proof of the classical isoperimetric inequality which is different from that given by Klain [10].

2. Basic definitions and cyclic rearrangement

We denote by $\mathbb{R}^2$ the usual 2-dimensional Euclidean space with the canonical inner product $\langle\cdot,\cdot\rangle$. A nonempty bounded set $M \subseteq \mathbb{R}^2$ is called a *convex domain* if it has a nonempty interior and for any points $x, y \in M$, the line segment with endpoints x, y is contained in M. If a convex domain M is centrally symmetric with respect to the origin o, then it is said to be *o-symmetric*.

Let M, N be two convex domains. The Minkowski sum of M and N is defined by

$$M + N = \{x + y \mid x \in M,\ y \in N\}.$$

Given a real number λ, the product λM is defined by

$$\lambda M = \{\lambda x \mid x \in M\}$$

and it is customary to write simply $-M$ instead of $(-1)M$. The convex domain

$$M_C = \tfrac{1}{2}(M + (-M))$$

is called the *central symmetrisation* of M.

The nonempty set $S \subseteq \mathbb{R}^2$ is called a *star-shaped set* with respect to a point $c \in S$, if for all $z \in S$ the line segment with endpoints z, c lies entirely in S, and c is called a *centre* of S. Let S^1 be the unit circle in $\mathbb{R}^2$ centred at the point c. The *radial function* $\rho_S : S^1 \to [0, \infty]$ of a star-shaped set S is a nonnegative function on the unit circle, defined by

$$\rho_S(u) = \max\{a \in \mathbb{R} \mid au \in S\}.$$

If ρ_S is a positive continuous function, then the star-shaped set S is called a *star body*.

In order to make this note complete, we shall describe the cyclic rearrangement for star bodies in almost the same terms as Klain [10]. Let $\{u, v\}$ be a set of unit vectors which forms a basis of $\mathbb{R}^2$, and denote by φ the angle between u and v; without loss of generality, let $0 < \varphi \leq \frac{1}{2}\pi$. Suppose that S is a star body such that there exist a point c and two directions u, v satisfying

$$\rho_S(u) = \rho_S(-u) \overset{\Delta}{=} a, \quad \rho_S(v) = \rho_S(-v) \overset{\Delta}{=} b. \tag{2.1}$$

Klain [10] defined the *cyclic rearrangement* of S with respect to the point c and the basis $\{u, v\}$, denoted by $\boxtimes S$, to be the set formed by the four part dissection and rearrangement shown in Figure 1.

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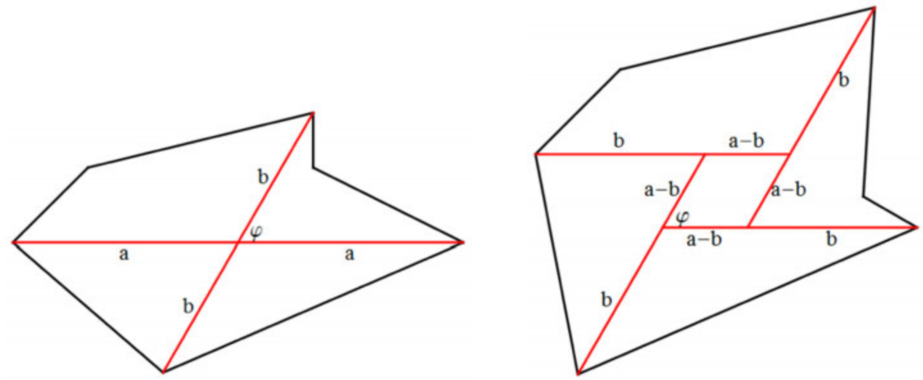

(a) Star body. (b) Star body's cyclic rearrangement.

FIGURE 1. Star body and its cyclic rearrangement.

A detailed definition of $\boxtimes S$ runs as follows. Let Q_i, $i = 1, 2, 3, 4$, denote the four quadrants of $\mathbb{R}^2$ induced by the basis $\{u, v\}$, listed counterclockwise starting with the positive quadrant as shown in Figure 1(a). Let ϕ denote reflection across the line passing through the point c and $u + v$. Set $S_i = S \cap Q_i$, $i = 1, 2, 3, 4$, and

$$\hat{S}_1 = \phi S_1 + \frac{a-b}{2}(u - v), \quad \hat{S}_2 = S_2 + \frac{a-b}{2}(u + v),$$
$$\hat{S}_3 = \phi S_3 + \frac{a-b}{2}(-u + v), \quad \hat{S}_4 = S_4 + \frac{a-b}{2}(-u - v).$$

