J. Aust. Math. Soc. **99** (2015), 341–349
doi:10.1017/S144678871500021X

ON ISOMORPHISMS OF VERTEX-TRANSITIVE CUBIC GRAPHS

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(Received 28 October 2014; accepted 28 April 2015; first published online 13 August 2015)

Communicated by B. Alspach

Abstract

We study the isomorphism problem of vertex-transitive cubic graphs which have a transitive simple group of automorphisms.

2010 *Mathematics subject classification*: primary 05C25.

Keywords and phrases: coset graph, isomorphisms, GI-graphs, vertex-transitive.

1. Introduction

Determining whether two given graphs are isomorphic is fundamental for understanding graphs and for determining isomorphism classes of graphs. Vertex-transitive graphs form an important class of graphs and have received considerable attention in the literature, see [5, 14], for instance. A vertex-transitive graph Γ is generically defined by a vertex-transitive group G of automorphisms, but it is very difficult to determine the full automorphism group $\mathrm{Aut}\Gamma$.

Given two G-vertex-transitive graphs, one would expect to determine their isomorphisms by the information of the group G. For example, for Cayley graphs, such an approach was initiated by a conjecture of Ádám in 1967 [1] and has been extensively studied over the past decades, see, for example, [2, 4, 7, 9, 13, 18, 20, 21, 23, 24] and many more references listed in the survey [19]. Since lots of vertex-transitive graphs are not Cayley graphs, it is a natural next step to extend the study of the isomorphism problem for Cayley graphs to vertex-transitive graphs. For instance, Dobson [8] studied the isomorphism problem for metacirculants which are not necessarily Cayley graphs, and Tyshkevich and Tan [27] extended Babai's lemma, see Theorem 2.2. Babai's lemma, also obtained by Alspach and Parsons [3] independently, is a group-theoretic criterion for deciding whether two Cayley graphs are isomorphic which has been frequently used in the study of the isomorphism problem of Cayley graphs.

This work was partially supported by an ARC Discovery grant and the NSFC (grant numbers 11326057, 11171292 and 11271208).

Let $\Gamma = (V, E)$ be a G-vertex-transitive graph, and let α be a vertex and S be the set of elements of G which maps α to a vertex that is adjacent to α. Then Γ is uniquely determined by the triple (G, G_α, S). Letting $H = G_\alpha$ and identifying the vertex set V with $[G : H]$, the action of G on V is equivalent to the action of G on $[G : H]$ by right multiplication. In particular, if $\alpha \in V$ is identified with $H \in [G : H]$, then the neighbourhood $\Gamma(\alpha)$ consists of Hg with $g \in S$, and, moreover, $Hx \sim Hy$ if and only if $yx^{-1} \in S$. This defines a *coset graph* representation of Γ, which is denoted by $\mathsf{Cos}(G, H, HSH)$. Obviously, for an automorphism $\tau \in \mathrm{Aut}(G)$, we have $\mathsf{Cos}(G, H, HSH) \cong \mathsf{Cos}(G, H^\tau, H^\tau S^\tau H^\tau)$.

Definition 1.1. A G-vertex-transitive graph $\Gamma = \mathsf{Cos}(G, H, HSH)$ is called a *GI-graph* ('GI' stands for 'Group automorphism inducing Isomorphism') of G if for any graph $\Sigma = \mathsf{Cos}(G, H, HS'H)$ whenever $\Gamma \cong \Sigma$, there exists $\tau \in \mathrm{Aut}(G)$ such that $H^\tau = H$ and $(HSH)^\tau = HS'H$. (Note that the automorphism τ acts on the vertex set $[G : H]$ and fixes the vertex H.)

In particular, if $H = 1$ then Γ is a Cayley graph of G and is called a *CI-graph* if it is a GI-graph.

We consider the isomorphism problem of vertex-transitive cubic graphs and first state a conjecture.

Conjecture 1.2. A connected G-vertex-transitive cubic graph is a GI-graph of G.

For Cayley graphs, this conjecture can be restated as: 'every finite group is a connected 3-CI-group', that is, 'every connected cubic Cayley graph is a CI-graph', which is true for most simple groups by [10] and is open in general, see [19, Problem 6.3(2)]. For non-Cayley graphs, by Dobson's result in [8], the conjecture is true for metacirculants. The main result of this paper shows that Conjecture 1.2 is true for many families of simple groups.

