SUB-MANIFOLDS OF A LOCALLY PRODUCT RIEMANNIAN MANIFOLD

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Summary. In this paper, we have obtained the conditions that a sub-manifold of a locally product Riemannian manifold may also be locally product. Generalizations of the equations of Gauss, Mainardi-Codazzi and Ricci have been obtained for the locally product sub-manifold of the locally product Riemannian manifold.

1. Sub-manifold. Let us consider an m-dimensional Riemannian manifold V_m, endowed with the Riemannian metric tensor a. Let $\bar{D}$ be the Reimannian connection in V_m. Let there be defined in V_m a vector valued linear function F, satisfying

$$F(F(K)) = K, \tag{1.1}$$

$$(\bar{D}_K F)(L) = 0, \tag{1.2}$$

$$'F(K, L) \stackrel{\text{def}}{=\!=} a(F(K), L) = 'F(L, K), \tag{1.3}$$

for arbitrary vector fields K, L, ... in V_m. Then F is said to give an almost product structure to V_m and V_m is said to be an almost product Riemannian manifold.

Let V_n be an orientable sub-manifold of V_m with the immersion $\xi : V_n \to V_m$ whose differential $B : T_p(V_n) \to T_{\xi(p)}(V_m)$ is injective at each point p of V_n, where $T_p(V_n)$ and $T_{\xi(p)}(V_m)$ denote the tangent spaces of V_n and V_m at p respectively. Then the induced metric tensor g of V_n is given by

$$g(X, Y) = a(BX, BY), \tag{1.4}$$

where X, Y are arbitrary C^∞ vector fields of V_n.

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Let N_x, $x = n+1, \ldots, m$ be a set of mutually orthogonal C^∞ unit vector fields to the sub-manifold V_n in such a way that $B(X)$ and N_x, for arbitrary X, form a positive sense of V_m and $B(X)$ a positive sense of V_n. Then we have

$$(1.5)\text{a} \qquad a(BX, N_x) = 0 ,$$

$$(1.5)\text{b} \qquad a(N_x, N_y) = \delta_{xy} .$$

The transform $F(BX)$ of BX and $F(N_x)$ of N_x by F can be expressed as a tangential and a normal part. Thus if repeated x, y, z also imply summation, we have

$$(1.6) \qquad F(BX) = Bf(X) + H_x(X) N_x ,$$

$$(1.7) \qquad F(N_x) = B K_x + L_{xy} N_y .$$

We will now study the C^∞ vector valued linear function f, the C^∞ linear function H_x, the C^∞ vector field K_x and the C^∞ function L_{xy}, defined in the sub-manifold V_n.

THEOREM 1.1. Let

$$(1.8) \qquad (G(X))(Y) \stackrel{\text{def}}{=\!=} g(X, Y) .$$

Then

$$(1.9) \qquad H_x(X) = (G(K_x))(X) = g(K_x, X) .$$

$$(1.10)\text{a} \qquad 'f(X, Y) \stackrel{\text{def}}{=\!=} g(f(X), Y)$$

is symmetrical in X, Y:

$$(1.10)\text{b} \qquad 'f(X, Y) = 'f(Y, X) .$$

Also L_{xy} is symmetrical in xy:

$$(1.10)\text{c} \qquad L_{xy} = L_{yx} .$$

Proof. Using (1.5)b and (1.3) in (1.6), we obtain

$$H_x(X) = a(F(BX), N_x) = 'F(BX, N_x) .$$

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Also from (1.8), (1.4), (1.3) and (1.7), we obtain

$$g(K_x, X) = (G(K_x)(X) = a(FN_x, BX) = 'F(N_x, BX) = 'F(BX, N_x) .$$

Comparing the last two equations, we obtain (1.9).

From (1.3), (1.4) and (1.10)a, we have

(1.11) $'F(BX, BY) = a(F(BX), BY) = g(f(X), Y) = 'f(X, Y)$.

Symmetry of $'F$ establishes the symmetry of $'f$.

