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ESTIMATES FOR MARCINKIEWICZ INTEGRALS IN BMO AND CAMPANATO SPACES

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Abstract. In this paper, the authors consider the behavior on $\mathrm{BMO}(\mathbb{R}^n)$ and Campanato spaces for the higher-dimensional Marcinkiewicz integral operator which is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\, dy \right|^2 \frac{dt}{t^3}\right)^{1/2},$$

where Ω is homogeneous of degree zero, has mean value zero and is integrable on the unit sphere. Under certain weak regularity condition on Ω, the authors prove that if f belongs to $\mathrm{BMO}(\mathbb{R}^n)$ or to a certain Campanato space, then $[\mu_\Omega(f)]^2$ is either infinite everywhere or finite almost everywhere, and in the latter case, some kind of boundedness is also obtained. The corresponding Lusin area integral is also considered.

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1. Introduction. As an analogue of the classical Littlewood-Paley g-function on $\mathbb{R}$, Marcinkiewicz [**10**] introduced the following integral which is now called the Marcinkiewicz integral and is defined by

$$\mu(f)(x) = \left(\int_0^\pi \frac{|F(x+t)+F(x-t)-2F(x)|^2}{t^3}\, dt\right)^{1/2},$$

where $x \in [0, 2\pi]$, f is an integrable function of period 2π and $F(x) = \int_0^x f(t)\, dt$. Marcinkiewicz in [**10**] conjectured that μ is bounded on $L^p([0, 2\pi])$ for any $p \in (1, \infty)$, which was proved by Zygmund in [**18**]. Stein [**17**] generalized the above Marcinkiewicz integral to a higher dimensional case. Let Ω be homogeneous of degree zero in $\mathbb{R}^n$ for

$n \geq 2$, integrable and have mean value zero on the unit sphere S^{n-1}, namely,

$$\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0.$$

The higher dimensional Marcinkiewicz integral μ_Ω is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty \left|\int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\, dy\right|^2 \frac{dt}{t^3}\right)^{1/2} \tag{1.1}$$

for $x \in \mathbb{R}^n$. Stein [**17**] proved that if $\Omega \in \mathrm{Lip}_\beta(S^{n-1})$ for some $\beta \in (0, 1]$, then μ_Ω is bounded on $L^p(\mathbb{R}^n)$ for $p \in (1, 2]$ and bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. Recently, Al-Salman, Al-Qassem, Cheng and Pan in [**1**] proved that $\Omega \in L(\log L)^{1/2}(S^{n-1})$ is a sufficient condition such that μ_Ω is bounded on $L^p(\mathbb{R}^n)$ for $p \in (1, \infty)$. Fan and Sato [**11**] proved that $\Omega \in L\log L(S^{n-1})$ is a sufficient condition such that μ_Ω is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. On the other hand, there are many works concerning the behavior on $\mathrm{BMO}(\mathbb{R}^n)$ and on Campanato spaces for μ_Ω. Han [**7**] proved that if $\Omega \in \mathrm{Lip}_\beta(S^{n-1})$ for some $\beta \in (0, 1]$, then for $f \in \mathrm{BMO}(\mathbb{R}^n)$, $\mu_\Omega f$ is either infinite almost everywhere or finite almost everywhere, and in the latter case,

$$\|\mu_\Omega(f)\|_{\mathrm{BMO}(\mathbb{R}^n)} \leq C\|f\|_{\mathrm{BMO}(\mathbb{R}^n)},$$

where $C > 0$ is independent of f. Qiu in [**14**] considered the boundedness in Campanato spaces for μ_Ω, and proved that if $\Omega \in \mathrm{Lip}_\beta(S^{n-1})$ for some $\beta \in (0, 1]$, then for $p \in (1, \infty)$, $\alpha \in [-n/p, \min\{1/2, \beta\})$ and $f \in \mathcal{E}^{\alpha,p}(\mathbb{R}^n)$, $\mu_\Omega f$ is either infinite almost everywhere or finite almost everywhere, and in the latter case,

$$\|\mu_\Omega f\|_{\mathcal{E}^{\alpha,p}(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{E}^{\alpha,p}(\mathbb{R}^n)},$$

where $\mathcal{E}^{\alpha,p}(\mathbb{R}^n)$ is the Campanato space and $C > 0$ is independent of f; see Definition 1.3 below. Ding, Lu and Xue in [**6**] improved the results of Han and Qiu, and proved that the above results of Han and Qiu hold when Ω satisfies some Dini-type condition.

In [**16**], Torchinsky and Wang introduced the Marcinkiewicz integral $\mu_{\Omega,S}$ corresponding to the Lusin area function, which is defined by

$$\mu_{\Omega,S}(f)(x) = \left(\int\int_{\Gamma(x)} \left|\int_{|y-z|\leq t} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)\, dz\right|^2 \frac{dy\, dt}{t^{n+3}}\right)^{1/2}, \tag{1.2}$$

where $x \in \mathbb{R}^n$ and $\Gamma(x) = \{(y, t) \in \mathbb{R}^{n+1}_+ : |y-x| < t\}$. Torchinsky and Wang [**16**] proved that for $p \in [2, \infty)$,

$$\|\mu_{\Omega,S}(f)\|_p \leq C\|f\|_p,$$

where $C > 0$ is independent of f. Here and in what follows, for any $p \in (1, \infty)$ and $f \in L^p(\mathbb{R}^n)$, we use $\|f\|_p$ to denote the $L^p(\mathbb{R}^n)$ norm of f. Sakamoto and Yabuta [**15**] considered the behavior on the Campanato spaces $\mathcal{E}^{\alpha,p}(\mathbb{R}^n)$, and proved that if $\Omega \in \mathrm{Lip}_\beta(S^{n-1})$ with $\beta \in (0, 1]$, and $f \in \mathcal{E}^{\alpha,p}(\mathbb{R}^n)$ with $p \in (1, \infty)$ and $\alpha \in [-n/p, \beta)$, then $\mu_{S,\Omega}$ is either infinite almost everywhere or finite almost everywhere, and in the latter case,

$$\|\mu_{\Omega,S}(f)\|_{\mathcal{E}^{\alpha,p}(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{E}^{\alpha,p}(\mathbb{R}^n)}$$

with $C > 0$ independent of f.

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The purpose of this paper is to improve the results about the behavior on BMO($\mathbb{R}^n$) space and on Campanato spaces $\mathcal{E}^{\alpha,p}(\mathbb{R}^n)$ for μ_Ω and $\mu_{\Omega,S}$ when Ω satisfies certain Dini-type regularity condition. To be precise, motivated by [**12**], under the hypothesis that Ω satisfies certain weak Dini-type regularity condition, we will prove that if $f \in \mathrm{BMO}(\mathbb{R}^n)$ or $f \in \mathcal{E}^{\alpha,p}(\mathbb{R}^n)$ for suitable indexes α and p, $[\mu_\Omega(f)]^2$ ($[\mu_{\Omega,S}(f)]^2$, respectively) is either infinite everywhere or finite almost everywhere, and in the latter case, some kind of boundedness for $[\mu_\Omega(f)]^2$ ($[\mu_{\Omega,S}(f)]^2$, respectively) is also presented. Our results are new even for the case $\Omega \in \mathrm{Lip}_\beta(S^{n-1})$ for some $\beta \in (0, 1]$.

To state the main results of this paper, we first recall some necessary definitions and notation.

Definition 1.1. ([**9**]) A locally integrable function f is said to belong to BMO($\mathbb{R}^n$) if there exists some constant $C_1 > 0$ such that for any ball $B \subset \mathbb{R}^n$,

$$\frac{1}{|B|}\int_B |f(x) - m_B(f)|\,dx \le C_1,$$

where $m_B(f)$ denotes the mean value of f over B, that is, $m_B(f) = \frac{1}{|B|}\int_B f(x)\,dx$. The minimal constant C_1 is defined to be the BMO($\mathbb{R}^n$) norm of f and denoted by $\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}$.

Definition 1.2. A locally integrable function f is said to belong to BLO($\mathbb{R}^n$) if there exists a constant $C_2 > 0$ such that for any ball $B \subset \mathbb{R}^n$,

$$\frac{1}{|B|}\int_B \left[f(x) - \inf_{y\in B} f(y)\right] dx \le C_2.$$

The minimal constant C_2 is defined to be the BLO($\mathbb{R}^n$) norm of f and denoted by $\|f\|_{\mathrm{BLO}(\mathbb{R}^n)}$.

The space BLO($\mathbb{R}^n$) was first introduced by Coifman and Rochberg in [**3**]. It should be pointed out that BLO($\mathbb{R}^n$) is not a linear space and $\|\cdot\|_{\mathrm{BLO}(\mathbb{R}^n)}$ is not a norm. However, it is easy to see

$$L^\infty(\mathbb{R}^n) \subset \mathrm{BLO}(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n).$$

Definition 1.3. Let $\alpha \in (-\infty, 1]$ and $p \in (0, \infty)$. A locally integrable function f is said to belong to the *Campanato space* $\mathcal{E}^{\alpha,p}(\mathbb{R}^n)$ if there exists some constant $C_3 > 0$ such that for any ball $B \subset \mathbb{R}^n$,

$$\frac{1}{|B|^{\alpha/n}}\left(\frac{1}{|B|}\int_B |f(x) - m_Q(f)|^p\,dx\right)^{1/p} \le C_3.$$

The minimal constant C_3 is defined to be the $\mathcal{E}^{\alpha,p}(\mathbb{R}^n)$ norm of f and denoted by $\|f\|_{\mathcal{E}^{\alpha,p}(\mathbb{R}^n)}$.

