48
Theoretical foundations

48.1 Generalities and dispersion relations

The fundamental concepts behind QCD spectral sum rules are the operator product expansion (OPE) and quark-hadron duality through the dispersion relation obeyed by the Green's functions due to their analytic properties. For illustrating the discussions, we shall consider, for definiteness, the generic two-point correlator:

$$\Pi_H(q^2) = i \int d^4x e^{iqx} \langle 0|\mathbf{T} J_H(x) J_H^\dagger(0)|0\rangle \ , \tag{48.1}$$

where $J_H(x)$ is a local gauge-invariant operators built from quark and/or gluon fields. In most applications, $J_H(x)$ are Noether currents associated to the global transformations of flavour degrees of freedom, such the vector $\bar\psi\gamma_\mu\psi$ or axial-vector $\bar\psi\gamma_\mu\gamma_5\psi$ current, but can also be the operators of gluon fields describing the gluonium $\mathbf{Tr}\, G_{\mu\nu}G^{\mu\nu}$, or operators describing the baryons $\psi\Gamma_1\psi\Gamma_2\psi$, the hybrids $\bar\psi\gamma_\mu\lambda_a G_a^{\mu\nu}\psi$ or weak matrix elements $\bar\psi\Gamma_1\psi\bar\psi\Gamma_2\psi \ldots$. Thanks to its analyticity property, it has been shown [624] that $\Pi(q^2)$ obeys the well-known Källen–Lehmann dispersion relation or *Hilbert representation*:

$$\Pi_H(q^2) = \int_{t_<}^{\infty} \frac{dt}{t - q^2 - i\epsilon} \frac{1}{\pi} \mathrm{Im}\Pi_H(t) + P(q^2) \tag{48.2}$$

where $P(q^2)$ represents subtraction terms, which are, in general, polynomial in q^2, with its degree depending on the convergence properties of the *spectral function* $\mathrm{Im}\Pi_H(t)$ for $t \to \infty$:

$$P(q^2) \equiv a + bq^2 + \cdots \ ; \tag{48.3}$$

$t_<$ is the hadronic threshold, which we shall take to be zero for simplifying the notation. The previous representation is a *QCD spectral sum rule*, which shows the *duality* between the LHS calculable theoretically in QCD, using the OPE, provided that $-q^2$ is much larger than Λ^2, with the RHS, where the *spectral function* $\mathrm{Im}\Pi_H(t)$ can be measured experimentally. In the case of the electromagnetic current:

$$J^\mu_{\mathrm{em}}(x) = \frac{2}{3}\bar u(x)\gamma^\mu u(x) - \frac{1}{3}\bar d(x)\gamma^\mu d(x) - \frac{1}{3}\bar s(x)\gamma^\mu s(x) + \cdots \tag{48.4}$$

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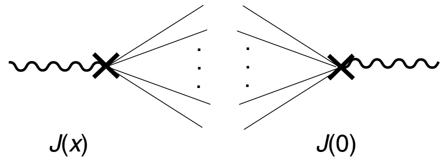

Fig. 48.1. Hadronic spectral function of Eq. (48.6).

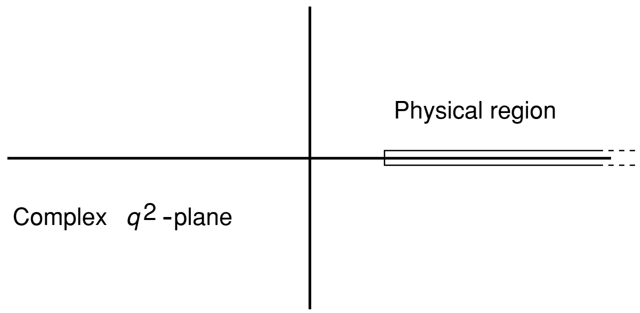

Fig. 48.2. The complex q^2-plane.