From (1.5)b, (1.7), (1.3), and the symmetry of $'F$ we have

$$L_{xy} = a(F(N_x), N_y) = 'F(N_x, N_y) = 'F(N_y, N_x) = a(F(N_y), N_x) = L_{xy} .$$

This proves (1.10)c.

Let D be the induced Riemannian connexion with respect to V_n. Then it is well known that

(1.12) $$\bar{D}_{BY}BX = B D_Y X + 'V_x(X, Y)N_x ,$$

where V_x is the symmetric bilinear mapping of V_n. Also we have

(1.13)a $$\bar{D}_{BX} N_y = - B V_x(X) + \theta_{xy} (X) N_y ,$$

where

(1.13)b $$g(V_x(X), Y) = 'V_x(X, Y) ,$$

and θ_{xy} is skew-symmetric in xy:

(1.13)c $$\theta_{xy} + \theta_{yx} = 0 .$$

THEOREM 1.2. <u>We have</u>

(1.14)a $$f(f(X)) + H_x(X) K_x = X ,$$

(1.14)b $$H_x(F(X)) + L_{xy} H_y(X) = 0 ,$$

(1.14)c $$L_{xy} L_{yz} + H_z(K_x) = \delta_{xz} ,$$

(1.14)d $$f(K_x) + L_{xy} K_y = 0 .$$

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Proof. Using (1.1) in (1.6), we obtain

$$B(X) = Bf(f(X)) + H_x(f(X))N_x + H_x(X)\, B(K_x) + H_y(X)\, L_{yx} N_x .$$

Tangential and normal parts of this equation yield (1.14)a, b.

Again from (1.1) and (1.7), we obtain

$$N_x = Bf(K_x) + H_y(K_x)N_y + L_{xy} B(K_y) + L_{xz} L_{zy} N_y .$$

Tangential and normal parts of this equation yield (1.14)c, d.

THEOREM 1.3. We also have

(1.15)a $$'F(F(N_x), N_y) \overset{\text{def}}{=\!=} a(F(N_x), F(N_y)) = K_x K_y + L_{xz} L_{zy} ,$$

(1.15)b $$'F(F(BX), BY) \overset{\text{def}}{=\!=} a(F(BX), F(BY)) =$$

$$g(f(X), f(Y)) + H_x(X)\, H_x(Y) ,$$

(1.15)c $$'F(F(BX), N_x) = 'F(B(X), F(N_x)) =$$

$$a(F(BX), F(N_x)) = f(X, K_x) + L_{xy}\, H_y(X) .$$

Proof. These equations follow immediately from (1.6) and (1.7).

THEOREM 1.4. $D_X f'$, $D_X H_x$ and $X \cdot L_{xy}$ are given by

(1.16)a $$(D_X 'f)(Y, Z) = H_x(Y)'V_x(X, Z) + H_x(Z)'V_x(X, Y) ,$$

(1.16)b $$(D_X H_x)(Y) = L_{xz}\, 'V_z(X, Y) + {}_{xz}\theta(X) H_z(Y) - 'V_x(X, f(Y)) .$$

(1.16)c $$X \cdot L_{xy} = -'V_y(X, K_x) - 'V_x(X, K_y) + \theta_{xz} L_{zy} + \theta_{yz} L_{zx} .$$

Proof. From (1.2) and (1.12), we have

$$\bar{D}_{BX}(F(BY)) = F(\bar{D}_{BX} BY) = F(BD_X Y) + V_x(X, Y)F(N_x) .$$

Substituting from (1.6) and (1.7) in this equation, we have

(1.17) $$B(D_X f)(Y) + V_y(X, f(Y))N_y + (D_X H_y)(Y)\, N_y + H_y(D_X Y)N_y - H_x(Y)BV_x(X)$$

$$+ H_x(Y)\; {}_{xy}\theta(X) N_y = H_y(D_X Y)N_y + 'V_x(X, Y)BK_x + 'V_x(X, Y)\, L_{xy}\, N_y .$$

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The tangential part of the above equation is

$$B(D_X f)(Y) = \underset{x}{H}(Y)\, B\, \underset{x}{V}(X) + {}'\underset{x}{V}(X, Y)\, B\underset{x}{K} .$$

Using (1.4), (1.13)b and (1.9) in this equation, we obtain (1.16)a. The normal part of (1.17) yields (1.16)b.