The Campanato space $\mathcal{E}^{\alpha,p}(\mathbb{R}^n)$ was first introduced by Campanato in [**2**]. For $\alpha \in (0, 1]$, let $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ be the space of Lipschitz functions, that is,

$$\mathrm{Lip}_\alpha(\mathbb{R}^n) = \left\{f : \|f\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} = \sup_{x,\,y\in\mathbb{R}^n,\,x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty\right\}.$$

It is known that if $\alpha \in (0, 1]$ and $p \in [1, \infty)$, then

$$\mathcal{E}^{\alpha,p}(\mathbb{R}^n) = \mathrm{Lip}_\alpha(\mathbb{R}^n)$$

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with equivalent norms; if $\alpha = 0$, then $\mathcal{E}^{\alpha,p}(\mathbb{R}^n)$ coincides with $\mathrm{BMO}(\mathbb{R}^n)$; and if $\alpha \in (-n/p, 0)$, then $\mathcal{E}^{\alpha,p}(\mathbb{R}^n)$ coincides with the Morrey space $L^{p,\,n+p\alpha}(\mathbb{R}^n)$; see also [**2**].

Motivated by the definition of $\mathrm{BLO}(\mathbb{R}^n)$, we introduce the following space $\mathcal{E}_*^{\alpha,p}(\mathbb{R}^n)$, which is a subspace of $\mathcal{E}^{\alpha,p}(\mathbb{R}^n)$.

DEFINITION 1.4. Let $\alpha \in (-\infty, 1]$ and $p \in (0, \infty)$. A locally integrable function f is said to belong to $\mathcal{E}_*^{\alpha,p}(\mathbb{R}^n)$ if there exists some constant $C_4 > 0$ such that for any ball $B \subset \mathbb{R}^n$,

$$\frac{1}{|B|^{\alpha/n}}\left(\frac{1}{|B|}\int_B \left[f(x) - \inf_{y\in B} f(y)\right]^p dx\right)^{1/p} \le C_4.$$

The minimal constant C_4 is defined to be the $\mathcal{E}_*^{\alpha,p}(\mathbb{R}^n)$ norm of f and denoted by $\|f\|_{\mathcal{E}_*^{\alpha,p}(\mathbb{R}^n)}$.

We point out that $\mathcal{E}_*^{\alpha,p}(\mathbb{R}^n)$ is not a linear space and $\|\cdot\|_{\mathcal{E}_*^{\alpha,p}(\mathbb{R}^n)}$ is not a norm.

In what follows, C always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_1, do not change in different occurrences. For $f \sim g$, we mean that the ratio f/g is both bounded and bounded away from zero by constants independent of the relevant variables in f and g. The notation $f \lesssim g$ is defined in a similar way. For any index $p \in [1, \infty]$, we denote by p' its conjugate index, that is, $1/p + 1/p' = 1$. For each bounded measurable set E, denote by χ_E the characteristic function of E.

2. Estimates for μ_Ω. In this section, we will consider the behavior on $\mathrm{BMO}(\mathbb{R}^n)$ space and on Campanato spaces for μ_Ω defined by (1.1). Let Ω be homogeneous of degree zero and belong to the space $L^q(S^{n-1})$ for some $q \in [1, \infty)$. Denote by ω_q the $L^q(S^{n-1})$-modulus of continuity of Ω, that is, for any $\delta > 0$,

$$\omega_q(\delta) = \sup_{\{\rho:\ |\rho - I| \le \delta\}} \left[\int_{S^{n-1}} |\Omega(\rho x') - \Omega(x')|^q \, d\sigma(x')\right]^{1/q}, \tag{2.1}$$

where ρ is a rotation on S^{n-1}, I is the identity on S^{n-1} and $|\rho - I| = \sup_{x'\in S^{n-1}} |(\rho - I)x'|$. For the case $q = 1$, we denote $\omega_q(\delta)$ simply by $\omega(\delta)$. We will prove that if Ω satisfies certain Dini-type regularity, then for $f \in \mathrm{BMO}(\mathbb{R}^n)$ or $f \in \mathcal{E}^{\alpha,p}(\mathbb{R}^n)$ for some suitable indexes α and p, $[\mu_\Omega(f)]^2$ is either infinite everywhere or finite almost everywhere. Precisely, we have the following results.

THEOREM 2.1. *Let Ω be homogeneous of degree zero, integrable on S^{n-1} and have mean value zero. Suppose that $\Omega \in L(\log L)^\gamma(S^{n-1})$ for some $\gamma \in (2, \infty)$ and the $L^1(S^{n-1})$-modulus of continuity of Ω satisfies*

$$\int_0^1 \omega(\delta) \log\left(2 + \frac{1}{\delta}\right) \frac{d\delta}{\delta} < \infty. \tag{2.2}$$

Then for any $f \in \mathrm{BMO}(\mathbb{R}^n)$, $\mu_\Omega(f)$ is either infinite everywhere or finite almost everywhere, and in the latter case,

$$\left\|\left[\mu_\Omega(f)\right]^2\right\|_{\mathrm{BLO}(\mathbb{R}^n)} \le C\|f\|^2_{\mathrm{BMO}(\mathbb{R}^n)},$$

where $C > 0$ is independent of f.

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THEOREM 2.2. *Let Ω be homogeneous of degree zero and have mean value zero on S^{n-1}. Suppose that for some $p \in (1, \infty)$, Ω belongs to the space $L^{p'}(S^{n-1})$ and the $L^{p'}(S^{n-1})$-modulus of continuity satisfies*

$$\int_0^1 \frac{\omega_{p'}(\delta)}{\delta}\,d\delta < \infty. \tag{2.3}$$

If $\alpha \in (-\infty, 0)$ and $p \in (n, \infty)$, or $\alpha \in (-1, 0)$ and $p \in (1, \infty)$, then for any $f \in \mathcal{E}^{\alpha,p}(\mathbb{R}^n)$, μ_Ω is either infinite everywhere or finite almost everywhere, and in the latter case,

$$\left\|\left[\mu_\Omega(f)\right]^2\right\|_{\mathcal{E}_*^{2\alpha,p/2}(\mathbb{R}^n)} \le C\|f\|^2_{\mathcal{E}^{\alpha,p}(\mathbb{R}^n)},$$

where $C > 0$ is independent of f.

THEOREM 2.3. *Let Ω be homogeneous of degree zero, integrable on S^{n-1} and have mean value zero. Suppose that the $L^1(S^{n-1})$-modulus of continuity of Ω satisfies*

$$\int_0^1 \frac{\omega(\delta)}{\delta^{1+\epsilon}}\,d\delta < \infty \tag{2.4}$$

for some $\epsilon \in (0, 1]$, then for any $\alpha \in (0, \epsilon/2)$ and $f \in \mathcal{E}^{\alpha,p}(\mathbb{R}^n)$, $\mu_\Omega(f)$ is either infinite everywhere or finite almost everywhere, and in the latter case,

$$\left\|\left[\mu_\Omega(f)\right]^2\right\|_{\mathrm{Lip}_{2\alpha}(\mathbb{R}^n)} \le C\|f\|^2_{\mathrm{Lip}_\alpha(\mathbb{R}^n)},$$

where $C > 0$ is independent of f.

Even for the case that $\Omega \in \mathrm{Lip}_\beta(S^{n-1})$ with $\beta \in (0, 1]$, Theorem 2.1, Theorem 2.2 and Theorem 2.3 are new.

Observe that for any fixed ball B and $x \in B$, if $\inf_{y\in B} \mu_\Omega(f)(y) < \infty$, then

$$\mu_\Omega(f)(x) - \inf_{y\in B} \mu_\Omega(f)(y) \le \left\{\left[\mu_\Omega(f)(x)\right]^2 - \inf_{y\in B}\left[\mu_\Omega(f)(y)\right]^2\right\}^{1/2}.$$

From Theorem 2.1 and Theorem 2.2 we can easily deduce the following results.

COROLLARY 2.1. *Under the assumption of Theorem 2.1, we have that for any $f \in \mathrm{BMO}(\mathbb{R}^n)$, $\mu_\Omega(f)$ is either infinite everywhere or finite almost everywhere, and in the latter case,*

$$\|\mu_\Omega(f)\|_{\mathrm{BLO}(\mathbb{R}^n)} \le C\|f\|_{\mathrm{BMO}(\mathbb{R}^n)},$$

where $C > 0$ is independent of f.

COROLLARY 2.2. *Under the assumption of Theorem 2.2, we have that if $\alpha \in (-\infty, 0)$ and $p \in (n, \infty)$, or $\alpha \in (-1, 0)$ and $p \in (1, \infty)$, then for any $f \in \mathcal{E}^{\alpha,p}(\mathbb{R}^n)$, $\mu_\Omega(f)$ is either infinite everywhere or finite almost everywhere, and in the latter case,*

$$\|\mu_\Omega(f)\|_{\mathcal{E}_*^{\alpha,p}(\mathbb{R}^n)} \le C\|f\|_{\mathcal{E}^{\alpha,p}(\mathbb{R}^n)},$$

where $C > 0$ is independent of f.

We remark that it will be interesting to find some other applications of Theorem 2.1 through Theorem 2.3.

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To prove our theorems, we need some preliminary lemmas.

LEMMA 2.1. ([**8**]) *Let* $\alpha \in (0,\ 1)$ *and* $p \in (1,\ \infty]$. *Then*

$$\|f\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \sim \sup_{B\subset\mathbb{R}^n} |B|^{-\alpha/n} \left\{ \frac{1}{|B|} \int_B |f(y) - m_B(f)|^p\, dy \right\}^{1/p},$$

where for $p = \infty$, *the formula should be interpreted appropriately.*

LEMMA 2.2. ([**1**]) *Let* Ω *be homogeneous of degree zero, integrable on* S^{n-1} *and have mean value zero. Suppose that* $\Omega \in L(\log L)^{1/2}(S^{n-1})$. *Then* μ_Ω *is bounded on* $L^p(\mathbb{R}^n)$ *for any* $p \in (1,\ \infty)$.

LEMMA 2.3. *Let* Ω *be homogeneous of degree zero and belong to the space* $L^q(S^{n-1})$ *for certain* $q \in [1,\ \infty)$. *If there exists a constant* $0 < a_0 < 1/2$ *such that* $|x| < a_0 R$, *then*

$$\left[\int_{R\le|y|\le 2R} \left|\frac{\Omega(y-x)}{|y-x|^{n-1}} - \frac{\Omega(y)}{|y|^{n-1}}\right|^q dy\right]^{1/q} \le CR^{n/q-(n-1)} \left\{ \frac{|x|}{R} + \int_{|x|/2R}^{|x|/R} \frac{\omega_q(\delta)}{\delta}\, d\delta \right\},$$

where ω_q *is as in* (2.1).