Theorem 1.3. *Let G be a simple group of Lie type of odd characteristic. Then each connected G-vertex-transitive cubic graph is a GI-graph of G.*

We will actually prove the conclusion of Theorem 1.3 for more families of simple groups, namely, for those families of simple groups satisfying Hypothesis 3.4.

2. Isomorphisms of vertex-transitive graphs

By the description before Definition 1.1, any vertex-transitive graph can be represented as a coset graph. Conversely, for any group G and a subgroup $H < G$, we can construct graphs which are G-vertex-transitive. Let G be a group, and let H be a subgroup of G. For a subset $S \subset G \backslash H$, define the *coset graph* $\Gamma = \mathsf{Cos}(G, H, HSH)$ to be the graph with vertex set $[G : H] = \{Hg \mid g \in G\}$ such that $\{Hx, Hy\}$ is an edge if and only if $yx^{-1} \in HSH$. Then each element $g \in G$ induces an automorphism of Γ by right multiplication:

$$g : Hx \mapsto Hxg \quad \text{where } x \in G.$$

https://doi.org/10.1017/S144678871500021X Published online by Cambridge University Press

Moreover, the following statements for coset graphs are well known.

(a) Γ is an undirected graph if and only if $HSH = HS^{-1}H$, where $S^{-1} = \{s^{-1} \mid s \in S\}$.
(b) G acts transitively on the set $[G:H]$, with kernel being the core of H in G, and G is a subgroup of $\mathrm{Aut}\Gamma$ if and only if H is core-free in G.
(c) Γ is connected if and only if $\langle H, S\rangle = G$.
(d) Γ is G-arc-transitive if and only if $HSH = HgH$ where $g \in G$ such that $g^2 \in H$.

Define a subgroup of the automorphism group $\mathrm{Aut}(G)$:

$$\mathrm{Aut}(G,H) = \{\sigma \in \mathrm{Aut}(G) \mid H^\sigma = H\}.$$

Then each element τ of $\mathrm{Aut}(G,H)$ induces an automorphism of Γ or an isomorphism between $\Gamma = \mathsf{Cos}(G,H,HSH)$ and $\mathsf{Cos}(G,H,HS^\tau H)$.

Now let $\Sigma = \mathsf{Cos}(G,H,HS'H)$ be isomorphic to $\Gamma = \mathsf{Cos}(G,H,HSH)$, and let $V = [G:H]$. Then V is the vertex set of Γ and Σ, $\mathrm{Aut}\Sigma \cong \mathrm{Aut}\Gamma$. Each isomorphism τ from Σ to Γ is a permutation of V, and maps G to G^τ. Thus $G, G^\tau \le \mathrm{Aut}\Gamma \le \mathrm{Sym}(V)$ are permutationally isomorphic. Let

$$\hat{G}_{/H} = \{\hat{g} \mid g \in G\} \quad \text{where } \hat{g}\colon Hx \to Hxg \text{ for all } Hx \in V.$$

For any two vertices $Hx, Hy \in V$, the element $\hat{g} = \hat{x}^{-1}\hat{y}$ maps Hx to Hy, and thus $\hat{G}_{/H}$ acts transitively on V. For convenience, $\hat{G}_{/H}$ will be simply written as $\hat{G}$.

Each element $\sigma \in \mathrm{Aut}(G,H)$ induces a permutation $\overline{\sigma}$ on $[G:H]$ by the natural action: $\overline{\sigma}\colon Hx \to Hx^\sigma$. We define

$$\overline{\mathrm{Aut}}(G,H) = \{\overline{\sigma} \mid \sigma \in \mathrm{Aut}(G,H)\},$$

which is a subgroup of $\mathrm{Sym}(V)$ fixing the vertex $\alpha = H$.

For a subgroup H of a group G, denote by $\mathrm{N}_G(H)$ and $\mathbf{C}_G(H)$ the normalizer and the centralizer of H in G, respectively, that is,

$$\mathrm{N}_G(H) = \{g \in G \mid g^{-1}Hg = H\} \quad \text{and} \quad \mathbf{C}_G(H) = \{g \in G \mid gh = hg \text{ for all } h \in H\}.$$

Then we have the following basic property, which is due to Godsil [12, Lemma 2.1].

LEMMA 2.1. *Let* $\Gamma = \mathsf{Cos}(G,H,HSH)$ *be a coset graph, and let* $\hat{G} = \hat{G}_{/H}$ *and* $V = [G:H]$. *Then* $\mathrm{N}_{\mathrm{Sym}(V)}(\hat{G}) = \hat{G}\overline{\mathrm{Aut}}(G,H)$.

