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THE FUNDAMENTAL PRIME IDEALS OF A NOETHERIAN PRIME PI RING

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Let R be a noetherian prime PI ring and let P be a prime ideal of R. There is a set of prime ideals, the *fundamental prime ideals*, associated with the injective hull of R/P and denoted by Fund(P). The set Fund(P) is finite, by a result of Müller. In this paper we give a natural description of Fund(P) in terms of the trace ring of R which strengthens Müller's result by establishing a uniform bound for the size of Fund(P) for all primes P in the ring.

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0. Introduction

Let R be a prime noetherian PI ring, let P be a prime ideal of R, and let E denote the R-injective hull of $(R/P)_R$. Müller has shown that the set of *fundamental prime ideals* of E, Fund(P), is finite. (If P is maximal then the set of fundamental prime ideals is precisely the set of annihilators of irreducible subfactors of E.) In this note we show that Fund(P) has a natural formulation in terms of the trace ring of R, and we present a "finite" procedure for determining Fund(P). Müller's result follows from this description. In fact, we are able to prove the following: If R is a prime noetherian PI ring then there exists a positive integer n such that $|\text{Fund}(Q)| \leqq n$ for all prime ideals Q of R.

1. The second layer condition

Although our main interest here concerns prime ideals and injective modules for noetherian polynomial identity algebras, many of the basic ideas are valid in a wider context and we start by describing this context. (For unexplained terminology in what follows, consult [**5**, **8**], or the specific references provided later—or at first reading assume that all rings have a polynomial identity and so are fully bounded.)

To begin, let R be a noetherian ring, and let P be a prime ideal of R. If Q is also a prime ideal, and if there exists an R–R-bimodule factor of $Q \cap P/QP$ which forms an $R/Q-R/P$-bimodule that is torsion free on each side, then there is said to be a *right link* (or *direct link*) from Q to P, denoted by $Q \rightsquigarrow P$. A subset X of spec R is said to be *closed under right links* if for each $Q \in X$ and each $Q' \rightsquigarrow Q$ it follows that $Q' \in X$. The *right clique* of P is the smallest subset of spec R which contains P and is closed under right links.

Next we consider injective modules over R. First, if M is a right R-module, denote by

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$E(M)_R$ the R-injective envelope of M. Now let U be a uniform right ideal of R/P. Observe that $E(R/P)_R \cong \bigoplus_{i=1}^{n} E(U)_R$, where n is the Goldie rank of R/P. Hence in order to study $E(R/P)_R$ it is enough to study $E(U)_R$, an indecomposable injective module. Conversely, if R is an FBN-ring then each indecomposable injective right module is isomorphic to $E(V)_R$ for some uniform right ideal V of a prime factor of R, see [6].

Definition 1.1. Let R be a noetherian ring. Then R is said to satisfy the *right second layer condition* if for each prime ideal P of R the following property holds: If $M_R \hookrightarrow E(R/P)_R$ is a finitely generated R-module then there exists a series $0 = M_0 \subset \cdots \subset M_n = M$ of right R-modules such that each M_i/M_{i-1} is isomorphic to a right ideal of R/P_i for some P_i in the right clique of P. (In particular, $MP_n \dots P_1 = 0$.) Further, R is said to satisfy the *second layer condition* if it satisfies the right and left second layer conditions. Fully bounded noetherian rings, and in particular noetherian PI rings satisfy the second layer condition; see [5].

2. Fundamental primes and ideal links

Throughout this section let R be a noetherian ring satisfying the second layer condition and let P be a prime ideal of R. Choose a uniform module U that is isomorphic to a right ideal of R/P and set $E = E(U)_R$, the R-injective hull of U. We start by defining the fundamental series and fundamental prime ideals of E and summarize the results that we need. Most of what we say is extracted from [4] and [5, Chapter 9].