Lemma 2.3 can be found in [**5**] for the case $q = 1$. For the case $q > 1$, by an argument similar to that used in [**5**], one can easily obtain the corresponding result.

Proof of Theorem 2.1. To prove Theorem 2.1, it suffices to verify that for any $f \in \mathrm{BMO}(\mathbb{R}^n)$, if there exists $y_0 \in \mathbb{R}^n$ such that $\mu_\Omega(f)(y_0) < \infty$, then for any ball $B \subset \mathbb{R}^n$ with $B \ni y_0$,

$$\frac{1}{|B|}\int_B \left([\mu_\Omega(f)(x)]^2 - \inf_{y\in B} [\mu_\Omega(f)(y)]^2 \right) dx \lesssim \|f\|^2_{\mathrm{BMO}(\mathbb{R}^n)}.$$

Without loss of generality, we may assume that $\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} = 1$, and for each fixed ball B as above, let r be its radius. Set

$$[\mu^r_\Omega(f)(x)]^2 = \int_0^{8r} \left| \int_{|x-z|\le t} \frac{\Omega(x-z)}{|x-z|^{n-1}} f(z)\, dz \right|^2 \frac{dt}{t^3} \tag{2.5}$$

and

$$[\mu^\infty_\Omega(f)(x)]^2 = \int_{8r}^{\infty} \left| \int_{|x-z|\le t} \frac{\Omega(x-z)}{|x-z|^{n-1}} f(z)\, dz \right|^2 \frac{dt}{t^3}. \tag{2.6}$$

By the vanishing moment of Ω, we see that for any $y \in B$,

$$[\mu^r_\Omega(f)(y)]^2 = \{\mu^r_\Omega([f - m_B(f)]\chi_{10B})(y)\}^2 \le \{\mu_\Omega([f - m_B(f)]\chi_{10B})(y)\}^2,$$

which via Lemma 2.2 gives us that

$$\int_B [\mu^r_\Omega(f)(x)]^2\, dx \lesssim \int_{10B} |f(x) - m_B(f)|^2\, dx \lesssim |B|.$$

Observe that for any $y \in \mathbb{R}^n$, $\mu^\infty_\Omega(f)(y) \le \mu_\Omega(f)(y)$ and that for any $x \in B$,

$$[\mu^\infty_\Omega(f)(x)]^2 - \inf_{y\in B} [\mu^\infty_\Omega(f)(y)]^2 \le \sup_{y\in B} \left| [\mu^\infty_\Omega(f)(x)]^2 - [\mu^\infty_\Omega(f)(y)]^2 \right|.$$

Thus, to finish the proof of this theorem, it suffices to prove that for any $x,\ y \in B$,

$$\left|\left[\mu_\Omega^\infty(f)(x)\right]^2 - \left[\mu_\Omega^\infty(f)(y)\right]^2\right| \lesssim 1. \tag{2.7}$$

For each $y \in \mathbb{R}^n$ and $t > 0$, set

$$E_f(y,\ t) = \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)\, dz. \tag{2.8}$$

It is easy to see that

$$\begin{aligned}\left|\left[\mu_\Omega^\infty(f)(x)\right]^2 - \left[\mu_\Omega^\infty(f)(y)\right]^2\right| &\le \left|\int_{8r}^\infty |E_f(x,\ t)|^2 \frac{dt}{t^3} - \int_{8r}^\infty |E_f(y,\ t)|^2 \frac{dt}{t^3}\right| \\ &\le \int_{8r}^\infty \left[|E_f(x,\ t)| + |E_f(y,\ t)|\right] |E_f(x,\ t) - E_f(y,\ t)| \frac{dt}{t^3}.\end{aligned} \tag{2.9}$$

A known inequality says that for $s,\ t > 0$ and $\gamma \ge 1$,

$$st^\gamma \le s \log^\gamma(2+s) + e^t \tag{2.10}$$

(see Lemma 2.2 in **[13]**), which together with the vanishing moment of Ω further tells us that for $y \in \mathbb{R}^n$ and $t > 0$,

$$\begin{aligned}|E_f(y,\ t)| &= \left|\sum_{k=-\infty}^{0} \int_{B(y,\ 2^k t)\setminus B(y,\ 2^{k-1}t)} \frac{\Omega(y-z)}{|y-z|^{n-1}} \left[f(z) - m_{B(y,\ 2^k t)}(f)\right] dz\right| \\ &\le \sum_{k=-\infty}^{0} \int_{B(y,\ 2^k t)\setminus B(y,\ 2^{k-1}t)} \frac{|\Omega(y-z)|}{|y-z|^{n-1}} \left|f(z) - m_{B(y,\ 2^k t)}(f)\right| dz \\ &\lesssim \sum_{k=-\infty}^{0} (2^k t)^{1-n} \int_{B(y,\ 2^k t)\setminus B(y,\ 2^{k-1}t)} |\Omega(y-z)| \log(2 + |\Omega(y-z)|)\, dz \\ &\quad + \sum_{k=-\infty}^{0} (2^k t)^{1-n} \int_{B(y,\ 2^k t)} \exp\left(\frac{|f(z) - m_{B(y,\ 2^k t)}(f)|}{C_5}\right) dz \\ &\lesssim t,\end{aligned} \tag{2.11}$$

where in the last inequality, we have invoked the John-Nirenberg inequality, which states that there are positive constants C_5 and C_6 such that for all $b \in \mathrm{BMO}(\mathbb{R}^n)$,

$$\sup_B \frac{1}{|B|} \int_B \exp\left(\frac{|b(z) - m_B(b)|}{C_5 \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\right) dz \le C_6.$$

For each fixed $x \in B$ and $t \ge 8r$, set

$$H_f(x,\ y,\ t) = \left|E_f(x,\ t) - E_f(x,\ 8r) - \left[E_f(y,\ t) - E_f(y,\ 8r)\right]\right|.$$

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It follows from (2.9) and (2.11) that for any $x,\ y \in B$,

$$
\begin{aligned}
&\left|\left[\mu_\Omega^\infty(f)(x)\right]^2-\left[\mu_\Omega^\infty(f)(y)\right]^2\right|\\
&\quad\lesssim \int_{8r}^\infty \left|E_f(x,\ t)-E_f(y,\ t)\right|\frac{dt}{t^2}\\
&\quad\lesssim \int_{8r}^\infty \left|E_f(x,\ 8r)\right|\frac{dt}{t^2}+\int_{8r}^\infty \left|E_f(y,\ 8r)\right|\frac{dt}{t^2}+\int_{8r}^\infty H_f(x,\ y,\ t)\frac{dt}{t^2}\\
&\quad\lesssim 1+\int_{8r}^\infty H_f(x,\ y,\ t)\frac{dt}{t^2}.
\end{aligned}
$$

Therefore, the proof of inequality (2.7) can be reduced to proving that for any $x,\ y \in B$,

$$
\int_{8r}^\infty H_f(x,\ y,\ t)\frac{dt}{t^2}\lesssim 1. \tag{2.12}
$$

We now prove (2.12). Write

$$
\begin{aligned}
H_f(x,\ y,\ t) &\le \int_{\substack{8r<|x-z|\le t\\ 8r<|y-z|\le t}}\left|\frac{\Omega(x-z)}{|x-z|^{n-1}}-\frac{\Omega(y-z)}{|y-z|^{n-1}}\right||f(z)-m_B(f)|\,dz\\
&\quad+\int_{8r<|x-z|\le t,\ |y-z|>t}\frac{|\Omega(x-z)|}{|x-z|^{n-1}}|f(z)-m_B(f)|\,dz\\
&\quad+\int_{8r<|y-z|\le t,\ |x-z|>t}\frac{|\Omega(y-z)|}{|y-z|^{n-1}}|f(z)-m_B(f)|\,dz\\
&\quad+\int_{8r<|x-z|\le t,\ |y-z|\le 8r}\frac{|\Omega(x-z)|}{|x-z|^{n-1}}|f(z)-m_B(f)|\,dz\\
&\quad+\int_{8r<|y-z|\le t,\ |x-z|\le 8r}\frac{|\Omega(y-z)|}{|y-z|^{n-1}}|f(z)-m_B(f)|\,dz\\
&= H_{f,\,1}(x,\ y,\ t)+H_{f,\,2}(x,\ y,\ t)+H_{f,\,3}(x,\ y,\ t)\\
&\quad+H_{f,\,4}(x,\ y,\ t)+H_{f,\,5}(x,\ y,\ t).
\end{aligned}
$$

Again by (2.10) and the John-Nirenberg inequality, we obtain that for any $x,\ y \in B$,

$$
\begin{aligned}
\int_{8r}^\infty H_{f,\,2}(x,\ y,\ t)\frac{dt}{t^2} &\lesssim \int_{\mathbb{R}^n\setminus B(x,\,4r)}\frac{|\Omega(x-z)|}{|x-z|^{n-1}}|f(z)-m_B(f)|\left(\int_{|x-z|}^{|y-z|}\frac{dt}{t^2}\right)dz\\
&\lesssim r\int_{\mathbb{R}^n\setminus B(x,\,4r)}\frac{|\Omega(x-z)|}{|x-z|^{n+1}}|f(z)-m_B(f)|\,dz\\
&\lesssim r\sum_{k=2}^\infty \int_{B(x,\,2^{k+1}r)\setminus B(x,\,2^k r)}\frac{|\Omega(x-z)|}{|x-z|^{n+1}}\left|f(z)-m_{B(x,\,2^{k+1}r)}(f)\right|dz\\
&\quad+r\sum_{k=2}^\infty \left|m_{B(x,\,2^{k+1}r)}(f)-m_B(f)\right|\int_{B(x,\,2^{k+1}r)\setminus B(x,\,2^k r)}\frac{|\Omega(x-z)|}{|x-z|^{n+1}}\,dz\\
&\lesssim 1,
\end{aligned}
$$