Denote by R the rhombus centred at the point c, with sides of length $a - b$ and parallel to u and v. Then, $\boxtimes S$ is defined by Figure 1(b), that is

$$\boxtimes S = \hat{S}_1 \cup \hat{S}_2 \cup \hat{S}_3 \cup \hat{S}_4 \cup R.$$

It is clear that $\boxtimes S$ is a compact simply connected set, but not necessarily a star-shaped set. Further, for the length L and area A,

$$L(\boxtimes S) = L(S), \quad A(\boxtimes S) = A(S) + (a - b)^2|\sin\varphi|.$$

If we can do a cyclic rearrangement for $\boxtimes S$, then we denote it by $\boxtimes_2 S = \boxtimes(\boxtimes S)$. In general, $\boxtimes_k S$ represents the kth cyclic rearrangement of S.

3. Proof of Theorem 1.1

In this section, we will first give some propositions and then prove the main result.

PROPOSITION 3.1. *Let S be a star body. If S is centrally symmetric with respect to a point c, then so is $\boxtimes S$.*

PROOF. Without loss of generality, we can choose the centre c as the origin. If S is a disk, then the result is obvious. If S is not a disk, then as in (2.1) there exist two

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directions u, v such that $a = \rho_S(u) = \rho_S(-u)$, $b = \rho_S(v) = \rho_S(-v)$ and $a \neq b$. Let A_i be the boundary points of S in directions $\pm u$ and $\pm v$, where $i = 1, 2, 3, 4$. Then, in the affine coordinate system $c - uv$, the A_i can be expressed as $A_1(a, 0)$, $A_2(0, b)$, $A_3(-a, 0)$ and $A_4(0, -b)$. Denote by B_i the points corresponding to A_i under the cyclic rearrangement. From the definition of the cyclic rearrangement and a simple computation, it follows that B_i can be expressed as

$$B_1\Big(\frac{a-b}{2}, \frac{a+b}{2}\Big), \quad B_2\Big(\frac{a+b}{2}, -\frac{a-b}{2}\Big), \quad B_3\Big(-\frac{a-b}{2}, -\frac{a+b}{2}\Big), \quad B_4\Big(-\frac{a+b}{2}, \frac{a-b}{2}\Big),$$

which implies that $\boxtimes S$ is also centrally symmetric with respect to the origin. □

Corollary 3.2. *If K is an o-symmetric convex domain, then $\boxtimes K$ is also o-symmetric.*

Remark 3.3. Notice that $\boxtimes K$ is not necessarily a convex domain even if K is an o-symmetric convex domain. For example, take K to be an ellipse, u the direction of its semi-major axis and v an arbitrary direction which is not its semi-major axis or semi-minor axis.

The next proposition will show that the convexity is maintained on special directions in the process of the cyclic rearrangement for an o-symmetric convex domain.

Proposition 3.4. *If K is an o-symmetric convex domain, then we can find a cyclic rearrangement such that $\boxtimes K$ is also an o-symmetric convex domain.*

Proof. From Corollary 3.2, it follows that K is o-symmetric. Without loss of generality, we may assume that K is not a disk. Denote by ρ_K the radial function of K. Since ρ_K is a function on S^1, there exist two directions u, v such that $\rho_K(u) = (\rho_K)_{\max}$ and $\rho_K(v) = (\rho_K)_{\min}$. Since K is o-symmetric, u and v are directions of the maximum width and minimum width. From [13, page 33], one can see that there are four points A, B, C, D on the boundary of K such that AC and BD are on the directions of the maximum width and minimum width and perpendicular to supporting lines at A, C and B, D (see Figure 2). Thus, the convexity is maintained in the process of cyclic rearrangement with respect to the directions u and v. □

The final proposition helps to determine the final shape of an o-symmetric convex domain by a sequence of cyclic rearrangements.

Proposition 3.5. *Let K be an o-symmetric convex domain and u, v the respective directions of the maximum and minimum of its radial function. If the cyclic rearrangement $\boxtimes K$ of K with respect to the basis $\{u, v\}$, is equal to K, then K must be a disk.*

Proof. Let $a = \rho_K(u) = (\rho_K)_{\max}$, $b = \rho_K(v) = (\rho_K)_{\min}$. From the proof of Proposition 3.4, it follows that $\boxtimes K$ is also o-symmetric. If K is not a disk, then $a \neq b$. From the definition of cyclic rearrangement, $L(\boxtimes K) = L(K)$ and

$$A(\boxtimes K) = A(K) + (a-b)^2|\sin\varphi| > A(K),$$

which contradicts $\boxtimes K = K$. Thus, K must be a disk. □

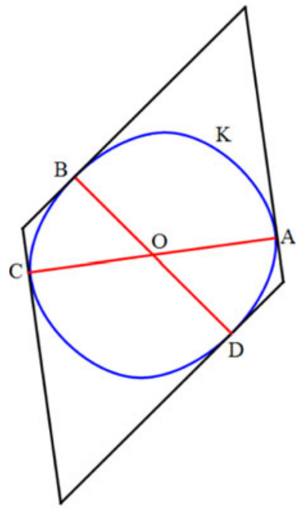

FIGURE 2. Centrally symmetric convex domain and its circumscribed parallelogram.