Let Y be a set of prime ideals of R. A semiprime ideal S of R is called *Y-semiprime* if each prime ideal minimal over S is in Y. Define $X_1(P) = P$, and, for $n \geqq 1$,

$$X_{n+1}(P) = \{Q \in \operatorname{spec}(R) \mid Q \rightsquigarrow Q_n \text{ for some } Q_n \in X_n(P)\}.$$

Thus $X = X(P) = \bigcup_{n=1}^{\infty} X_n(P)$ is the right clique of P. The set X satisfies the *incomparability* condition: that is, if $Q_1, Q_2 \in X$ and $Q_1 \subseteq Q_2$ then $Q_1 = Q_2$. The *fundamental series* $\{F_n\}$ of E is defined as follows: $F_1 = \operatorname{ann}_E(P)$, and for $n \geqq 1$, F_{n+1} is the full inverse image in E of the sum of annihilators in E/F_n of all X_{n+1}-semiprime ideals of R. It is not difficult to see that $E = \bigcup_{n=1}^{\infty} F_n$ [4, Lemma 5.4; 5, 9.1.2]. We denote by $\operatorname{Fund}_{n+1}(P)$ the set of assassinator prime ideals of E/F_n and set $\operatorname{Fund}(P) = \bigcup_{n=1}^{\infty} \operatorname{Fund}_n(P)$, the *fundamental prime ideals* of P. Thus $\operatorname{Fund}(P) \subseteq$ right clique(P). However, in general $\operatorname{Fund}(P) \neq$ right clique(P): this is demonstrated in a spectacular manner by a result of Müller [10, Theorem 7] which states that if R is a noetherian PI ring then $\operatorname{Fund}(P)$ is finite, while even for a prime noetherian PI ring right clique(P) may be infinite [9, p. 242].

Our original intention was to gain a better understanding of Müller's result in the prime case by finding some known finite set of prime ideals that would contain $\operatorname{Fund}(P)$. In order to do this we develop a description of the fundamental prime ideals that is internal to the ring R. If $A \subseteq B$ are ideals of R, and if Q, P are prime ideals of R such that B/A is naturally an $R/Q - R/P$-bimodule that is torsion free on both sides, then we say that there is an *ideal link* from Q to P via B/A and write $Q \approx P$. More generally, if R, S are prime noetherian rings and ${}_R B_S$ is a bimodule that is torsion free

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and finitely generated on both sides then we say that there is a *bond* from R to S. The following lemma gives some of the well-known properties concerning torsion and bonds.

Lemma 2.1. *Let R and S be noetherian rings satisfying the second layer condition. Let B be an R–S-bimodule that is finitely generated on each side.*

(i) *Suppose that S is prime. If B_S is a torsion S-module then* $\mathrm{ann}(B)_S \neq 0$.

(ii) *If B is faithful on each side then the classical Krull dimensions of R and S are equal.*

(iii) *Suppose that R and S are prime rings and have the same classical Krull dimension. If B_S is not a torsion module then some factor bimodule of B forms a bond from R to S.*

Proof. (i) This follows from [**5**, 5.1.2].

(ii) This is [**5**, 8.2.8].

(iii) Suppose that B_S is not torsion, let ${}_RT_S$ be the right torsion bimodule, and let $\bar{B} = B/T$. Let ${}_R\bar{A}_S$ be the left torsion bimodule of $\bar{B}$. Since $\bar{B}_S$ is torsion free $\bar{A}_S$ must be faithful, but since ${}_R\bar{A}$ is torsion ${}_R\bar{A}$ cannot be faithful. In view of (ii) and the assumption that the classical Krull dimensions of R and S coincide, we see that $\bar{A} = 0$. Hence $\bar{B}$ is a bond from R to S.

The next result is essentially in [**5**, 9.1.2] or [**4**, Lemma 5.4] although in neither place is it precisely stated in this form.