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and similarly,

$$\int_{8r}^{\infty} H_{f,3}(x, y, t)\frac{dt}{t^2} \lesssim 1.$$

A trivial computation tells us that for $x, y \in B$,

$$H_{f,4}(x, y, t) \le \int_{8r<|x-z|\le 10r} \frac{|\Omega(x-z)|}{|x-z|^{n-1}}|f(z)-m_B(f)|\,dz \lesssim r,$$

which leads to

$$\int_{8r}^{\infty} H_{f,4}(x, y, t)\frac{dt}{t^2} \lesssim 1,$$

and similarly,

$$\int_{8r}^{\infty} H_{f,5}(x, y, t)\frac{dt}{t^2} \lesssim 1.$$

To estimate $H_{f,1}(x, y, t)$, we employ the inequality that for $t>0$, $A>0$ and $\eta>1$,

$$A \le t + t^{1-\eta}A^{\eta},$$

and write

$$\begin{aligned}
&\int_{8r}^{\infty} H_{f,1}(x, y, t)\frac{dt}{t^2}\\
&\lesssim \int_{\mathbb{R}^n\setminus B(x,8r)} \left|\frac{\Omega(x-z)}{|x-z|^{n-1}}-\frac{\Omega(y-z)}{|y-z|^{n-1}}\right| |f(z)-m_B(f)| \left(\int_{|x-z|}^{\infty}\frac{dt}{t^2}\right)dz\\
&\lesssim \sum_{k=3}^{\infty}\int_{B(x,2^{k+1}r)\setminus B(x,2^k r)} \left|\frac{\Omega(x-z)}{|x-z|^{n-1}}-\frac{\Omega(y-z)}{|y-z|^{n-1}}\right| \frac{|f(z)-m_{B(x,2^{k+1}r)}(f)|}{|x-z|}dz\\
&\quad+\sum_{k=3}^{\infty}k\int_{B(x,2^{k+1}r)\setminus B(x,2^k r)} \left|\frac{\Omega(x-z)}{|x-z|^{n-1}}-\frac{\Omega(y-z)}{|y-z|^{n-1}}\right| \frac{dz}{|x-z|}\\
&\lesssim \sum_{k=3}^{\infty}k^{1-\gamma}\int_{B(x,2^{k+1}r)\setminus B(x,2^k r)} \frac{|\Omega(x-z)|}{|x-z|^{n}}\left|f(z)-m_{B(x,2^{k+1}r)}(f)\right|^{\gamma}dz\\
&\quad+\sum_{k=3}^{\infty}k^{1-\gamma}\int_{B(x,2^{k+1}r)\setminus B(x,2^k r)} \frac{|\Omega(y-z)|}{|y-z|^{n}}\left|f(z)-m_{B(x,2^{k+1}r)}(f)\right|^{\gamma}dz\\
&\quad+\sum_{k=3}^{\infty}k\int_{B(x,2^{k+1}r)\setminus B(x,2^k r)} \left|\frac{\Omega(x-z)}{|x-z|^{n-1}}-\frac{\Omega(y-z)}{|y-z|^{n-1}}\right| \frac{1}{|x-z|}dz\\
&= \mathrm{G}_1+\mathrm{G}_2+\mathrm{G}_3.
\end{aligned}$$

An argument similar to (2.11) via (2.10) tells us that

$$\begin{aligned}
\mathrm{G}_1 &\lesssim \sum_{k=3}^{\infty}k^{1-\gamma}(2^k r)^{-n}\left\{\int_{B(x,2^{k+1}r)}|\Omega(x-z)||\log^{\gamma}(2+|\Omega(x-z)|)\,dz\right.\\
&\qquad\left.+\int_{B(x,2^{k+1}r)}\exp\left(\frac{|f(z)-m_{B(x,2^{k+1}r)}(f)|}{C_5}\right)dz\right\}\\
&\lesssim 1,
\end{aligned}$$

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where we used the facts that $\gamma > 2$ and $\Omega \in L(\log L)^{\gamma}(S^{n-1})$. Similarly, we have

$$\mathrm{G}_2 \lesssim 1.$$

On the other hand, Lemma 2.3 together with (2.2) now tells us that

$$\mathrm{G}_3 \lesssim \sum_{k=3}^{\infty} k(2^k r)^{-1} 2^k r\left[\frac{r}{2^k r} + \int_{2^{-k-1}}^{2^{-k}} \omega(\delta)\,\frac{d\delta}{\delta}\right] \lesssim 1 + \int_0^1 \omega(\delta)\log\left(2+\frac{1}{\delta}\right)\frac{d\delta}{\delta} \lesssim 1.$$

Combining the estimates for G_1, G_2 and G_3 leads to

$$\int_{8r}^{\infty} H_{f,1}(x,\, y,\, t)\,\frac{dt}{t^2} \lesssim 1,$$

and hence the estimate (2.7) holds. This completes the proof of Theorem 2.1.

Proof of Theorem 2.2. Similarly to the proof of Theorem 2.1, to prove Theorem 2.2, it suffices to verify that for any $f \in \mathcal{E}^{\alpha,p}(\mathbb{R}^n)$ with $\|f\|_{\mathcal{E}^{\alpha,p}(\mathbb{R}^n)} = 1$, if there exists $y_0 \in \mathbb{R}^n$ such that $\mu_\Omega(f)(y_0) < \infty$, then for any ball $B \subset \mathbb{R}^n$ with $B \ni y_0$,

$$\left(\frac{1}{|B|}\int_B \left\{\left[\mu_\Omega(f)(x)\right]^2 - \inf_{y\in B}\left[\mu_\Omega(f)(y)\right]^2\right\}^{p/2} dx\right)^{2/p} \lesssim |B|^{2\alpha/n}.$$

Let r be the radius of B, $\mu_\Omega^r(f)$ and $\mu_\Omega^\infty(f)$ be the same as in (2.5) and (2.6), respectively. Since

$$\begin{aligned}
&\left\{\left[\mu_\Omega(f)(x)\right]^2 - \inf_{y\in B}\left[\mu_\Omega(f)(y)\right]^2\right\}^{p/2}\\
&\quad\le \left\{\left[\mu_\Omega^r(f)(x)\right]^2 + \left[\mu_\Omega^\infty(f)(x)\right]^2 - \inf_{y\in B}\left[\mu_\Omega^\infty(f)(y)\right]^2\right\}^{p/2}\\
&\quad\le \left(\mu_\Omega^r(f)(x) + \left\{\left[\mu_\Omega^\infty(f)(x)\right]^2 - \inf_{y\in B}\left[\mu_\Omega^\infty(f)(y)\right]^2\right\}^{1/2}\right)^p,
\end{aligned}$$

by the vanishing moment of Ω, we can write

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B \left\{\left[\mu_\Omega(f)(x)\right]^2 - \inf_{y\in B}\left[\mu_\Omega(f)(y)\right]^2\right\}^{p/2} dx\right)^{2/p}\\
&\quad\lesssim \left(\frac{1}{|B|}\int_B \left\{\mu_\Omega^r\left(\left[f - m_B(f)\right]\chi_{10B}\right)(x)\right\}^p dx\right)^{2/p}\\
&\qquad+ \left(\frac{1}{|B|}\int_B \left\{\left[\mu_\Omega^\infty(f)(x)\right]^2 - \inf_{y\in B}\left[\mu_\Omega^\infty(f)(y)\right]^2\right\}^{p/2} dx\right)^{2/p}.
\end{aligned}$$

It follows from Lemma 2.2 that

$$\begin{aligned}
\left(\frac{1}{|B|}\int_B \left\{\mu_\Omega^r\left(\left[f - m_B(f)\right]\chi_{10B}\right)(x)\right\}^p dx\right)^{2/p} &\lesssim \left(\frac{1}{|B|}\int_{10B} |f(x) - m_B(f)|^p\,dx\right)^{2/p}\\
&\lesssim |B|^{2\alpha/n}.
\end{aligned}$$

Thus, the proof of Theorem 2.2 is now reduced to proving that for any $x,\ y \in B$,

$$\left|\left[\mu_\Omega^\infty(f)(x)\right]^2 - \left[\mu_\Omega^\infty(f)(y)\right]^2\right| \lesssim r^{2\alpha}. \tag{2.13}$$

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Let $E_f(y, t)$ be the same as in (2.8). If $\alpha \in (-1, 0)$, a standard computation gives us that for any $y \in B$ and $t \geq 0$,

$$
\begin{aligned}
|E_f(y, t)| &= \Big| \sum_{k=-\infty}^{0} \int_{B(y, 2^k t)\setminus B(y, 2^{k-1} t)} \frac{\Omega(y-z)}{|y-z|^{n-1}} \big[f(z) - m_{B(y, 2^k t)}(f)\big]\, dz \Big| \\
&\leq \sum_{k=-\infty}^{0} \int_{B(y, 2^k t)\setminus B(y, 2^{k-1} t)} \frac{|\Omega(y-z)|}{|y-z|^{n-1}} \big|f(z) - m_{B(y, 2^k t)}(f)\big|\, dz \\
&\lesssim \sum_{k=-\infty}^{0} (2^k t)^{1-n} \left(\int_{B(y, 2^k t)} |\Omega(y-z)|^{p'}\, dz \right)^{1/p'} \\
&\quad \times \left(\int_{B(y, 2^k t)} \big|f(z) - m_{B(y, 2^k t)}(f)\big|^p\, dz \right)^{1/p} \\
&\lesssim t^{\alpha+1}.
\end{aligned} \tag{2.14}
$$

On the other hand, if $p \in (n, \infty)$, it follows from the Hölder inequality that

$$
\begin{aligned}
\big|E_f(y, t)\big| &= \int_{|y-z|\leq t} \frac{|\Omega(y-z)|}{|y-z|^{n-1}} \big|f(z) - m_{B(y, t)}(f)\big|\, dz \\
&\leq \left(\int_{B(y, t)} \frac{|\Omega(y-z)|^{p'}}{|y-z|^{(n-1)p'}}\, dz \right)^{1/p'} \left(\int_{B(y, t)} |f(z) - m_{B(y, t)}(f)|^p\, dz \right)^{1/p} \\
&\lesssim t^{1+\alpha}.
\end{aligned} \tag{2.15}
$$