PROOF OF THEOREM 1.1. If K is centrally symmetric, we may assume that K is o-symmetric since we can choose the centre of symmetry of K as the origin. From Corollary 3.2 and Proposition 3.4, it follows that there are two sequences of directions u_i and v_i such that $\rho_{K_i}(u_i) = (\rho_{K_i})_{\max}$ and $\rho_{K_i}(v_i) = (\rho_{K_i})_{\min}$, where $K_i = \boxtimes_i K$, $i = 1, 2, \ldots$. Hence, one can find a sequence of convex bodies K_i with the same perimeter and increasing area such that $K_i = \boxtimes_i K$, $i = 1, 2, \ldots$. The diameter $D(K_i)$ of K_i satisfies

$$D(K_i) \leq \frac{L(K_i)}{2} = \frac{L(K)}{2}.$$

So, from the Blaschke selection theorem and Proposition 3.5, the process of cyclic rearrangement for an o-symmetric convex domain K will go on until it turns into a disk with the same perimeter as K (that is, the sequence K_i converges to a disk with the same perimeter as K in the Hausdorff metric on convex domains).

If K is not centrally symmetric, we first construct the central symmetrisation K_C for K, that is, $K_C = \frac{1}{2}(K + (-K))$. By properties of central symmetrisation [14, Solutions 6–9(c)(d), pages 237–239], one has $A(K_C) \geq A(K)$ and $L(K_C) = L(K)$. (Notice that the proofs of these properties depend on geometric constructions which avoid using the Brunn–Minkowski inequality.) Together with the definition of cyclic rearrangement, this yields a sequence of convex bodies K_i defined by $K_1 = K_C$ and $K_{i+1} = \boxtimes_i K_C$ for $i = 1, 2, \ldots$, with the same perimeter and increasing area which converges to a disk with the same perimeter as K, by applying the argument in the first part of the proof to the sequence of centrally symmetric bodies $\boxtimes_i K_C$. □

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4. A short proof of the isoperimetric inequality

As an application of Theorem 1.1, we can give a proof of the classical isoperimetric inequality by Klain's cyclic rearrangement.

THEOREM 4.1. *If K is a planar region with perimeter $L(K)$ and area $A(K)$, then*

$$L(K)^2 - 4\pi A(K) \geq 0, \tag{4.1}$$

and the equality in (4.1) holds if and only if K is a disk.

PROOF. Since the convex hull for a planar set does not decrease its area and does not increase its perimeter, without loss of generality, we assume that K is a convex domain. It follows from Theorem 1.1 that we can obtain a sequence of convex domains K_i with the same perimeter and increasing area which converges to a disk of radius $L(K)/(2\pi)$, where $K_i = \boxtimes_i K$ when K is centrally symmetric or $K_1 = K_C$, $K_{i+1} = \boxtimes_i K_C$ when K is not centrally symmetric, and

$$a_i = \rho_{K_i}(u_i) = (\rho_{K_i})_{\max}, \quad b_i = \rho_{K_i}(v_i) = (\rho_{K_i})_{\min}, \quad i = 1, 2, \ldots.$$

Denote by φ_i the angle between u_i and v_i. Then

$$L(K)^2 - 4\pi A(K) \geq L(K_1)^2 - 4\pi A(K_1) \geq \cdots \geq L(K_n)^2 - 4\pi A(K_n), \tag{4.2}$$

where $L(K_i) = L(K)$, $i = 1, 2, \ldots, n$, and

$$A(K_n) = \begin{cases} A(K) + \displaystyle\sum_{i=1}^{n} (a_i - b_i)^2 |\sin\varphi_i|, & \text{if } K \text{ is centrally symmetric,} \\ A(K_C) + \displaystyle\sum_{i=1}^{n-1} (a_i - b_i)^2 |\sin\varphi_i|, & \text{if } K \text{ is not centrally symmetric.} \end{cases}$$

Since K_i converges to a disk, $L(K_n)^2 - 4\pi A(K_n) \to 0$ as $n \to \infty$. Obviously, if K is a disk, the equality in (4.1) holds. Conversely, it is obvious that (4.1) is strict, since at least one inequality sign of (4.2) is strict when K is not a disk. □

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