Lemma 2.2. *Let R be a noetherian ring with the second layer condition and let P be a prime ideal of R. Let U be a uniform right ideal of R/P and set $E = E(U)_R$. Then:*

(i) *If M is a finitely generated submodule of E with $M \subseteq F_n$ then there exist* $\mathrm{Fund}(P)$*-semiprime ideals $S_n, \ldots, S_2$ of R such that $MS_n \ldots S_2P = 0$.*

(ii) *If M is a finitely generated right submodule of E then there exist prime ideals $P_t, \ldots, P_1 \in \mathrm{Fund}(P)$ such that $MP_t \ldots P_1 = 0$.*

Proof. (i) The proof is by induction on n. If $n = 1$ then $MP = 0$. If $n > 1$ and M is generated by $m_1, \ldots, m_s$ then there are $\mathrm{Fund}(P)$-semiprime ideals $T_1, \ldots, T_s$ such that $m_iT_i \subseteq F_{n-1}$ and it follows that $MT \subseteq F_{n-1}$ where $T = T_1 \cap \cdots \cap T_s$ is a $\mathrm{Fund}(P)$-semiprime ideal of R. The result now follows by induction, since MT is finitely generated.

(ii) This follows from (i) since each $\mathrm{Fund}(P)$-semiprime ideal contains a product of prime ideals belonging to $\mathrm{Fund}(P)$.

Theorem 2.3. *Let R be a noetherian ring satisfying the second layer condition and let P be a prime ideal of R. Then $Q \in \mathrm{Fund}(P)$ if and only if $Q \approx P$.*

Proof. Let $Q \in \mathrm{ass}(F_{n+1}/F_n)$ and choose $e \in F_{n+1} \backslash F_n$ such that $eQ \subseteq F_n$. There exist $\mathrm{Fund}(P)$-semiprime ideals $S_n, \ldots, S_2$ such that $eQS_n \ldots S_2P = 0$ by Lemma 2.2(i). Set

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$B = QS_n \ldots S_2 \cap S_n \ldots S_2 P$ and $A = QS_n \ldots S_2 P$ and note that B/A is an $R/Q - R/P$-bimodule. If B/A is torsion as an R/P-module then there exists an ideal Y of R with $P \subsetneqq Y$ such that $BY \subseteq A$, by Lemma 2.1(i). In this case we see that $eBY = 0$, and it follows that $eB = 0$, since $\operatorname{ann}_E(Y) = 0$. Now $C = (Q \cap S_n)(S_n \cap S_{n-1}) \ldots (S_2 \cap P)$ is a product of n Fund(P)-semiprime ideals and $C \subseteq B$, so $eC = 0$. However this implies that $e \in F_n$, by [**4**, Lemma 5.4]. Thus B/A is not a torsion R/P-module and so, by Lemma 2.1(iii), $Q \approx P$.

Conversely, suppose that $Q \approx P$ via an ideal link B/A. Choose a right ideal K of R maximal such that $B \cap K = A$. Then there is an embedding of B/A as an essential submodule of the right R-module R/K.

Since B/A is finitely generated and torsion free as an R/P-module it embeds into a finite direct sum of copies of E [**5**, 2.2.15], and hence R/K also embeds in this way. Hence, by Lemma 2.2(ii), there are prime ideals $P_1, \ldots, P_n \in \operatorname{Fund}(P)$ such that $(R/K)P_n \ldots P_1 = 0$, and so $P_n \ldots P_1 \subseteq K$. Thus $P_n \ldots P_1 B \subseteq B \cap K = A$, so that $P_n \ldots P_1 \subseteq Q$. Therefore, $P_i \subseteq Q$ for some $1 \leqq i \leqq n$. However $\text{cl.}K\dim(R/Q) = \text{cl.}K\dim(R/P) = \text{cl.}K\dim(R/P_i)$, by Lemma 2.1(ii), so $Q = P_i$ and $Q \in \operatorname{Fund}(P)$.

Although the following result may be well known, we give a proof since we have not been able to locate a specific reference.