Therefore, for any $x, y \in B$,

$$
\begin{aligned}
&\Big| \big[\mu_\Omega^\infty(f)(x)\big]^2 - \big[\mu_\Omega^\infty(f)(y)\big]^2 \Big| \\
&\quad \lesssim \int_{8r}^{\infty} \big|E_f(x, t) - E_f(y, t)\big| \frac{dt}{t^{2-\alpha}} \\
&\quad \lesssim \int_{8r}^{\infty} \big|E_f(x, 8r)\big| \frac{dt}{t^{2-\alpha}} + \int_{8r}^{\infty} \big|E_f(y, 8r)\big| \frac{dt}{t^{2-\alpha}} + \int_{8r}^{\infty} H_f(x, y, t) \frac{dt}{t^{2-\alpha}} \\
&\quad \lesssim 1 + \int_{8r}^{\infty} H_f(x, y, t) \frac{dt}{t^{2-\alpha}},
\end{aligned}
$$

where $H_f(x, y, t)$ is the same as in the proof of Theorem 2.1. Again decompose $H_f(x, y, t)$ into

$$
\begin{aligned}
H_f(x, y, t) \leq{}& H_{f,1}(x, y, t) + H_{f,2}(x, y, t) + H_{f,3}(x, y, t) \\
&+ H_{f,4}(x, y, t) + H_{f,5}(x, y, t).
\end{aligned}
$$

Applying the Hölder inequality and Lemma 2.3 together with (2.3), we obtain that for any $x, y \in B$,

$$
\begin{aligned}
&\int_{8r}^{\infty} H_{f,1}(x, y, t) \frac{dt}{t^{2-\alpha}} \\
&\quad \lesssim \int_{\mathbb{R}^n\setminus B(x, 8r)} \left| \frac{\Omega(x-z)}{|x-z|^{n-1}} - \frac{\Omega(y-z)}{|y-z|^{n-1}} \right| |f(z) - m_B(f)| \left(\int_{|x-z|}^{\infty} \frac{dt}{t^{2-\alpha}} \right) dz
\end{aligned}
$$

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$$\lesssim \sum_{k=3}^{\infty}(2^k r)^{\alpha-1}\left(\int_{B(x,\,2^{k+1}r)\setminus B(x,\,2^k r)}\left|\frac{\Omega(x-z)}{|x-z|^{n-1}}-\frac{\Omega(y-z)}{|y-z|^{n-1}}\right|^{p'}dz\right)^{1/p'}$$
$$\times\left(\int_{B(x,\,2^{k+1}r)}|f(z)-m_B(f)|^p\,dz\right)^{1/p}$$
$$\lesssim \sum_{k=3}^{\infty}(2^k r)^{\alpha-1}(2^k r)^{n/p'-(n-1)}\left\{2^{-k}+\int_{2^{-k-1}}^{2^{-k}}\frac{\omega_{p'}(\delta)}{\delta}\,d\delta\right\}(2^k r)^{n/p}(2^k r)^{\alpha}$$
$$\lesssim r^{2\alpha}\sum_{k=3}^{\infty}\left\{2^{k(2\alpha-1)}+2^{2\alpha k}\int_{2^{-k-1}}^{2^{-k}}\frac{\omega_{p'}(\delta)}{\delta}\,d\delta\right\}$$
$$\lesssim r^{2\alpha}.$$

As for the term $H_{f,2}(x, y, t)$, another application of the Hölder inequality yields

$$\int_{8r}^{\infty} H_{f,2}(x, y, t)\,\frac{dt}{t^{2-\alpha}}$$
$$\lesssim \int_{\mathbb{R}^n\setminus B(x,\,4r)}\frac{|\Omega(x-z)|}{|x-z|^{n-1}}|f(z)-m_B(f)|\left(\int_{|x-z|}^{|y-z|}\frac{dt}{t^{2-\alpha}}\right)dz$$
$$\lesssim r\int_{\mathbb{R}^n\setminus B(x,\,4r)}\frac{|\Omega(x-z)|}{|x-z|^{n+1-\alpha}}|f(z)-m_B(f)|\,dz$$
$$\lesssim r\sum_{k=2}^{\infty}\int_{B(x,\,2^{k+1}r)\setminus B(x,\,2^k r)}\frac{|\Omega(x-z)|}{|x-z|^{n+1-\alpha}}\left|f(z)-m_{B(x,\,2^{k+1}r)}(f)\right|dz$$
$$+\,r\sum_{k=2}^{\infty}(2^k r)^{\alpha}\int_{B(x,\,2^{k+1}r)\setminus B(x,\,2^k r)}\frac{|\Omega(x-z)|}{|x-z|^{n+1-\alpha}}\,dz$$
$$\lesssim r\sum_{k=2}^{\infty}(2^k r)^{\alpha-n-1}\left(\int_{B(x,\,2^{k+1}r)\setminus B(x,\,2^k r)}|\Omega(x-z)|^{p'}\,dz\right)^{1/p'}$$
$$\times\left(\int_{B(x,\,2^{k+1}r)\setminus B(x,\,2^k r)}\left|f(z)-m_{B(x,\,2^{k+1}r)}(f)\right|^p dz\right)^{1/p}$$
$$+\,r\sum_{k=2}^{\infty}(2^k r)^{\alpha-n-1}(2^k r)^{\alpha}(2^k r)^{n/p}\left(\int_{B(x,\,2^{k+1}r)}|\Omega(x-z)|^{p'}\,dz\right)^{1/p'}$$
$$\lesssim r^{2\alpha}.$$

Similarly to the estimate $H_{f,2}(x, y, t)$, we have that

$$\int_{8r}^{\infty} H_{f,3}(x, y, t)\,\frac{dt}{t^{2-\alpha}}\lesssim r^{2\alpha}.$$

On the other hand, it is easy to verify that for $x, y\in B$,

$$H_{f,4}(x, y, t)\le\int_{8r<|x-z|\le 10r}\frac{|\Omega(x-z)|}{|x-z|^{n-1}}\left|f(z)-m_B(f)\right|dz\lesssim r^{1+\alpha},$$

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which gives us that

$$\int_{8r}^{\infty} H_{f,4}(x, y, t) \frac{dt}{t^{2-\alpha}} \lesssim r^{2\alpha};$$

and in the same way,

$$\int_{8r}^{\infty} H_{f,5}(x, y, t) \frac{dt}{t^{2-\alpha}} \lesssim r^{2\alpha}.$$

Combining the estimates for $H_{f,j}(x, y, t)$ $(1 \le j \le 5)$ leads to the estimate (2.13), which completes the proof of Theorem 2.2.

Proof of Theorem 2.3. By Lemma 2.1, similarly to the proof of Theorem 2.1, it suffices to prove that for any $f \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$ with $\|f\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} = 1$ and $\alpha \in (0, \epsilon/2)$, if there exists $y_0 \in \mathbb{R}^n$ such that $\mu_\Omega(f)(y_0) < \infty$, then for any ball $B \subset \mathbb{R}^n$ with $B \ni y_0$,

$$\int_B \left\{ \left[\mu_\Omega(f)(x)\right]^2 - \inf_{y\in B} \left[\mu_\Omega(f)(y)\right]^2 \right\} dx \lesssim |B|^{1+2\alpha/n}.$$

Let r be the radius of B and $\mu_\Omega^\infty(f)$ be as in (2.6). As in the proof of Theorem 2.2, we need only prove that for any $x, y \in B$,

$$\left| \left[\mu_\Omega^\infty(f)(x)\right]^2 - \left[\mu_\Omega^\infty(f)(y)\right]^2 \right| \lesssim r^{2\alpha}. \tag{2.16}$$

With the notation $E_f(y, t)$, $H_f(x, y, t)$ and $H_{f,j}(x, y, t)$ for $j \in \{1, 2, 3, 4, 5\}$ as in the proof of Theorem 2.1, a standard computation gives us that for any $y \in B$ and $t \ge 8r$,

$$\left|E_f(y, t)\right| \le \int_{|y-z|\le t} \frac{|\Omega(y-z)|}{|y-z|^{n-1}} |f(z) - f(y)|\, dz \le \int_{|y-z|\le t} \frac{|\Omega(y-z)|}{|y-z|^{n-1-\alpha}}\, dz \lesssim t^{\alpha+1}.$$

Therefore, for any $x, y \in B$, we can write

$$\left| \left[\mu_\Omega^\infty(f)(x)\right]^2 - \left[\mu_\Omega^\infty(f)(y)\right]^2 \right| \lesssim 1 + \int_{8r}^{\infty} H_f(x, y, t) \frac{dt}{t^{2-\alpha}}.$$

From Lemma 2.3 and (2.4) we deduce that for any $x, y \in B$,

$$\begin{aligned}
&\int_{8r}^{\infty} H_{f,1}(x, y, t) \frac{dt}{t^{2-\alpha}} \\
&\quad \lesssim \int_{\mathbb{R}^n \setminus B(x, 8r)} \left| \frac{\Omega(x-z)}{|x-z|^{n-1}} - \frac{\Omega(y-z)}{|y-z|^{n-1}} \right| |f(z) - m_B(f)| \left(\int_{|x-z|}^{\infty} \frac{dt}{t^{2-\alpha}} \right) dz \\
&\quad \lesssim \sum_{k=3}^{\infty} (2^k r)^{\alpha-1} \int_{B(x, 2^{k+1}r)\setminus B(x, 2^k r)} \left| \frac{\Omega(x-z)}{|x-z|^{n-1}} - \frac{\Omega(y-z)}{|y-z|^{n-1}} \right| |f(z) - m_B(f)|\, dz \\
&\quad \lesssim \sum_{k=3}^{\infty} (2^k r)^{\alpha} \left\{ 2^{-k} + \int_{2^{-k-1}}^{2^{-k}} \frac{\omega(\delta)}{\delta}\, d\delta \right\} (2^k r)^{\alpha} \\
&\quad \lesssim r^{2\alpha} \sum_{k=3}^{\infty} \left\{ 2^{k(2\alpha-1)} + 2^{2\alpha k} \int_{2^{-k-1}}^{2^{-k}} \frac{\omega(\delta)}{\delta}\, d\delta \right\} \\
&\quad \lesssim r^{2\alpha}.
\end{aligned}$$