The equation (1.16)c can be obtained similarly.

2. Locally product sub-manifold.

THEOREM 2.1. *If in a sub-manifold of a locally product Riemannian manifold the matrix* $((\underset{xy}{L}))$ *is equal to*

$$\text{diag}\ (c_{n+1}\ ,\ c_{n+2}\ ,\ \text{----}\ ,\ c_m)\ ,$$

where c's *are constants, the submanifold is also a locally product manifold.*

Proof. Let $\underset{xy}{L} = \begin{cases} \underset{x}{c} & \text{if } x = y \\ o & \text{if } x \neq y \end{cases}$. Then from (1.14)c, we have

$$\underset{y}{H}(\underset{x}{K}) = o\ ,\quad y \neq x\ ,$$

which is equivalent to

$$g(\underset{y}{K}\ ,\ \underset{x}{K}) = o\ ,\quad y \neq x\ ,$$

in consequence of (1.9). These are at the most $\frac{1}{2}(m-n)(m-n+1)$ independent equations in m-n unknowns $\underset{x}{K}$. But

$$(m-n) - \frac{1}{2}(m-n)\ (m-n+1) = -\frac{1}{2}(m-n)\ (m-n-p)\ ,$$

which is negative. Hence the equations do not have non-trivial solutions. Consequently,

$$\underset{x}{K} = o\ ,\quad \underset{x}{H} = o\ .$$

In consequence of these equations, the equations (1.14)a, c and (1.16)a, b assume the forms

$$f(f(X)) = X\ , \tag{2.1}$$

$$(\underset{x}{c})^2 = (\underset{xx}{L})^2 = 1\ ,\quad x \text{ not summed}\ , \tag{2.2}$$

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(2.3) $$(D_X f)(Y) = o ,$$

(2.4) $$'V_x(X, Y) = \pm 'V(X, f(Y)) .$$

The equations (2.1), (2.3) and the fact that 'f is symmetric, prove the statement.

Note (2.1). If the conditions of the above theorem are satisfied, the equations (1.6) and (1.7) assume the forms

$$F(BX) = Bf(X) ,$$

$$F(N_x) = \pm N_x .$$

We will now relax the conditions of Theorem (2.1).

THEOREM 2.2. *The necessary and sufficient condition that the sub-manifold* V_n *of a locally product manifold* V_m *be locally product is*

(2.5)a $$K_x = o ,$$

or

(2.5) $$H_x(X) = o , \text{ for arbitrary } X.$$

Proof. Let the condition (2.5) be satisfied. Then the equations (1.14)a and (1.16)a reduce to

(2.6) $$f(f(X)) = X , \quad (D_X f)(Y) = o .$$

These two equations together with the symmetry of 'f prove that the sub-manifold is a locally product manifold.

Conversely, let (2.6) be satisfied. Then (1.14)a reduces to

$$H_x(X) K_x = o ,$$

which yields the condition (2.6). Hence, we have the statement.

COROLLARY 2.1. *If the sub-manifold* V_n *of the locally product manifold* V_m *be locally product,*

(2.7)a $$F(BX) = Bf(X) ,$$

(2.7)b $$F(N_x) = L_{xy} N_y ,$$

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$$(2.8)\text{a} \qquad f(f(X)) = X ,$$

$$(2.8)\text{b} \qquad L_{xy} L_{yz} = \delta_{xz} ,$$

$$(2.9)\text{a} \qquad (D_X f)Y = o ,$$

$$(2.9)\text{b} \qquad L_{xz}\ 'V_z(X, Y) = 'V_x(X, f(Y)) ,$$

$$(2.9)\text{c} \qquad X \cdot L_{xy} = \theta_{xz} + L_{zy} + \theta_{yz} L_{zx} .$$

Proof. Putting $\underset{x}{H}(X) = o$ and $\underset{x}{K} = o$ in (1.6) and (1.7) we obtain (2.7). Putting these values in (1.14)a, c, and (1.16) we obtain (2.8) and (2.9) respectively.