Proposition 2.4. *Let R be a noetherian ring and let x be a central element of R. If P and Q are prime ideals of R such that $Q \approx P$ and $x \in P$ then $x \in Q$ and $\bar{Q} \approx \bar{P}$ in $\bar{R} = R/xR$.*

Proof. Let B/A be an ideal link from Q to P. Since $xB = Bx \subseteq BP \subseteq A$ it follows that $x \in Q$. Now $B \cap x^m R \subseteq BxR \subseteq A$, for some $m \geqq 1$, by the AR property for xR, so $B \cap x^m R = A \cap x^m R$. Choose $n \geqq 0$ such that $B \cap x^n R \neq A \cap x^n R$, while $B \cap x^{n+1} R = A \cap x^{n+1} R$. Note that the non-zero bimodule $(B \cap x^n R) + A/A$ gives an ideal link from Q to P. Set $B' = \{r \mid x^n r \in B\}$ and $A' = \{r \mid x^n r \in A\}$. Then multiplication by the central element x^n induces an R–R-bimodule isomorphism between B'/A' and $(B \cap x^n R) + A/A$. Thus B'/A' gives an ideal link from Q to P. If $b' = xr \in B' \cap xR$ then $x^n b' = x^{n+1} r \in B \cap x^{n+1} R = A \cap x^{n+1} R \subseteq A$, so $b' \in A'$ and $B' \cap xR \subseteq A'$. Hence $B' + xR/A' + xR \cong B'/A'$ gives an ideal link from Q to P and the result follows.

An ideal I of a noetherian ring R is said to be *polycentral* if it can be generated by a sequence of elements $x_1, \ldots, x_n$ such that for each $i = 1, \ldots, n$ the element x_i is central modulo the ideal $\sum_{j=1}^{i-1} x_j R$. Obviously, a centrally generated ideal is polycentral.

Corollary 2.5. *Let R be a noetherian ring and let I be a polycentral ideal of R. If P and Q are prime ideals of R such that $Q \approx P$ and $I \subseteq P$ then $I \subseteq Q$ and $\bar{Q} \approx \bar{P}$ in $\bar{R} = R/I$.*

3. Lying over and contraction for ideal links

Let $R \hookrightarrow S$ be an embedding of rings. Let Q be a prime ideal of R, and let $\tilde{Q}$ be a

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prime ideal of S. Following a standard convention, we say that $\tilde{Q}$ *lies over* Q provided Q is a minimal prime ideal over $\tilde{Q}\cap R$.

Theorem 3.1. *Let $R \hookrightarrow S$ be an embedding of noetherian rings such that S is a finitely generated right R-module. Suppose further that R and S satisfy the second layer condition. Let P and Q be prime ideals of R such that $Q \approx P$. Then there exist prime ideals $\tilde{Q}$ lying over Q and $\tilde{P}$ lying over P such that $\tilde{Q}\approx\tilde{P}$.*

Proof. The proof may be obtained by retracing the steps of the proofs of [**7**, 5.1(i)] and [**7**, 5.3] after first replacing the direct link $(Q_\alpha\cap Q_\beta)/A$ considered there by an arbitrary ideal link C/A. We provide an adaptation of this approach.

To begin, choose an ideal K of S maximal such that $K\cap R\subseteq P\cap Q$ and such that $Q/K\cap R\approx P/K\cap R$. To prove the proposition it suffices to show that the conclusion remains true for the embedding $R/K\cap R\hookrightarrow S/K$. Hence we may assume without loss of generality that $K=0$.

We next demonstrate a fundamental property of S. Let C/A be an ideal link in R from Q to P, and let I be a nonzero ideal of S. Suppose that $I\cap C\subseteq A$. Then $(I\cap R)\cdot C$ and $C\cdot(I\cap R)$ are both contained in A, which in turn implies that $I\cap R\subseteq P\cap Q$, and it is easy to see that $Q/I\cap R$ is ideal linked to $P/I\cap R$ via $(C+I\cap R)/(A+I\cap R)$. This statement contradicts our choice of K above. We may therefore conclude that $I\cap C\nsubseteq A$.