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On the other hand, a trivial computation yields

$$\begin{aligned}
&\int_{8r}^{\infty} H_{f,2}(x, y, t)\,\frac{dt}{t^{2-\alpha}} + \int_{8r}^{\infty} H_{f,3}(x, y, t)\,\frac{dt}{t^{2-\alpha}} \\
&\lesssim \int_{\mathbb{R}^n\setminus B(x,4r)} \frac{|\Omega(x-z)|}{|x-z|^{n-1}}|f(z)-m_B(f)|\left(\int_{|x-z|}^{|y-z|}\frac{dt}{t^{2-\alpha}}\right)dz \\
&\quad + \int_{\mathbb{R}^n\setminus B(y,4r)} \frac{|\Omega(y-z)|}{|y-z|^{n-1}}|f(z)-m_B(f)|\left(\int_{|y-z|}^{|x-z|}\frac{dt}{t^{2-\alpha}}\right)dz \\
&\lesssim r^{2\alpha}.
\end{aligned}$$

Again by some simple calculations, we have that

$$\int_{8r}^{\infty} H_{f,4}(x, y, t)\,\frac{dt}{t^{2-\alpha}} + \int_{8r}^{\infty} H_{f,5}(x, y, t)\,\frac{dt}{t^{2-\alpha}} \lesssim r^{1+\alpha}\int_{8r}^{\infty}\frac{dt}{t^{2-\alpha}} \lesssim r^{2\alpha}.$$

Combining the estimates for $H_{f,i}(x, y, t)$ $(1 \le i \le 5)$ leads to the estimate (2.16), which completes the proof of Theorem 2.3.

3. Estimates for area integrals. This section is devoted to the behavior on $\mathrm{BMO}(\mathbb{R}^n)$ space and Campanato spaces $\mathcal{E}^{\alpha,p}(\mathbb{R}^n)$ for $\mu_{\Omega,S}$ defined by (1.2). We will prove that the operator $\mu_{\Omega,S}$ enjoys the properties which are similar to those of the operator μ_Ω in Section 2. Our results can be stated as follows.

THEOREM 3.1. *Let Ω be homogeneous of degree zero, integrable on S^{n-1} and have mean value zero. Suppose that the $L^1(S^{n-1})$-modulus of continuity of Ω satisfies*

$$\int_0^1 \frac{\omega(\delta)}{\delta}\log^{\sigma}\left(2+\frac{1}{\delta}\right)d\delta < \infty \tag{3.1}$$

for some $\sigma \in (2, \infty)$. Then for any $f \in \mathrm{BMO}(\mathbb{R}^n)$, $\mu_{\Omega,S}(f)$ is either infinite everywhere or finite almost everywhere, and in the latter case,

$$\left\|\left[\mu_{\Omega,S}(f)\right]^2\right\|_{\mathrm{BLO}(\mathbb{R}^n)} \le C\|f\|^2_{\mathrm{BMO}(\mathbb{R}^n)},$$

where $C > 0$ is independent of f.

As for the behavior of $[\mu_{\Omega,S}(f)]^2$ on Campanato spaces, we have the following conclusions.

THEOREM 3.2. *Let Ω be homogeneous of degree zero, integrable on S^{n-1} and have mean value zero. Suppose that the $L^1(S^{n-1})$-modulus of continuity of Ω satisfies*

$$\int_0^1 \frac{\omega(\delta)}{\delta}\,d\delta < \infty.$$

If $\alpha \in (-1, 0)$ and $p \in [2, \infty)$, or $\alpha \in (-\infty, 0)$ and $p \in (n, \infty)$, then for any $f \in \mathcal{E}^{\alpha,p}(\mathbb{R}^n)$, $\mu_{\Omega,S}(f)$ is either infinite everywhere or finite almost everywhere, and in the latter case,

$$\left\|\left[\mu_{\Omega,S}(f)\right]^2\right\|_{\mathcal{E}_*^{2\alpha,p/2}(\mathbb{R}^n)} \le C\|f\|^2_{\mathcal{E}^{\alpha,p}(\mathbb{R}^n)},$$

where $C > 0$ is independent of f.

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THEOREM 3.3. *Let Ω be homogeneous of degree zero, integrable on S^{n-1} and have mean value zero. Suppose that the $L^1(S^{n-1})$-modulus of continuity of Ω satisfies*

$$\int_0^1 \frac{\omega(\delta)}{\delta^{1+\epsilon}}\, d\delta < \infty,$$

for some $\epsilon \in (0, 1]$. Then for any $\alpha \in (0, \epsilon/2)$ and $f \in \mathrm{Lip}_\alpha(S^{n-1})$, $\mu_{\Omega, S}(f)$ is either infinite everywhere or finite almost everywhere, and in the latter case,

$$\left\| \left[\mu_{\Omega, S}(f)\right]^2 \right\|_{\mathrm{Lip}_{2\alpha}(\mathbb{R}^n)} \le C \|f\|^2_{\mathrm{Lip}_\alpha(\mathbb{R}^n)},$$

where $C > 0$ is independent of f.

Similarly to Corollary 2.1 and Corollary 2.2, from Theorem 3.1 and Theorem 3.2, we can deduce the following results.

COROLLARY 3.1. *Under the hypothesis of Theorem* 3.1*, for any $f \in \mathrm{BMO}(\mathbb{R}^n)$, $\mu_\Omega(f)$ is either infinite everywhere or finite almost everywhere, and in the latter case,*

$$\|\mu_{\Omega, S}(f)\|_{\mathrm{BLO}(\mathbb{R}^n)} \le C \|f\|_{\mathrm{BMO}(\mathbb{R}^n)},$$

where $C > 0$ is independent of f.

COROLLARY 3.2. *Under the hypothesis of Theorem* 3.2*, we have that if $\alpha \in (-1, 0)$ and $p \in [2, \infty)$, or $\alpha \in (-\infty, 0)$ and $p \in (n, \infty)$, then for any $f \in \mathcal{E}^{\alpha, p}(\mathbb{R}^n)$, $\mu_{\Omega, S}(f)$ is either infinite everywhere or finite almost everywhere, and in the latter case,*

$$\|\mu_{\Omega, S}(f)\|_{\mathcal{E}_*^{\alpha, p}(\mathbb{R}^n)} \le C \|f\|_{\mathcal{E}^{\alpha, p}(\mathbb{R}^n)},$$

where $C > 0$ is independent of f.

To prove Theorem 3.1, we need recall the boundedness of area integral $\mu_{\Omega, S}$ in Lebesgue spaces; see [**4**].

LEMMA 3.1. *Let Ω be homogeneous of degree and have mean value zero. If $\Omega \in L\log L(S^{n-1})$, then for any $p \in [2, \infty)$, there exists a constant $C_p > 0$ such that for all $f \in L^p(\mathbb{R}^n)$,*

$$\|\mu_{\Omega, S}(f)\|_p \le C_p \|f\|_p.$$

Proof of Theorem 3.1. Similarly to the proof of Theorem 2.1, it suffices to verify that for any ball B and $f \in \mathrm{BMO}(\mathbb{R}^n)$ with $\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} = 1$, if there exists $y_0 \in \mathbb{R}^n$ such that $\mu_{\Omega, S}(f)(y_0) < \infty$, then for any ball $B \subset \mathbb{R}^n$ with $B \ni y_0$,

$$\frac{1}{|B|} \int_B \left(\left[\mu_{\Omega, S}(f)(x)\right]^2 - \inf_{y \in B} \left[\mu_{\Omega, S}(f)(y)\right]^2 \right) dx \lesssim 1.$$

Let r be the radius of B and set

$$\left[\mu^r_{\Omega, S}(f)(x)\right]^2 = \int_0^{8r} \int_{|u|<1} \left| \int_{|x-z+tu| \le t} \frac{\Omega(x - z + tu)}{|x - z + tu|^{n-1}} f(z)\, dz \right|^2 \frac{du\, dt}{t^3}$$

and

$$\left[\mu^{\infty}_{\Omega,S}(f)(x)\right]^2 = \int_{8r}^{\infty}\int_{|u|<1}\left|\int_{|x-z+tu|\le t}\frac{\Omega(x-z+tu)}{|x-z+tu|^{n-1}}f(z)\,dz\right|^2\frac{du\,dt}{t^3}.$$

Write

$$\int_B\left[\mu_{\Omega,S}(f)(x)\right]^2dx = \int_B\left[\mu^{r}_{\Omega,S}(f)(x)\right]^2dx + \int_B\left[\mu^{\infty}_{\Omega,S}(f)(x)\right]^2dx.$$

Note that (3.1) implies that $\Omega \in L\log L(S^{n-1})$. It then follows from Lemma 3.1 and the vanishing moment of Ω that

$$\int_B\left[\mu^{r}_{\Omega,S}\big([f-m_B(f)]\chi_{10B}\big)(x)\right]^2dx \lesssim \int_{10B}|f(x)-m_B(f)|^2\,dx \lesssim |B|.$$

As in the proof of Theorem 2.1, if suffices to prove that for any $x,\ y\in B$,

$$\left|\left[\mu^{\infty}_{\Omega,S}(f)(x)\right]^2-\left[\mu^{\infty}_{\Omega,S}(f)(y)\right]^2\right| \lesssim 1. \tag{3.2}$$