COROLLARY 2.2. When the sub-manifold V_n of the locally product manifold V_m is locally product, $F(BX)$ is tangential to V_n and $F(\underset{x}{N})$ is a c^∞ set of mutually orthogonal unit normal vectors to V_n.

Proof. Putting $\underset{x}{H}(X) = o$ and $\underset{x}{K} = o$ in (1.15) and using (2.8)b and (2.7)a, we obtain

$$(2.10)\text{a} \qquad a(F(\underset{x}{N}) , F(\underset{y}{N})) = \delta_{xy} ,$$

$$(2.10)\text{b} \qquad a(F(BX) , F(BY)) = g(f(X), f(Y)) = g(X, Y) ,$$

$$(2.10)\text{c} \qquad a(F(BX) , F(\underset{x}{N})) = 0 .$$

These equations prove the statement.

Note (2.2). The equation (2.10)b proves that the magnitude of $F(BX)$ is the same as the magnitude of BX in V_m.

3. Curvature tensor. In this section, we confine ourselves to V_m only. But the results hold also for V_n when V_n is also a locally product Riemannian manifold.

THEOREM 3.1. Let S be the curvature tensor of the locally product Riemannian manifold V_m, that is, it is a vector valued trilinear function of V_m, and let

$$(3.1) \qquad \bar{L} \overset{\text{def}}{=\!=} F(L) , \quad L \in V_m .$$

Then

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(3.2) $$\bar{S}(L, M, N) = S(L, M, \bar{N}) .$$

Proof. We have

$$S(L, M, N) = \bar{D}_L \bar{D}_M N - \bar{D}_M \bar{D}_L N - D_{[L, M]} N .$$

Using (3.1) and (4.2) in this equation, we obtain

$$S(L, M, \bar{N}) = \bar{D}_L \bar{D}_M (F(N)) - \bar{D}_M \bar{D}_L (F(N)) - \bar{D}_{[L, M]} (F(N))$$

$$= F(\bar{D}_L \bar{D}_M N - \bar{D}_M \bar{D}_L N - \bar{D}_{[L, M]} N)$$

$$= \bar{S}(L, M, N) ,$$

which proves the statement.

COROLLARY 3.1. We have

(3.3) $$S(L, M, N) = \bar{S}(L, M, \bar{N}) .$$

Proof. Using (1.1) in (3.2), we obtain (3.3).

COROLLARY 3.2. Put

(3.4) $$T(L, M, N, P) = a(S(L, M, N), P) .$$

Then

(3.5) $$T(L, M, N. P) = T(L, M, \bar{N} \bar{P}) .$$

Proof. From (3.3), (1.3) and (3.4), we have

$$T(L, M, N. P) = a(S(L, M, N), P) = a(\bar{S}(L, M, N), P)$$

$$= a(S(L, M, \bar{N}), \bar{P}) = T(L, M, \bar{N}, \bar{P})$$

which is the equation (3.5).

COROLLARY 3.3. Let the Ricci tensor Ric which is the bilinear scalar function, be defined by

(3.6) $$Ric(L, M) \stackrel{def}{=\!=} (C_1^1 S)(L, M) ,$$

where C_1^1 is the contraction (Mishra, 1965). Then

(3.7)a $$Ric(L, M) = (C_1^1 \bar{S})(L, \bar{M}) ,$$

(3.7)b $$Ric(L, \bar{M}) = (C_1^1 \bar{S})(L, M) ,$$

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(3.7)c $$\mathrm{Ric}(\bar{L}, M) = (C_1^1 \bar{S})(\bar{L}, \bar{M}) ,$$

(3.7)d $$\mathrm{Ric}(\bar{L}, \bar{M}) = (C_1^1 \bar{S})(\bar{L}, M) .$$

Proof. (3.7)a follows at once from (3.3) and (3.6). Remaining equations follow from (3.7)a by barring L and M in (3.7)a.