the spectral function is related to the $e^+e^- \to$ hadrons total cross-section $\sigma_H(t)$ through the optical theorem:

$$\sigma_H(t) = \frac{4\pi^2\alpha}{t}\frac{1}{\pi}\mathrm{Im}\Pi_{\mathrm{em}}(t) , \tag{48.5}$$

with:

$$-3\theta(q)\frac{t}{\pi}\mathrm{Im}\Pi_{\mathrm{em}}(t) = \sum_\Gamma \langle 0|J^\mu_{\mathrm{em}}(0)|\Gamma\rangle\langle\Gamma|J^\mu_{\mathrm{em}}(0)^\dagger|0\rangle(2\pi)^3\delta^{(4)}(q - p_\Gamma) , \tag{48.6}$$

where the sum runs over all possible physical states, and the integration over the corresponding phase space is understood. This is represented in Fig. 48.1.

In this case, the lowest possible state is the two pions. Therefore, the function $\Pi(q^2)$ is analytic in the complex q^2-plane but for a cut near the real axis $4m_\pi^2 \le q^2 \le \infty$ shown in Fig. 48.2.

48.2 Explicit derivation of the dispersion relation

In so doing, we consider the lowest order two-point function:

$$\begin{aligned}\Pi^{\mu\nu}(q^2) &\equiv i\int d^4x e^{iqx}\langle 0|\mathbf{T}J^\mu_V(x)\left(J^\nu_V\right)^\dagger(0)|0\rangle \\ &= -(g^{\mu\nu}q^2 - q^\mu q^\nu)\Pi(q^2) ,\end{aligned} \tag{48.7}$$

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shown in Fig. 8.31 but built from the electromagnetic current:

$$J_V^\mu = e\bar{\psi}\gamma^\mu\psi \;, \tag{48.8}$$

where ψ is a massive quark field with mass m. We follow the same procedure as for the pseudoscalar current for evaluating this lowest-order diagram. It is easy to show that the renormalized two-point function subtracted at $q^2 = 0$ if we choose on-shell renormalization reads:

$$\Pi_{\text{R}}(q^2) \equiv \Pi(q^2) - \Pi(0) = -\frac{\alpha}{\pi}\int_0^1 dx\, 2x(1-x)\log\left(1 - \frac{q^2}{m^2 - i\epsilon}x(1-x)\right). \tag{48.9}$$

With the change of variables $y = 1 - 2x$, and using the fact that the resulting integral is symmetric when $y \to -y$, we get:

$$\Pi_{\text{R}}(q^2) = \frac{\alpha}{\pi}\int_0^1 dy(1-y^2)\log\left[1 - \frac{q^2}{4m^2 - i\epsilon}(1-y^2)\right]. \tag{48.10}$$

Integrating by parts this equation using the identity: $1 - y^2 = \frac{\partial}{\partial y}(y - \frac{1}{3}y^3)$, one obtains the integral:

$$\Pi_{\text{R}}(q^2) = \frac{\alpha}{\pi}\int_0^1 dy\, 2y\left(y - \frac{1}{3}y^3\right)\frac{q^2}{4m^2 - q^2(1-y^2) - i\epsilon} \;. \tag{48.11}$$

With a new change of variables: $t = 4m^2/(1-y^2)$, we finally obtains the representation of the renormalized two-point function:

$$\frac{\Pi_{\text{R}}(q^2)}{q^2} = \frac{\alpha}{\pi}\int_{4m^2}^\infty \frac{dt}{t}\frac{1}{t - q^2 - i\epsilon}\frac{1}{3}\left(1 + \frac{2m^2}{t}\right)\sqrt{1 - \frac{4m^2}{t}} \;. \tag{48.12}$$

The reason why this representation is interesting is that it is in fact *a dispersion relation*. We have succeeded in rewriting the initial Feynman parametric representation in Eq. (48.9) as a dispersion relation by simple changes of variables. Using the identity:

$$\frac{1}{t - q^2 - i\epsilon} = \text{PP}\frac{1}{t - q^2} + i\pi\delta(t - q^2) \;, \tag{48.13}$$

we immediately see that:

$$\frac{1}{\pi}\text{Im}\Pi(t) = \frac{\alpha}{\pi}\frac{1}{3}\left(1 + \frac{2m^2}{t}\right)\sqrt{1 - \frac{4m^2}{t}}\theta(t - 4m^2) \;. \tag{48.14}$$