By the inequality (2.11) and the vanishing moment condition of Ω on S^{n-1}, we can write

$$\begin{aligned}
&\left|\left[\mu^{\infty}_{\Omega,S}(f)(x)\right]^2-\left[\mu^{\infty}_{\Omega,S}(f)(y)\right]^2\right|\\
&\le \int_{8r}^{\infty}\int_{|u|<1}\left|\left[E_f(x+tu,\ t)\right]^2-\left[E_f(y+tu,\ t)\right]^2\right|\frac{du\,dt}{t^3}\\
&\le \int_{8r}^{\infty}\int_{|u|<1}\left|E_f(x+tu,\ t)-E_f(y+tu,\ t)\right|\frac{du\,dt}{t^2}\\
&\lesssim \int_{8r}^{\infty}\int_{|u|<1}\left|E_{f_1}(x+tu,\ t)\right|\frac{du\,dt}{t^2}+\int_{8r}^{\infty}\int_{|u|<1}\left|E_{f_1}(y+tu,\ t)\right|\frac{du\,dt}{t^2}\\
&\quad+\int_{8r}^{\infty}\int_{|u|<1}\left|E_{f_2}(x+tu,\ t)-E_{f_2}(y+tu,\ t)\right|\frac{du\,dt}{t^2}\\
&\lesssim \int_{8r}^{\infty}\int_{|u|<1}\left|E_{f_1}(x+tu,\ t)\right|\frac{du\,dt}{t^2}+\int_{8r}^{\infty}\int_{|u|<1}\left|E_{f_1}(y+tu,\ t)\right|\frac{du\,dt}{t^2}\\
&\quad+\int_{8r}^{\infty}\int_{|u|<1}\int_{\substack{|y-z+tu|>t\\|x-z+tu|\le t}}\frac{|\Omega(x-z+tu)|}{|x-z+tu|^{n-1}}|f_2(z)|\,dz\,\frac{du\,dt}{t^2}\\
&\quad+\int_{8r}^{\infty}\int_{|u|<1}\int_{\substack{|y-z+tu|\le t\\|x-z+tu|>t}}\frac{|\Omega(y-z+tu)|}{|y-z+tu|^{n-1}}|f_2(z)|\,dz\,\frac{du\,dt}{t^2}\\
&\quad+\int_{8r}^{\infty}\int_{|u|<1}\int_{\substack{|y-z+tu|\le t\\|x-z+tu|\le t}}\left|\frac{\Omega(x-z+tu)}{|x-z+tu|^{n-1}}-\frac{\Omega(y-z+tu)}{|y-z+tu|^{n-1}}\right||f_2(z)|\,dz\,\frac{du\,dt}{t^2}\\
&=\sum_{j=1}^{5}\mathrm{U}_j(x,\ y),
\end{aligned}$$

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where $f_1 = [f - m_B(f)]\chi_{10B}$ and $f_2 = [f - m_B(f)]\chi_{\mathbb{R}^n\setminus 10B}$. A change of variable now tells us that

$$\begin{aligned}
\mathrm{U}_1(x,\ y) &\le \int_{8r}^{\infty}\int_{|u-x|<t}\int_{|u-z|\le t}\frac{|\Omega(z-u)|}{|z-u|^{n-1}}|f_1(z)|\,dz\,\frac{du\,dt}{t^{n+2}}\\
&\lesssim \int_{8r}^{\infty}\int_{10B}\left(\int_{|u-z|\le t}\frac{|\Omega(z-u)|}{|z-u|^{n-1}}\,du\right)|f_1(z)|\,dz\,\frac{dt}{t^{n+2}}\\
&\lesssim \int_{8r}^{\infty}\int_{10B}|f(z)-m_B(f)|\,dz\,\frac{dt}{t^{n+1}}\\
&\lesssim 1
\end{aligned}$$

and

$$\mathrm{U}_2(x,\ y) \lesssim 1.$$

The estimates for $\mathrm{U}_3(x,\ y)$ and $\mathrm{U}_4(x,\ y)$ are similar, and we only consider $\mathrm{U}_3(x,\ y)$. Write

$$\begin{aligned}
\mathrm{U}_3(x,\ y) &\lesssim \int_{8r}^{\infty}\int_{|u-x|<t}\int_{\substack{|z-u|\le t\\ |y-z+u-x|>t}}\frac{|\Omega(z-u)|}{|z-u|^{n-1}}|f_2(z)|\,dz\,\frac{du\,dt}{t^{n+2}}\\
&\lesssim \int_{8r}^{\infty}\int_{|u-x|<t}\int_{\substack{|z-u|\le t,\ 2|u-z|>|z-x|\\ |y-z+u-x|>t}}\frac{|\Omega(z-u)|}{|z-u|^{n-1}}|f_2(z)|\,dz\,\frac{du\,dt}{t^{n+2}}\\
&\quad+\int_{8r}^{\infty}\int_{|u-x|<t}\int_{\substack{8r<|z-u|\le t,\ 2|u-z|\le|z-x|\\ |y-z+u-x|>t}}\frac{|\Omega(z-u)|}{|z-u|^{n-1}}|f_2(z)|\,dz\,\frac{du\,dt}{t^{n+2}}\\
&\quad+\int_{8r}^{\infty}\int_{|u-x|<t}\int_{\substack{|z-u|\le 8r,\ 2|u-z|\le|z-x|\\ |y-z+u-x|>t}}\frac{|\Omega(z-u)|}{|z-u|^{n-1}}|f_2(z)|\,dz\,\frac{du\,dt}{t^{n+2}}\\
&= \mathrm{U}_{31}(x,\ y)+\mathrm{U}_{32}(x,\ y)+\mathrm{U}_{33}(x,\ y).
\end{aligned}$$

For $x,\ y \in B$, $z \in \mathbb{R}^n \setminus 10B$ and $2|u-z| > |z-x|$, we have $|u-z| > 2|x-y|$ and so

$$\begin{aligned}
\mathrm{U}_{31}(x,\ y) &\lesssim \int_{\mathbb{R}^n\setminus 10B}\int_{2|u-z|>|z-x|}\frac{|\Omega(z-u)|}{|z-u|^{n-1}}|f_2(z)|\left(\int_{|z-u|}^{|y-z+u-x|}\frac{dt}{t^{n+2}}\right)du\,dz\\
&\lesssim \int_{\mathbb{R}^n\setminus 10B}\int_{2|u-z|>|z-x|}\frac{|\Omega(z-u)|}{|z-u|^{n-1}}|f_2(z)|\\
&\qquad\times\left|\frac{1}{|u-z|^{n+1}}-\frac{1}{|y-z+u-x|^{n+1}}\right|du\,dz\\
&\lesssim r\int_{\mathbb{R}^n\setminus 10B}\int_{2|u-z|>|z-x|}\frac{|\Omega(z-u)|}{|z-u|^{2n+1}}|f_2(z)|\,du\,dz\\
&\lesssim r\int_{\mathbb{R}^n\setminus 10B}\frac{|f_2(z)|}{|z-x|^{n+1}}\,dz\\
&\lesssim 1.
\end{aligned}$$

On the other hand, if $z \in \mathbb{R}^n \setminus 10B$, $u \in \mathbb{R}^n$ with $|z-u| > 8r$, $|u-x| < t$, $2|z-u| \le |z-x|$ and $|y-z+u-x| > t$, then for $x,\ y \in B$,

$$t < |y-z+u-x| \le |z-u| + |x-y| \le |z-u| + 2r \le 2|z-u|,$$

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which in turn gives

$$|x-z| \le |x-u| + |u-z| < t + \frac{1}{2}|x-z| < 2|z-u| + \frac{1}{2}|x-z|.$$

Therefore,

$$\begin{aligned}
\mathrm{U}_{32}(x,\ y) &\lesssim \int_{\mathbb{R}^n\setminus 10B}\int_{\substack{2|u-z|\le|z-x|\le 4|u-z|\\ |u-z|>8r}} \frac{|\Omega(z-u)|}{|z-u|^{n-1}}|f_2(z)|\left(\int_{|z-u|}^{|y-z+u-x|}\frac{dt}{t^{n+2}}\right)du\,dz\\
&\lesssim \int_{\mathbb{R}^n\setminus 10B}\int_{\substack{2|u-z|\le|z-x|\le 4|u-z|\\ |u-z|>8r}} \frac{|\Omega(z-u)|}{|z-u|^{n-1}}|f_2(z)|\\
&\qquad\times\left|\frac{1}{|u-z|^{n+1}}-\frac{1}{|y-z+u-x|^{n+1}}\right|du\,dz\\
&\lesssim r\int_{\mathbb{R}^n\setminus 10B}\int_{2|u-z|\le|z-x|\le 4|u-z|}\frac{|\Omega(z-u)|}{|z-u|^{2n+1}}|f_2(z)|\,du\,dz\\
&\lesssim r\int_{\mathbb{R}^n\setminus 10B}\frac{|f_2(z)|}{|z-x|^{n+1}}\,dz\\
&\lesssim 1.
\end{aligned}$$

Similarly, we can obtain that for any $x \in B$,

$$\begin{aligned}
\mathrm{U}_{33}(x,\ y) &\lesssim \int_{8r}^{\infty}\int_{|u-x|<t}\int_{2|u-z|\le|z-x|,\ |z-u|\le 8r}\frac{|\Omega(z-u)|}{|z-u|^{n-1}}|f_2(z)|\,dz\,\frac{du\,dt}{t^{n+2}}\\
&\lesssim \int_{\mathbb{R}^n\setminus 10B}\int_{\substack{|z-u|\le 8r\\ |x-z|\le 2|x-u|}}\frac{|\Omega(z-u)|}{|z-u|^{n-1}}|f_2(z)|\left(\int_{|x-u|}^{\infty}\frac{1}{t^{n+2}}\,dt\right)du\,dz\\
&\lesssim \int_{\mathbb{R}^n\setminus 10B}\int_{\substack{|z-u|\le 8r\\ |x-z|\le 2|x-u|}}\frac{|\Omega(z-u)|}{|z-u|^{n-1}}|f_2(z)|\frac{1}{|x-u|^{n+1}}\,du\,dz\\
&\lesssim r\int_{\mathbb{R}^n\setminus 10B}\frac{|f_2(z)|}{|x-z|^{n+1}}\,dz\\
&\lesssim 1.
\end{aligned}$$