We now show that S is uniform as an S–S-bimodule. Let C/A again be an ideal link from Q to P, and let I, J be ideals of S such that $I\neq 0$ but $I\cap J=0$. By the previous paragraph, $I\cap C\nsubseteq A$. So let $C'=I\cap C$, and let $A'=I\cap A$. Then C'/A' is an ideal link from Q to P. However, $J\cap C'=J\cap I\cap C=0\subseteq A'$. Hence the preceding paragraph shows that $J=0$. We conclude that S is a uniform S–S-bimodule.

To describe the assassinators of S, first let $H\in \operatorname{ass} S_S$. Then there exists a nonzero ideal I of S such that $IH=0$. Let C/A be the ideal link in R from Q to P. As shown above, $I\cap C\nsubseteq A$, and $I\cap C/I\cap A$ is an ideal link from Q to P. Since $(I\cap C/I\cap A)\cdot(H\cap R)=0$, we see that $H\cap R\subseteq P$. A left-sided argument shows that if $G\in \operatorname{ass}{}_S S$, then $G\cap R\subseteq Q$. Further, since S is a uniform S–S-bimodule, and since S satisfies the second layer condition, it follows from [**5**, 8.3.7] that $\operatorname{ass} S_S$ consists of a single prime ideal H and $\operatorname{ass}{}_S S$ consists of a single prime ideal G.

Now we describe the minimal prime ideals of R and S. From the previous paragraph we let $\{G\}=\operatorname{ass}{}_S S$. Now form a left affiliated series of S–R-bimodules for ${}_S S_R$, say $0=S_0\subset\cdots\subset S_n=S$, and for each i let $G_i=\operatorname{ann}{}_S(S_i/S_{i-1})$. Observe that G_1 must be G. Next, form a right affiliated series of S–R-bimodules for ${}_S S_R$, say $0=S'_0\subset\cdots\subset S'_m=S$, and for each j let $T_j=\operatorname{ann}(S'_j/S'_{j-1})_R$. Since $\operatorname{ass}{}_S S=\{G\}$, the S–R-sub-bimodule $M=\{s\in S: Gs=0\}$ is essential as a left S-submodule of ${}_S S$. Hence ${}_S(M\cap S'_1)_R$ forms a bond from S/G to R/T_1. Further, by [**5**, 8.3.1], each G_i is in the left clique of G, and each T_j is in the right clique of T_1. Therefore, by repeated applications of Lemma 2.1(ii), we see that the factors $S/G_1,\ldots,S/G_n$, $R/T_1,\ldots,R/T_m$ all have the same classical Krull dimension. Noting that $G_1G_2\ldots G_n=T_mT_{m-1}\ldots T_1=0$, we see therefore that $\{G_1,\ldots,G_n\}$ is precisely the set of minimal prime ideals of S and that $\{T_1,\ldots,T_m\}$ is precisely the set of minimal prime ideals of R. Further, again by Lemma 2.1(ii), it must be the case that the minimal prime ideals of R and S are closed with respect to ideal links.