Equation (48.12) is a particular case of the general dispersion relation written in Eq. (48.2), when the arbitrary polynomial is just a constant, and we have got rid of the constant because the on-shell renormalized Π_{R} is defined as:

$$\Pi_{\text{R}}(q^2) = \Pi(q^2) - \Pi(0) = \int_0^\infty \frac{dt}{t}\frac{q^2}{t - q^2 - i\epsilon}\frac{1}{\pi}\text{Im}\Pi(t) \;. \tag{48.15}$$

It is perhaps worth insisting on the fact that asymptotically:

$$\lim_{t\to\infty} \frac{1}{\pi}\mathrm{Im}\Pi(t) \Longrightarrow \frac{\alpha}{\pi}\frac{1}{3} ; \tag{48.16}$$

i.e. the electromagnetic spectral function goes to a constant. In fact, it is this constant which fixes the value of the lowest-order contribution to the β–function associated with the charge renormalization in QED.

48.3 General proof of the dispersion relation

We shall now sketch a proof of the dispersion relation property for two-point functions in full generality following [361]. The key of the proof lies in the definition of the time-ordered product implicit in Eq. (48.1):

$$\mathcal{T}(J_H(x)J_H(0)^\dagger = \theta(x)J_H(x)J_H(0)^\dagger + \theta(-x)J_H(0)^\dagger J_H(x) , \tag{48.17}$$

and the use of translation invariance. The function $\theta(x)$ is the Heaviside function: $\theta(x) = 1$ if $x_0 > 0$ and $\theta(x) = 0$ if $x_0 < 0$, which has the integral representation:

$$\theta(x) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} dw \, \frac{e^{iwx}}{w - i\epsilon} . \tag{48.18}$$

First we insert a complete set of states $\sum_\Gamma |\Gamma\rangle\langle\Gamma|$ between the two currents in the T–product definition. This leads to matrix elements of the type $\langle 0|J_H(x)|\Gamma\rangle$ to which we apply translation invariance:

$$\langle 0|J(x)|\Gamma\rangle = \langle 0|\mathcal{U}^{-1}\mathcal{U}J_H(x)\mathcal{U}^{-1}\mathcal{U}|\Gamma\rangle , \tag{48.19}$$

where $\mathcal{U}$ is the unitary operator induced by translations in space–time:

$$\mathcal{U}(a)J_H(x)\mathcal{U}^{-1}(a) = J_H(x+a) \quad \text{and} \quad \mathcal{U}(a)|\Gamma\rangle = e^{ip_\Gamma\cdot a}|\Gamma\rangle , \tag{48.20}$$

and p_Γ denotes the sum of the energy–momenta of all the particles which define the state $|\Gamma\rangle$. The choice $a \equiv -x$ factors out the x-dependence of the matrix element into an exponential:

$$\langle 0|J_H(x)|\Gamma\rangle = e^{-ip_\Gamma\cdot x}\,\langle 0|J_H(0)|\Gamma\rangle . \tag{48.21}$$

All the particles in the state $|\Gamma\rangle$ are on-shell. This constrains the total energy–momentum p_Γ to be a time–like vector: $p_\Gamma^2 = t$ with $t \geq 0$. With these constraints on p_Γ we can insert the identity:

$$\int_0^\infty dt \int d^4p\,\theta(p)\,\delta(p^2 - t)\,\delta^{(4)}(p - p_\Gamma) = 1 , \tag{48.22}$$

inside the sum $\sum_\Gamma$ over the complete set of states. Interchanging the order of sum over Γ and integration over t and p, there appears naturally the definition of the spectral function

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associated with the J-operator

$$\sum_{\Gamma} \langle 0|J_H(0)|\Gamma\rangle\langle\Gamma|J_H(0)^{\dagger}|0\rangle\,(2\pi)^4\delta^{(4)}(p-p_{\Gamma}) \equiv 2\pi\rho(p^2)\,. \tag{48.23}$$