An important criterion for determining CI-graphs and GI-graphs is given in the next theorem, which was first obtained by Babai [4] and Alspach and Parsons [3] independently for CI-graphs, and extended by Tyshkevich and Tan [27] for GI-graphs.

THEOREM 2.2 [27]. *A* G*-vertex-transitive graph* $\Gamma = (V,E)$ *is a GI-graph of* G *if and only if subgroups of* $\mathrm{Aut}\Gamma$ *which are permutationally isomorphic to* G *(acting on* V*) are all conjugate in* $\mathrm{Aut}\Gamma$.

The CI-graph criterion is then stated as ‘a Cayley graph Γ of a group G is CI if and only if all regular subgroups of $\mathrm{Aut}\Gamma$ that are isomorphic to G are conjugate’. It has played an important role in the study of the isomorphism problem of Cayley graphs, see [19]. A special case for the criterion given in Theorem 2.2 is stated as follows.

Corollary 2.3. *Let Γ be a G-vertex-transitive graph such that G is normal in* $\mathrm{Aut}\Gamma$. *Then Γ is a GI-graph of G if and only if G is the only subgroup of* $\mathrm{Aut}\Gamma$ *that is permutationally isomorphic to G.*

In particular, if G is the only subgroup of $\mathrm{Aut}\Gamma$ *that is isomorphic to G, then Γ is a GI-graph.*

A result of Gross [16] shows that if a group X has a Hall π-subgroup of odd order, then all Hall π-subgroups of X are conjugate. Thus, we have the following consequence.

Corollary 2.4. *Let Γ be a graph of odd order. If* $\mathrm{Aut}\Gamma$ *has a Hall subgroup G which is of odd order and is vertex-transitive on Γ, then Γ is a GI-graph of G.*

By a result of Wielandt (see [25, page 166]), if a group X has a nilpotent Hall π-subgroup, then all Hall π-subgroups are conjugate. This gives the following consequence.

Corollary 2.5. *Let Γ be a graph such that* $\mathrm{Aut}\Gamma$ *has a nilpotent Hall subgroup G which is vertex-transitive on Γ. Then Γ is a GI-graph of G.*

In particular, a vertex-transitive graph of p-power order is a GI-graph of a Sylow p-subgroup of its automorphism group.

3. Proof of Theorem 1.3

First we quote a well-known result of elementary number theory, which is also a consequence of *Legendre's formula*, see, for instance, http://en.wikipedia.org/wiki/Factorial.

Lemma 3.1. *For any positive integer d and any prime p, the power $p^{[(d-1)/(p-1)]+1}$ does not divide $d!$.*

In order to prove Theorem 1.3, we first prove two group-theoretical lemmas. Recall that the *soluble radical* of a group X is the largest soluble normal subgroup of X, which is denoted by $R(X)$. Let $N:G$ be the semidirect product of G acting on N.

Lemma 3.2. *Let G be a nonabelian simple group acting on a group N by conjugation. Suppose that $|N|$ divides $|G|$. Then either $\langle G, N\rangle \cong G\times N$, or there exist a prime p and an integer d such that $p^d \mid |G|$ and $G \lesssim \mathrm{GL}(d,p)$.*

Proof. Suppose that $\langle G, N\rangle \ncong G\times N$. Let M be the soluble radical of N. Then M is a characteristic subgroup of N, and so M is normalized by G.

Suppose that $M\neq N$. Let $\overline{N} = N/M$, and $\overline{X} = X/M$. Then $\overline{X} = \overline{N}:G$, and each minimal normal subgroup of $\overline{N}$ is nonabelian. Suppose that G does not centralize $\overline{N}$. Then G does not centralize some minimal normal subgroup T^d of $\overline{X}$, where T is a nonabelian simple group and $d\geq 1$. Hence G permutes the d direct factors of T^d, and

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so $G \le S_d$. Since $|N|$ divides $|G|$, so does $|T^d|$. It follows that $2^{2d} = 4^d$ divides $|G|$, which is not possible by Lemma 3.1. Thus G centralizes $\overline{N}$, and so

$$X = M \cdot \overline{X} = M \cdot (\overline{N} : G) = M \cdot (\overline{N} \times G).$$

If G centralizes M, then G is normal in X and $\langle G, N\rangle \cong G \times N$, which contradicts our assumption.