COROLLARY 3.4. We have

(3.8)a $$\mathrm{Ric}(L, \bar{M}) = \mathrm{Ric}(\bar{L}, M) ,$$

(3.8)b $$\mathrm{Ric}(\bar{L}, \bar{M}) = \mathrm{Ric}(L, M) .$$

Proof. Using Bianchi's first identities in (3.7)b, we get

(3.9)a

$$\mathrm{Ric}(\bar{L}, M) = \mathrm{Ric}(M, \bar{L}) = (C_1^1 \bar{S})(M\ L) = -(C_2^1 \bar{S})(M. L) - (C_3^1 \bar{S})(M, L) .$$

But

(3.9)b $$C_3^1 \bar{S} = 0$$

(3.9)c $$(C_2^1 \bar{S})(M, L) = -(C_1^1 \bar{S})(L, M) .$$

Substituting from (3.9)b, c in (3.9)a, and using (3.7)b we obtain

$$\mathrm{Ric}(\bar{L}, M) = (C_1^1 \bar{S})(L, M) = \mathrm{Ric}(L, \bar{M}) .$$

Barring L in (3.8)a and using (1.1) we obtain (3.8)b.

COROLLARY 3.5. $\mathrm{Ric}(L, \bar{M})$ is proportional to $'F(L, M)$ if and only if the product space is an Einstein space.

Proof. We have, using (1.1)

$$'F(L, \bar{M}) = {}'F(\bar{M}, L) = {}'F(F(M), L) = a(M, L) .$$

Let

$$\mathrm{Ric}(L, \bar{M}) = k\ F(L, M) .$$

Barring M and using (1.1), we get

$$\mathrm{Ric}(L, M) = k\,'F(L, \bar{M}) = k\ a(L, M) .$$

This proves the statement.

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4. Gauss, Codazzi and Ricci equations. In this section, we shall assume that the sub-manifold V_n of a locally product manifold V_m is also locally product. In this case $\underset{x}{H}(X) = o$, $\underset{x}{K} = o$, in consequence of Theorem (2.2), and the equations (1.6) and (1.7) reduce to

$$F(BX) = Bf(X) , \tag{4.1}$$

$$F(\underset{x}{N}) = \underset{xy}{L}\,\underset{y}{N} . \tag{4.2}$$

THEOREM 4.1. The following are the generalizations of the equations of Gauss and Mainardi-Codazzi for the locally product sub-manifold V_n of a locally product enveloping manifold V_m

(4.3)a

$$\bar{K}(F(BX), F(BY), F(BZ), F(BW)) = K(f(X), f(Y), f(Z), f(W)) - {}'\underset{x}{V}(f(Y), f(Z))'\underset{x}{V}(f(X), f(Z)) + {}'V(f(X), f(Z))'\underset{x}{V}(f(Y), f(Z)),$$

(4.3)b

$$\bar{K}(F(BX), F(BY), F(BZ), F(\underset{x}{N})) = \{(D_{f(X)}{}'\underset{t}{V})(f(Y), f(Z)) - (D_{f(Y)}{}'\underset{t}{V})(f(X), f(Z)) + {}'V(f(Y), f(Z))\,\underset{zt}{\theta}(f(X)) - {}'\underset{z}{V}(f(X), f(Z))\,\underset{zt}{\theta}(f(Y))\}\,\underset{tx}{L} .$$

where

(4.4)a

$$\bar{K}(F(BX), F(BY), F(BZ), F(BW)) \stackrel{\text{def}}{=\!=} a(\bar{R}(F(BX), F(BY), F(BZ), F(BW)) ,$$

(4.4)b

$$\bar{K}(F(BX), F(BY), F(BZ), F(\underset{x}{N})) \stackrel{\text{def}}{=\!=} a(\bar{R}(F(BX), F(BY), F(BZ), F(\underset{x}{N})) .$$