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Next we prove that P and Q are minimal prime ideals of R: Let B_R be an R-submodule of S_R maximal such that $B \cap C = A$, where C/A is again an ideal link from Q to P. So C/A may be considered as an essential right R-submodule of $(S/B)_R$. Therefore, by [5, 7.1.2], $SP_1 \dots P_t \subseteq B$ for some prime ideals $P_1, \dots, P_t$ in the right clique of P. The choice of B now guarantees that $SP_1 \dots P_t \cap C \subseteq A$, and the original choice of K above thus implies that there exists no nonzero ideal L of S such that $L \subseteq SP_1 \dots P_t$. In other words, $S/SP_1 \dots P_t$ is faithful as a left S-module. Since $S/SP_1 \dots P_t$ is a right $(R/P_1 \dots P_t)$-module, it follows from Lemma 2.1(ii) that $\text{cl.}K\dim(R/P_1 \dots P_t) \geqq \text{cl.}K\dim(S) = \text{cl.}K\dim(R)$. Hence the classical Krull dimension of $R/P_1 \dots P_t$ is equal to the classical Krull dimension of R, and it follows that some P_i is a minimal prime ideal. We saw earlier that the set of minimal prime ideals of R was closed under ideal links. Hence the set $\{P_1, \dots, P_t, Q, P\}$ consists entirely of minimal prime ideals. In particular, P and Q are minimal.

To finish, let $\{G\} = \text{ass}\,_S S$ and $\{H\} = \text{ass}\, S_S$ as above. We have seen that $G \cap R \subseteq Q$ and $H \cap R \subseteq P$. Since P and Q are minimal we see then that G lies over P and that H lies over Q. Since $_S S_S$ is a uniform bimodule, $\{s \in S \mid Gs = 0\} \cap \{s \in S \mid sH = 0\} \neq 0$, and this ideal provides an ideal link in S from G to H. This fact completes the proof.

Recall that a ring embedding $R \hookrightarrow S$ is called a *finite centralizing extension* provided $S = Rc_1 + \cdots + Rc_n$ for some elements $c_1, \dots, c_n \in S$ such that $rc_i = c_i r$ for all $r \in R$ and each $1 \leqq i \leqq n$. Also recall that if $R \hookrightarrow S$ is a finite centralizing extension, and if $P \in \text{spec}\, S$, then $P \cap R \in \text{spec}\, R$. See [8, 10.2] for details.

Theorem 3.2. *Let $R \hookrightarrow S$ be a finite centralizing extension of noetherian rings. If P and Q are prime ideals of S such that $Q \approx P$, then $Q \cap R \approx P \cap R$.*

Proof. Let $S = Rc_1 + \cdots + Rc_n$, where each c_i centralizes R. For each $1 \leqq i \leqq n$, let $S_i = Rc_1 + \cdots + Rc_i$, and let $S_0 = 0$. Observe that each S_i is an R–R-sub-bimodule of $_R S_R$. We may define, for each i, an R–R-bimodule homomorphism $\theta_i : R \to S_i/S_{i-1}$ via $r \to rc_i + S_{i-1}$. It is straightforward to check that θ_i is an R–R-bimodule homomorphism, given that c_i centralizes R. Moreover, if $X_i = \ker \theta_i$, then X_i is a two-sided ideal of R and R/X_i is isomorphic to S_i/S_{i-1} as an R–R-bimodule.

Now let P and Q be prime ideals of S such that C/A is an ideal link in S from Q to P. Consider the two R–R-bimodule series $0 = S_0 \subset S_1 \subset \cdots \subset S_n = S$ and $0 \subset A \subset C \subset S$. By the Schreier Refinement Theorem, there exist an R–R-sub-bimodule B of S with $A \subsetneqq B \subset C$, and an $R-R$-bimodule subfactor W of S_i/S_{i-1}, for some i, such that W is isomorphic to B/A as an R–R-bimodule.

We claim that B/A, and hence W, is a bond from $R/Q \cap R$ to $R/P \cap R$. If B/A is not a bond then, without loss of generality, suppose that B/A has a nonzero sub-bimodule B'/A that is torsion as a right $R/P \cap R$-module. Then there exists an ideal X of R strictly bigger than $P \cap R$ such that $B'X \subseteq A$. But then $SXS \subseteq \text{ann}(B'/A)_S$, since S is a centralizing extension of R; and so $SXS \subseteq P$, thus contradicting $X \nsubseteq P \cap R$. Now the map θ_i above shows that W is isomorphic to an R–R-bimodule subfactor V of R/X_i. Since V will equal K/L for some pair of ideals K, L of R, it follows that $Q \cap R \approx P \cap R$.