It remains to estimate $\mathrm{U}_5(x,\ y)$. Note that if $|u-z| \le 8r$, then for any $x,\ y \in B$, $|y-z+u-x| \le 10r$. It then follows that

$$\begin{aligned}
\mathrm{U}_5(x,\ y) &\lesssim \int_{8r}^{\infty}\int_{|u-x|<t}\int_{\substack{|u-z|\le 8r\\ |y-z-x+u|\le t}}\left|\frac{\Omega(u-z)}{|u-z|^{n-1}}-\frac{\Omega(y-z-x+u)}{|y-z-x+u|^{n-1}}\right||f_2(z)|\,dz\,\frac{du\,dt}{t^{n+2}}\\
&\quad+\int_{8r}^{\infty}\int_{|u-x|<t}\int_{\substack{8r<|u-z|\le t\\ |y-z-x+u|\le t}}\left|\frac{\Omega(u-z)}{|u-z|^{n-1}}-\frac{\Omega(y-z-x+u)}{|y-z-x+u|^{n-1}}\right||f_2(z)|\,dz\,\frac{du\,dt}{t^{n+2}}\\
&\lesssim \int_{8r}^{\infty}\int_{|u-x|<t}\int_{|u-z|\le 8r}\frac{|\Omega(u-z)|}{|u-z|^{n-1}}|f_2(z)|\,dz\,\frac{du\,dt}{t^{n+2}}\\
&\quad+\int_{8r}^{\infty}\int_{|u-x|<t}\int_{|y-z-x+u|\le 10r}\frac{|\Omega(y-z-x+u)|}{|y-z-x+u|^{n-1}}|f_2(z)|\,dz\,\frac{du\,dt}{t^{n+2}}\\
&\quad+\int_{8r}^{\infty}\int_{|u-x|<t}\int_{\substack{8r<|u-z|\le t\\ |y-z-x+u|\le t}}\left|\frac{\Omega(u-z)}{|u-z|^{n-1}}-\frac{\Omega(y-z-x+u)}{|y-z-x+u|^{n-1}}\right||f_2(z)|\,dz\,\frac{du\,dt}{t^{n+2}}\\
&= \mathrm{U}_{51}(x,\ y)+\mathrm{U}_{52}(x,\ y)+\mathrm{U}_{53}(x,\ y).
\end{aligned}$$

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For $z \in \mathbb{R}^n \setminus 10B$ and $u \in \mathbb{R}^n$ with $|u-z| \le 8r$, we have that $|u-x| \sim |z-x|$. Thus,

$$\begin{aligned}
&\mathrm{U}_{51}(x, y) + \mathrm{U}_{52}(x, y) \\
&\quad \lesssim \int_{\mathbb{R}^n} |f_2(z)| \int_{|u-z|\le 8r} \frac{|\Omega(u-z)|}{|u-z|^{n-1}} \left(\int_{|z-x|}^{\infty} \frac{dt}{t^{n+2}} \right) du\, dz \\
&\qquad + \int_{\mathbb{R}^n} |f_2(z)| \int_{|y-z+u-x|\le 10r} \frac{|\Omega(y-z+u-x)|}{|y-z+u-x|^{n-1}} \left(\int_{|z-x|}^{\infty} \frac{dt}{t^{n+2}} \right) du\, dz \\
&\quad \lesssim r \int_{\mathbb{R}^n \setminus 10B} \frac{|f(z) - m_B(f)|}{|x-z|^{n+1}}\, dz \\
&\quad \lesssim 1.
\end{aligned}$$

To deal with $\mathrm{U}_{53}(x, y)$, note that

$$\begin{aligned}
\int_{\max\{|u-z|,\, |x-z|\}}^{\infty} \frac{dt}{t^{n+2}} &\sim \frac{1}{\max(|u-z|,\, |x-z|)^{n+1}} \\
&\lesssim \frac{1}{|x-z|^n \log^{\sigma}(2+|x-z|/r)} \frac{\log^{\sigma}(2+|u-z|/r)}{|u-z|}.
\end{aligned}$$

On the other hand, Lemma 2.3 via a trivial computation gives that

$$\begin{aligned}
&\int_{|u-z|>8r} \left| \frac{\Omega(u-z)}{|u-z|^{n-1}} - \frac{\Omega(y-z-x+u)}{|y-z-x+u|^{n-1}} \right| \frac{\log^{\sigma}(2+|u-z|/r)}{|u-z|}\, du \\
&\quad \lesssim \sum_{k=3}^{\infty} \left(k^{\sigma} 2^{-k} + k^{\sigma} \int_{2^{-k-1}}^{2^{-k}} \frac{\omega(\delta)}{\delta}\, d\delta \right) \\
&\quad \lesssim 1.
\end{aligned}$$

This in turn implies that

$$\begin{aligned}
\mathrm{U}_{53}(x, y) &\lesssim \int_{\mathbb{R}^n} |f_2(z)| \int_{|u-z|>8r} \left| \frac{\Omega(u-z)}{|u-z|^{n-1}} - \frac{\Omega(y-z-x+u)}{|y-z-x+u|^{n-1}} \right| \\
&\qquad \times \left(\int_{\max\{|u-z|,\, |x-z|\}}^{\infty} \frac{dt}{t^{n+2}} \right) du\, dz \\
&\lesssim \int_{\mathbb{R}^n} \frac{|f_2(z)|}{|x-z|^n \log^{\sigma}(2+|x-z|/r)}\, dz \\
&\lesssim \sum_{k=3}^{\infty} k^{-\sigma} \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B \setminus 2^k B} |f(z) - m_{2^{k+1}B}(f)|\, dz + \sum_{k=3}^{\infty} k^{-\sigma+1} \\
&\lesssim 1.
\end{aligned}$$

Combining the estimates for $\mathrm{U}_j(x, y)$ $(1 \le j \le 5)$ then gives the desired inequality (3.2), which completes the proof of Theorem 3.1.

Proof of Theorem 3.2. As in the proof of Theorem 2.2, it suffices to prove that for $f \in \mathcal{E}^{\alpha, p}(\mathbb{R}^n)$ with $\|f\|_{\mathcal{E}^{\alpha, p}(\mathbb{R}^n)} = 1$, if there exists $y_0 \in \mathbb{R}^n$ such that $\mu_{\Omega, S}(f)(y_0) < \infty$, then for any ball $B \subset \mathbb{R}^n$ with $B \ni y_0$, and any $x, y \in B$,

$$\left| \left[\mu_{\Omega, S}^{\infty}(f)(x) \right]^2 - \left[\mu_{\Omega, S}^{\infty}(f)(y) \right]^2 \right| \lesssim r^{2\alpha}.$$

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By the estimates (2.14) and (2.15), we have that for any $x,\ y \in B$,

$$\begin{aligned}
&\left|\left[\mu^{\infty}_{\Omega,\,S}(f)(x)\right]^2 - \left[\mu^{\infty}_{\Omega,\,S}(f)(y)\right]^2\right| \\
&\lesssim \int_{8r}^{\infty}\int_{|u|<1}\left|E_{f_1}(x+tu,\ t)\right|\frac{du\,dt}{t^{2-\alpha}} + \int_{8r}^{\infty}\int_{|u|<1}\left|E_{f_1}(y+tu,\ t)\right|\frac{du\,dt}{t^{2-\alpha}} \\
&\quad + \int_{8r}^{\infty}\int_{|u|<1}\int_{\substack{|y-z+tu|>t\\|x-z+tu|\le t}}\frac{|\Omega(x-z+tu)|}{|x-z+tu|^{n-1}}|f_2(z)|\,dz\,\frac{du\,dt}{t^{2-\alpha}} \\
&\quad + \int_{8r}^{\infty}\int_{|u|<1}\int_{\substack{|y-z+tu|\le t\\|x-z+tu|>t}}\frac{|\Omega(y-z+tu)|}{|y-z+tu|^{n-1}}|f_2(z)|\,dz\,\frac{du\,dt}{t^{2-\alpha}} \\
&\quad + \int_{8r}^{\infty}\int_{|u|<1}\int_{\substack{|y-z+tu|\le t\\|x-z+tu|\le t}}\left|\frac{\Omega(x-z+tu)}{|x-z+tu|^{n-1}} - \frac{\Omega(y-z+tu)}{|y-z+tu|^{n-1}}\right||f_2(z)|\,dz\,\frac{du\,dt}{t^{2-\alpha}} \\
&= \sum_{j=1}^{5}\mathrm{W}_j(x,\ y).
\end{aligned}$$

As in the proof of Theorem 3.1, we can deduce that

$$\mathrm{W}_1(x,\ y)+\mathrm{W}_2(x,\ y)\lesssim\left(\int_{8r}^{\infty}\frac{dt}{t^{n+1-\alpha}}\right)\int_{10B}|f(z)-m_B(f)|dz\lesssim r^{n+\alpha}\int_{8r}^{\infty}\frac{dt}{t^{n+1-\alpha}}\lesssim r^{2\alpha},$$

and that

$$\mathrm{W}_3(x,\ y)+\mathrm{W}_4(x,\ y)\lesssim r\int_{\mathbb{R}^n\setminus 10B}\frac{|f_2(z)|}{|x-z|^{n+1-\alpha}}\,dz + r\int_{\mathbb{R}^n\setminus 10B}\frac{|f_2(z)|}{|y-z|^{n+1-\alpha}}\,dz\lesssim r^{2\alpha}.$$

On the other hand, invoking the fact that

$$\int_{|u-z|>8r}\left|\frac{\Omega(u-z)}{|u-z|^{n-1}} - \frac{\Omega(y-z-x+u)}{|y-z-x+u|^{n-1}}\right|\frac{dz}{|u-z|}\lesssim\sum_{k=3}^{\infty}\left(2^{-k}+\int_{2^{-k-1}}^{2^{-k}}\frac{\omega(\delta)}{\delta}\,d\delta\right)\lesssim 1,$$

we have

$$\begin{aligned}
\mathrm{W}_5(x,\ y) &\lesssim r\int_{\mathbb{R}^n}\frac{|f_2(z)|}{|x-z|^{n+1-\alpha}}\,dz + \int_{\mathbb{R}^n}\frac{|f_2(z)|}{|x-z|^{n-\alpha}}\,dz \\
&\lesssim r\sum_{k=3}^{\infty}(2^k r)^{-n-1+\alpha}(2^k r)^n(2^k r)^{\alpha} + \sum_{k=3}^{\infty}(2^k r)^{-n+\alpha}(2^k r)^n(2^k r)^{\alpha} \\
&\lesssim r^{2\alpha},
\end{aligned}$$

which completes the proof of Theorem 3.2.

Theorem 3.3 can be proved by the argument used in the proof of Theorem 2.3, together with some estimates used in the proof of Theorem 3.2. We omit the details for brevity.

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