Proof. In consequence of (4.1) and (4.2) we have

$$\bar{R}(F(BX), F(BY), F(BZ)) = \bar{R}(Bf(X), Bf(Y), Bf(Z))$$

$$= \bar{D}_{Bf(X)}\bar{D}_{Bf(Y)}Bf(Z) - \bar{D}_{Bf(Y)}\bar{D}_{Bf(X)}Bf(Z) - \bar{D}_{[Bf(X), Bf(Y)]}Bf(Z) . \tag{4.5}$$

Now in consequence of (1.12) and (1.13)a,

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$$\bar{D}_{Bf(X)}\bar{D}_{Bf(Y)}Bf(Z) = \bar{D}_{Bf(X)}(BD_{f(Y)}f(Z) + {}'\underset{x}{V}(f(Y), f(Z))\underset{x}{N})$$

(4.6)
$$= BD_{f(X)}D_{f(Y)}f(Z) + \underset{x}{V}(f(X), D_{f(Y)}f(Z))\underset{x}{N}$$
$$+ f(X)\cdot({}'\underset{x}{V}(f(Y), f(Z))\underset{x}{N} - {}'\underset{x}{V}(f(Y), f(Z))B\underset{x}{V}(f(X))$$
$$+ {}'\underset{z}{V}(f(Y), f(Z)) \underset{zx}{\theta}(f(X))\underset{x}{N} \quad ,$$

and

(4.7) $$\bar{D}_{[Bf(X), Bf(Y)]}Bf(Z) = BD_{[f(X), f(Y)]}f(Z) + {}'\underset{x}{V}([f(X), f(Y)], f(Z))\underset{x}{N} .$$

Breaking (4.5) into tangential and normal parts and using (4.6) and (4.7), we get

(4.8) $$\text{Tan } \bar{R}(F(BX), F(BY), F(BZ)) = B\, R(f(X), f(Y), f(Z))$$
$$- {}'\underset{x}{V}(f(Y), f(Z))B\underset{x}{V}(f(X)) + {}'\underset{x}{V}(f(X), f(Z))B\, \underset{x}{V}(f(Y)) .$$

(4.9)a $$\text{nor } \bar{R}(F(BX), F(BY), F(BZ)) = \{\underset{x}{V}(f(X), D_{f(Y)}f(Z) -$$
$${}'\underset{x}{V}(f(Y), D_{f(X)}f(Z)) + f(X)\cdot({}'\underset{x}{V}(f(Y), f(Z))) - f(Y)\cdot({}'\underset{x}{V}(f(X), f(Z)))$$
$$+ {}'\underset{z}{V}(f(Y), f(Z)) \underset{zx}{\theta}(f(X)) - {}'\underset{z}{V}(f(X), f(Z)) \underset{zx}{\theta}(f(Y))\} \underset{x}{N} .$$

But

$$f(X)\cdot({}'\underset{x}{V}(f(Y), f(Z))) - {}'\underset{x}{V}(f(Y), D_{f(X)}f(Z)) = (D_{f(X)}\, {}'\underset{x}{V})(f(Y), f(Z))$$
$$+ {}'\underset{x}{V}(D_{f(X)}f(Y), f(Z)).$$

Therefore the equation (4.9)a assumes the form

$$\text{nor } \bar{R}(F(BX), F(BY), F(BZ)) = \{(D_{f(X)}\, {}'\underset{x}{V})(f(Y), f(Z)) +$$

(4.9)b
$$(D_{f(Y)}\, {}'\underset{x}{V})(f(X), f(Z)) +$$
$${}'\underset{z}{V}(f(Y), f(Z)) \underset{zx}{\theta}(f(X)) - {}'\underset{z}{V}(f(X), f(Z))$$
$$\underset{zx}{\theta}(f(Y))\} \underset{x}{N} .$$

Substituting from (4.8) in (4.9)b and using (4.1) and (4.2) in (4.4) we obtain (4.3).