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Remark. In [**3**, Proposition 5] it is shown that if $P, Q \in \operatorname{spec} S$ and $Q \rightsquigarrow P$ then either $Q \cap R \rightsquigarrow P \cap R$ or $Q \cap R = P \cap R$, for $R \hookrightarrow S$ a finite centralizing extension.

4. Rings satisfying a polynomial identity

In this section we apply the results of the previous two sections to the study of Fund(P) for a prime ideal P of a noetherian prime PI ring R. The main facts that we need concerning PI rings can be found in [**8**, Chapter 13, especially §9]. Recall that a prime PI ring R is contained in a possibly larger subring of its quotient ring, the *trace ring* of R, written $T = T(R)$. If R is noetherian then T is a finite centralizing extension of R. Noetherian PI rings satisfy the second layer condition so our earlier results are all available for use in this setting. Our first aim is to transfer to T the problem of finding ideal links in R. The advantage of working in the trace ring is that T is an integral extension of its centre Z, and, as we shall see in Proposition 4.1, there are only finitely many prime ideals of T with a given contraction to Z.

We start by introducing two more concepts of linkage between prime ideals of R. Let P and Q be prime ideals of R. Following [**3**] we say that P and Q are *trace-linked* if there are prime ideals $\tilde{P}$ and $\tilde{Q}$ of T such that $\tilde{P} \cap R = P$, $\tilde{Q} \cap R = Q$, and $\tilde{P} \cap Z = \tilde{Q} \cap Z$. The reader should be warned that this is *not* an equivalence relation on spec(R). Denote by Tr(P) the set of prime ideals that are trace-linked to P.

Proposition 4.1. *Let R be a noetherian prime PI ring with PI-degree n and suppose that the trace ring T of R can be generated as an R-module by m elements that centralize R. Then for any prime ideal P or R,*

$$|\mathrm{Tr}(P)| \leqq mn.$$

Proof. Let $P_1, \dots, P_r$ be the distinct prime ideals of T such that $P_i \cap R = P$. By [**11**, Theorem 3.4], $r \leqq m$. For any fixed i, the maximal number of prime ideals Q of T such that $P_i \cap Z = Q \cap Z$ is n. This follows from [**1**, Proposition 5] once one has reduced to the case that $P_i \cap Z$ is a maximal ideal of Z, by using [**2**, Lemma 2]. The prime ideals of R given by $Q \cap R$ for such primes Q form Tr(P), so $|\mathrm{Tr}(P)| \leqq mn$.

Let P and Q be prime ideals of the noetherian prime PI ring R. We say that there is a *trace-ideal-link* from Q to P, written $Q \approx_T P$, if there exist prime ideals $\tilde{P}$ and $\tilde{Q}$ of T such that $\tilde{P} \cap R = P$, $\tilde{Q} \cap R = Q$, and $\tilde{Q} \approx \tilde{P}$.

Lemma 4.2. (i) *$Q \approx P$ if and only if $Q \approx_T P$.*

(ii) *If $Q \approx_T P$ then $Q \in \mathrm{Tr}(P)$.*

Proof. (i) This is immediate from Theorems 3.1 and 3.2.

(ii) Let $\tilde{P}$ and $\tilde{Q}$ be prime ideals of T such that $\tilde{P} \cap R = P$, $\tilde{Q} \cap R = Q$, and $\tilde{Q} \approx \tilde{P}$. Then $\tilde{Q} \cap Z = \tilde{P} \cap Z$, since multiplication by central elements passes across the ideal link, and so Q and P are trace-linked.