The spectral function $\rho(p^2)$ is a scalar function of the Lorentz invariant p^2 and the masses of the particles in the states $|\Gamma\rangle$ only. By construction it is a real function and non-negative:

$$\rho(p^2)^* = \rho(p^2) \geq 0\,. \tag{48.24}$$

We can now rewrite the two-point function in Eq. (48.1) as follows:

$$\begin{aligned}\Pi_H(q^2) = &\int d^4x\, e^{iq\cdot x}\int_0^{\infty} dt\,\rho(t)\\ &\times\int \frac{d^4p}{(2\pi)^3}[i\theta(x)\,e^{-ip\cdot x}\theta(p)\delta(p^2-t) + i\theta(-x)\,e^{ip\cdot x}\theta(p)\delta(p^2-t)].\end{aligned} \tag{48.25}$$

Here, one can recognize the familiar functions of free field theory:

$$\Delta^{+}(x) = \int \frac{d^4p}{(2\pi)^3}\, e^{-ip\cdot x}\theta(p)\delta(p^2-t) \tag{48.26}$$

and:

$$\begin{aligned}\Delta^{-}(x) &= \int \frac{d^4p}{(2\pi)^3}\, e^{ip\cdot x}\theta(p)\delta(p^2-t)\\ &= \int \frac{d^4p}{(2\pi)^3}\, e^{-ip\cdot x}\theta(-p)\delta(p^2-t)\,;\end{aligned} \tag{48.27}$$

and therefore the Feynman propagator function:

$$\begin{aligned}\Delta_{\mathrm{F}}(x;t) &= i\theta(x)\Delta^{+}(x;t) + i\theta(-x)\Delta^{-}(x;t)\\ &= \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot x}}{t-p^2-i\epsilon}\,,\end{aligned} \tag{48.28}$$

where the last expression can be obtained using the representation in Eq. (48.18) of the θ–function (see e.g. ref. [625]). The two-point function $\Pi(q^2)$ appears then to be the Fourier transform of a scalar free-field propagating with an arbitrary mass squared t weighted by the spectral function density $\rho(t)$ and integrated over all possible values of t:

$$\Pi_H(q^2) = \int d^4x\, e^{iq\cdot x}\int_0^{\infty} dt\,\rho(t)\,\Delta_{\mathrm{F}}(x;t)\,. \tag{48.29}$$

Integrating over x and p results finally in the wanted representation:

$$\Pi_H(q^2) = \int_0^{\infty} dt\,\rho(t)\frac{1}{t-q^2-i\epsilon}\,. \tag{48.30}$$

With:

$$\Pi_H(q^2) = \mathrm{Re}\Pi_H(q^2) + i\mathrm{Im}\Pi_H(q^2)\,, \tag{48.31}$$

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and the use of the identity in Eq. (48.13), it follows that:

$$\rho(t) \equiv \frac{1}{\pi}\text{Im}\Pi_H(t)\ , \tag{48.32}$$

which identifies the spectral function with the imaginary part of the two-point function.

Notice that the formal manipulations above avoid the question of convergence of the principal value integral:

$$\text{Re}\Pi_H(q^2) = \text{PP}\int_0^\infty dt\ \frac{1}{t-q^2}\ \frac{1}{\pi}\text{Im}\Pi_H(t)\ . \tag{48.33}$$

The convergence of the integral in the UV limit ($t \to \infty$) depends on the behaviour of the spectral function at large t-values. When doing above the exchange of sum over Γ and integrations we have implicitly assumed good convergence properties; but in general the product of the distributions $\theta(x)$ and $\int_0^\infty dt\ \rho(t)\ \Delta^+(x;t)$ may not be a well-defined distribution. The ambiguity manifests by the presence of an arbitrary polynomial in q^2 in the RHS of the PP-integral:

$$\text{Re}\Pi(q^2) = \text{PP}\int_0^\infty dt\ \frac{1}{t-q^2}\ \frac{1}{\pi}\text{Im}\Pi_H(t) + P(q^2)\ . \tag{48.34}$$