THEOREM 4.2. <u>The following are the generalizations of the equations of Ricci for the locally product sub-manifold</u> V_n <u>of a locally product enveloping manifold</u> V_m.

$$\bar{K}(F(BX), F(BY), F(N_x), F(N_y)) = L_{yz}\{((f(X))\cdot L_{xz})f(Y) - ((f(Y))\cdot L_{xz})f(X)$$

$$(4.10) \qquad + L_{xt}\,'V_z(F(Y), V_t(f(X))) - L_{xt}\,'V_z(f(X), V_t(f(Y)))$$

$$+ L_{xt}(D_{f(X)}\theta_{tz})(f(Y)) - L_{xt}(D_{f(Y)}\theta_{tz})(f(X))$$

$$+ L_{xp}\theta_{pt}(f(Y))\theta_{tz}(f(X)) - L_{xp}\theta_{pt}(f(X))\theta_{tz}(f(Y))\} .$$

<u>Proof</u>. In consequence of (4.2), we have

$$\bar{R}(F(BX), F(BY), F(N_x)) = \bar{R}(Bf(X), Bf(Y), L_{xy}N_y) =$$

$$\bar{D}_{Bf(X)}\bar{D}_{Bf(Y)}(L_{xy}N_y) - \bar{D}_{Bf(Y)}\bar{D}_{Bf(X)}(L_{xy}N_y) - \bar{D}_{[Bf(X), Bf(Y)]}(L_{xy}N_y).$$

But

$$\bar{D}_{Bf(X)}\bar{D}_{Bf(Y)}(L_{xy}N_y) = \bar{D}_{Bf(X)}(((f(Y))\cdot L_{xy})N_y - L_{xy}V_y(f(Y)) + L_{xy}\theta_{yz}(f(Y))N_z)$$

$$= (f(X)\cdot(f(Y)\cdot L_{xy}))N_y + (f(Y)\cdot L_{xy})(-B\,V_y(f(X)) +$$

$$\theta_{yz}(F(X))N_z) - (L(X)\cdot L_{xy})B\,V_y(f(Y) -$$

$$L_{xy}(B\,D_{f(X)}(V_y(f(Y)) + {}'V_z(f(X), V_y(f(Y)))N_z +$$

$$(f(X)\cdot(L_{xy}\theta_{yz}(f(Y))))N_z +$$

$$L_{xy}\theta_{yt}(f(Y))(-V_t(f(X)) + \theta_{tz}(f(X))N_z) ,$$

and

$$\bar{D}_{[Bf(X), Bf(Y)]}(L_{xy}N_y) = ([f(X), f(Y)]\cdot L_{xz})N_z +$$

$$L_{xy}(-BV_y([f(X), f(Y)]) + (\theta_{yz}([f(X), f(Y)]))N_z .$$

Therefore

$$\bar{R}(F(BX), F(BY), F(N_x)) = L_{xy}(\theta_{yt}(f(X))BV_t(f(Y)) - \theta_{yt}(f(Y))BV_t(f(X))$$

$$+ B(D_{f(Y)} V_y)(f(X)) - B(D_{f(X)} V_y)(f(Y)))$$

$$+ \{ L_{xt}('V_z(f(Y), V_t(f(X)) - 'V_z(f(X), V_t f(Y)))$$

$$+ L_{xp}(\theta_{pt}(f(Y))\theta_{tz}(f(X)) - \theta_{pt}(f(X))\, \theta_{tz}(f(Y))$$

$$+ (f(X) \cdot L_{xz})(f(Y)) - (f(Y) \cdot L_{xz})(f(X))$$

$$+ L_{xt}((D_{f(x)}\theta_{tz})(f(Y)) - (D_{f(Y)}\theta_{tz})(f(X))\} N_z \ .$$

Substituting from this equation in

$$\bar{K}(F(BX), F(BY), F(N_x), F(N_y)) \overset{def}{=\!=\!=} a(\bar{R}(F(BX), F(BY), F(N_x)), F\ N_y)$$

and using (4.2) we obtain (4.10).

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