Remark. K. A. Brown has suggested the following alternative proof that $Q \approx_T P$

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implies $Q \approx P$: Let $Q, P \in \operatorname{spec} R$ and let $\tilde{Q}, \tilde{P}$ be prime ideals of T such that $\tilde{Q} \cap R = Q$ and $\tilde{P} \cap R = P$. Suppose that $\tilde{B}/\tilde{A}$ is an ideal link in T from $\tilde{Q}$ to $\tilde{P}$. There exists a nonzero ideal I of T such that $I \subseteq R$, and I must contain a regular central element c. (See [**8**, 13.9.6] and [**8**, 13.6.4].) Hence $\tilde{B}c/\tilde{A}c$ is an ideal link in R from Q to P.

Theorem 4.3. *If R is a noetherian prime PI ring and P is a prime ideal of R then* $\operatorname{Fund}(P) \subseteq \operatorname{Tr}(P)$.

Proof. Let $Q \in \operatorname{Fund}(P)$. There is an ideal link $Q \approx P$, by Theorem 2.3. Thus the result follows, by Lemma 4.2.

Corollary 4.4. *Let R be a noetherian prime PI ring with trace ring T. Suppose that T can be generated as an R-module by m elements that centralize R, and suppose that the PI-degree of R is n. If P is any prime ideal of R then $|\operatorname{Fund}(P)| \leqq mn$.*

Proof. This follows immediately from Theorem 4.3 and Proposition 4.1.

Of course, Theorem 4.3 suggests the obvious question as to whether $\operatorname{Fund}(P) = \operatorname{Tr}(P)$. We have not yet been able to answer the following related question: Let R be a noetherian prime ring that is finitely generated as a module over its centre A. Suppose that A is a local ring with maximal ideal M. If P and Q are maximal ideals of R we know that $P \cap A = M = Q \cap A$. Is there an ideal link from Q to P?

Leaving aside this problem, we finish the section by outlining a "finite" procedure for finding $\operatorname{Fund}(P)$ when P is a prime ideal of the noetherian prime PI ring R. By Lemma 4.2 and Theorem 2.3 we need only find the prime ideals of R that are trace-ideal-linked to P. Thus if $\tilde{P}_1, \ldots, \tilde{P}_r$ are the prime ideals of T such that $\tilde{P}_i \cap R = P$, then what is left to determine is which prime ideals of T are ideal linked to some $\tilde{P}_i$. This transfers the problem of determining $\operatorname{Fund}(P)$ to the problem of finding those prime ideals of T which are ideal linked to at least one $\tilde{P}$ in $\{\tilde{P}_1, \ldots, \tilde{P}_r\}$.

Now fix such a $\tilde{P} \in \{\tilde{P}_1, \ldots, \tilde{P}_r\}$. Recall for a prime ideal $\tilde{Q}$ of T that if $\tilde{Q} \approx \tilde{P}$ then $\tilde{Q} \cap Z = \tilde{P} \cap Z$. Let $M = \tilde{P} \cap Z$. Hence to find those prime ideals of T which are ideal linked to $\tilde{P}$, we need only consider the (finite) set of prime ideals of T which contract to M. Further, if $\mathscr{C} = A \backslash M$, then it is straightforward to check that $\tilde{Q} \approx \tilde{P}$ in T if and only if $\tilde{Q}\mathscr{C}^{-1} \approx \tilde{P}\mathscr{C}^{-1}$ in the localisation $T\mathscr{C}^{-1}$. Next, by Corollary 2.5, if we let $A = T\mathscr{C}^{-1}/TM\mathscr{C}^{-1}$, if we let θ be the natural map from T to A, then $\tilde{Q} \approx \tilde{P}$ in T if and only if $\theta(\tilde{Q}) \approx \theta(\tilde{P})$ in A. Observe that A is artinian, and observe that we have reduced the problem of finding prime ideals $\tilde{Q}$ linked to $\tilde{P}$ to the problem of locating ideal links in A. Finally, note that the ideal links in A will be completely determined by an A–A-bimodule composition series for A; see [**5**, pp. 142–144].

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