Notice that the coefficients of the arbitrary polynomial $P(q^2)$ have no discontinuities; in other words, the ambiguity of short-distance behaviour reflects only in the evaluation of the real part of the two-point function, not in the imaginary part. The physical meaning of these coefficients depends of course on the choice of the local operator $J_H(x)$ in the two-point function. In some cases the coefficients in question are fixed by low-energy theorems; e.g. if $\Pi(0)$ is known, we can trade the constant a in Eq. (48.34) for $\Pi(0)$:

$$\text{Re}\Pi_H(q^2) = \text{Re}\Pi_H(0) + \text{PP}\int_0^\infty \frac{dt}{t}\ \frac{q^2}{t-q^2}\ \frac{1}{\pi}\text{Im}\Pi_H(t) + bq^2 + \cdots\ . \tag{48.35}$$

while the constant b is related to its slope $\Pi'_H(0)$. In other cases the constants can be absorbed by renormalization constants. In general, it is always possible to get rid of the polynomial terms by taking an appropriate number of derivatives with respect to q^2. Various examples will be in the next chapter.

48.4 The QCD side of the sum rules

Using the SVZ expansion, one can express the two-point correlator in terms of the QCD condensates, where for large Euclidian q^2, one obtains:

$$\Pi_H = \sum_{D=0,2,4,\ldots} \frac{1}{(-s)^{D/2}} \sum_{\dim\ O=D} C^{(J)}(s,\mu)\ \langle O(\mu)\rangle\ , \tag{48.36}$$

where μ is an arbitrary scale that separates the long- and short-distance dynamics; $C^{(J)}$ are the Wilson coefficients calculable in perturbative QCD, while $\langle O\rangle$ are the non-perturbative quark and/or gluon condensates. The unit operator is the naïve perturbative contribution.

Table 48.1. *Values of perturbative QCD parameters used or obtained in the sum rules analysis (see chapter on α_s and on quark masses)*

Perturbative QCD parameters	Values	Sources
QCD coupling		
$\alpha_s(M_Z)$	0.118 ± 0.002	[139,16]
Quark running masses to $\mathcal{O}\left(\alpha_s^2\right)$		see Section 11.11
$\bar{m}_d(2\text{ GeV})$	(3.6 ± 0.6) MeV	average from different channels
$\bar{m}_d(2\text{ GeV})$	(6.5 ± 1.2) MeV	”
$\bar{m}_s(2\text{ GeV})$	(117.4 ± 23.4) MeV	”
$\bar{m}_c(m_c)$	(1.23 ± 0.05) GeV	average from the J/ψ and D, D^*
$\bar{m}_b(m_b)$	(4.24 ± 0.06) GeV	average from the Υ and B, B^*
Perturbative pole masses $\mathcal{O}\left(\alpha_s^2\right)$		see Section 11.12
M_c	(1.43 ± 0.04) GeV	average from the J/ψ and D, D^*
M_b	(4.66 ± 0.06) GeV	average from the Υ and B, B^*

One expects that, for enough large q^2 (usually of the order of 1–2 GeV^2), the first two-three lowest dimension condensates can give a good approximation of the QCD correlator. In practice, one usually truncates the series until the dimension-six condensates, which are already small corrections in the analysis. The well-known condensate is the quark $\langle\bar{\psi}\psi\rangle$ condensate responsible for the spontaneous breaking of chiral symmetry and is related to the pion and decay amplitude squared through the GMOR relation:

$$(m_u + m_d)\langle\bar{u}u + \bar{d}d\rangle = -2m_\pi^2 f_\pi^2 \qquad (48.37)$$

with $f_\pi = 93.2$ MeV. The other condensates are less known, and are not calculable from QCD first principles though one can determine them from phenomenological analysis. We summarize in Tables 48.1 and 48.2 the values of these QCD parameters which will be useful for the discussion in this part (Part X) of the book.

We have already anticipated the discussions of the theoretical input of the sum rules analysis in the previous chapters:

- In Part III, we discussed the different ingredients for treating and evaluating, within the $\overline{MS}$ scheme and using the renormalization group equation, the perturbative contributions to the unit operator. We have also given there and in Part VI the value of the running QCD coupling and the light and heavy quark masses in Sections 11.7, 11.11 and 11.12, entering the QCD Lagrangian and useful in the sum rules analysis.
- In Part VII, we have discussed the different non-perturbative contributions:
  - In Chapter 27, we have studied the operator product expansion (OPE) and classified the condensates versus their dimensions. We have also constructed renormalization group invariant condensates and given the values of some of the condensates which have been determined mainly from the sum rules.

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Table 48.2. *Values of the non-perturbative QCD (NPQCD) parameters used or obtained in the sum rules analysis*

Dimension	NPQCD parameters	Values	Sources
2	$(\alpha_s/\pi)\lambda^2$	$-(0.06\text{–}0.07)$ GeV^2	Chapter 30
3	$\frac{1}{2}\langle\bar{u}u+\bar{d}d\rangle(2\text{ GeV})$	$-[(254\pm15)\text{ MeV}]^3$	Chapter 27
	$\langle\bar{s}s\rangle/\langle\bar{u}u\rangle$	0.75 ± 0.12	non-normal ordered
		0.66 ± 0.10	normal ordered
4	$\langle\alpha_s G^2\rangle$	$(7\pm1)10^{-2}$ GeV^4	Chapter 27
5	$g\langle\bar{\psi}\sigma_{\mu\nu}\frac{\lambda^a}{2}\psi G_a^{\mu\nu}\rangle \equiv M_0^2\alpha_s^{1/3\beta_1}\langle\bar{\psi}\psi\rangle$	$M_0^2=(0.80\pm0.02)$ GeV^2	”
6	$g^3 f_{abc}\langle G^a G^b G^c\rangle$	$(1.2\text{ GeV}^2)\langle\alpha_s G^2\rangle$	”
	$\rho\alpha_s\langle\bar{\psi}\psi\rangle^2$	$(5.8\pm0.9)10^{-4}$ GeV^6	”

- In Chapter 28, we discuss in details the evaluation of the Wilson coefficients in the OPE. In so doing, we give as an explicit example the evaluation of the light quark pseudoscalar two-point function including dimension-six condensates. We also discuss the evaluation of the heavy quark correlators.
- In Part VIII, we give a compilation of different QCD two-point functions including radiative corrections to the unit operator and the contributions of different condensates.
- In Chapter 29, we discuss the modifications of the OPE due to IR and UV renormalons. IR renormalons introduce perturbative contributions, which lead to some ambiguities for defining the condensates at higher order of perturbation theories, though such ambiguities can be absorbed by the Wilson coefficients when computing the Green's functions. In practice, the IR renormalon effects are so tiny such that they do not affect in a significant way the phenomenology of the sum rules. UV renormalons have also been discussed so far, and affect the uncertainties of the PT series. Again, within the sum rules uncertainties, these effects are not quantifiable in the sum rules analysis.
- In Chapter 30, we have discussed the different scenarios beyond the SVZ expansion. In the following, we shall only discuss the modification due to the tachyonic gluon mass which modifies the OPE due to the presence of the new $D=2$ 'condensate', not present in the original SVZ-expansion owing to the fact that one cannot form a $D=2$ local gauge-invariant operator in QCD. We shall not discuss the effects of (direct) instantons which act like high-dimension operators and should be suppressed like other high-dimension condensates in the sum rule working region. However, some other schools expect that this contribution is dominant for the (pseudo)scalar channels but surprisingly are not there if one works with the longitudinal part of the axial-vector correlator, though the two are related to each others by Ward identity. However, the inclusion of the large instanton effects leads to inconsistencies in the scalar channel. Another confusion for the sum rule practitioners is the fact that the instanton liquid model does not use a novel OPE but provides an alternative way of parametrizing the condensates. However, the fact that the analysis is done in the coordinate rather than in the momentum space may probe a new region not explored in the momentum space. Interested readers may consult [386] where this method is explored in detail.

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