The group G thus does not centralize M. Let L be a characteristic subgroup of M which is minimal subject to the condition that G does not centralize L. Let $Y = L : G$. Let $K \lhd L$ be maximal subject to the condition that K is normalized by G. Then L/K is a minimal normal subgroup of $Y/K \cong (L/K) : G$. Now $L/K = \mathrm{Z}_p^d$ for some prime p and some integer d, and G acts nontrivially on $L/K = \mathrm{Z}_p^d$. Since G is simple, G is faithful on L/K, and hence $G \le \mathrm{Aut}(L/K) \cong \mathrm{GL}(d, p)$. Moreover, as $p^d \mid |N|$ and $|N| \mid |G|$, then $p^d \mid |G|$, as desired. □

A permutation group $G \le \mathrm{Sym}(\Omega)$ is said to be *quasiprimitive* if every nonidentity normal subgroup of G is transitive on Ω. The well-known O'Nan–Scott theorem for primitive permutation groups is extended in [22] for quasiprimitive groups. The *socle* of a group G is the product of all minimal normal subgroups, denoted by $\mathsf{soc}(G)$. The next lemma is about simple groups acting on groups, and involves quasiprimitive permutation groups.

Lemma 3.3. *Let G be a nonabelian simple group which acts on Ω transitively. If a group X with $G < X \le \mathrm{Sym}(\Omega)$ is quasiprimitive on Ω, then one of the following statements holds:*

(i) $\mathsf{soc}(X) \cong G^2$, *and G is regular on Ω;*

(ii) *X is almost simple;*

(iii) $G = \mathrm{PSL}(2, 7)$, $X = \mathrm{AGL}(3, 2) = \mathrm{Z}_2^3 : \mathrm{GL}(3, 2)$, *and both G and X are 2-transitive on Ω.*

Proof. Let N be the socle of X. Then $N = T_1 \times \cdots \times T_d = T^d$, where $T_i \cong T$ are simple and $d \ge 1$. Since $N \lhd X$, $N \cap G \lhd G$, and hence either $G \le N$ or $N \cap G = 1$.

First, assume that $G \le N$. Then N is a transitive permutation group on Ω. If $d = 1$, then X is almost simple, as in statement (ii). Suppose that $d > 1$. Then N has d normal subgroups M_i where $1 \le i \le d$ such that $N = M_i \times T_i$. Since G is simple and $G \cap M_i$ is normal in G, either $G \le M_i$ or $G \cap M_i = 1$. Since the intersection $M_1 \cap M_2 \cap \cdots \cap M_d = 1$, there is some M_j such that $M_j \cap G = 1$. Then $G \cong G/(M_j \cap G) \cong M_jG/M_j \le N/M_j \cong T$, namely, G is isomorphic to a subgroup of T.

For convenience, write $M = M_j$, and let $Y = MG$. Then $Y = M : G$. Suppose that G does not centralize M. Then G acts on M by conjugation faithfully, and so $G \le S_{d-1}$. Since 4 divides $|T|$, then $2^{2(d-1)} = 4^{d-1}$ divides $|G|$, which divides $(d-1)!$, which is not possible by Lemma 3.1. Thus G centralizes M, and $Y = M \times G$. Since G is transitive, the centralizer $M = \mathbf{C}_Y(G)$ is semiregular, and so $|M|$ divides $|G|$. Since $G \lesssim T$, we conclude that $M \cong G \cong T$. Hence $N = T_1 \times T_2$, and T_i is regular on Ω. As $G \cong T$ is transitive, G is regular, as in statement (i).

Next, suppose that $N \cap G = 1$. Assume further that $N = \mathrm{Z}_p^d$ is abelian. Then N is regular on Ω. Hence $|N| = p^d$ divides $|G|$, and G_α is a subgroup of the simple group G of index p^d. Inspecting the classification of such simple groups of Guralnick [15], either:

(a) $G = \mathrm{A}_{p^d}$, or $\mathrm{PSL}(2, 11)$ with $p^d = 11$, or M_{11} with $p^d = 11$, or M_{23} with $p^d = 23$, or $\mathrm{PSL}(n, q)$ with $p^d = (q^n - 1)/(q - 1)$; or

(b) $(G, G_\alpha) = (\mathrm{PSU}(4, 2), 2^4 : \mathrm{A}_5)$, and $p^d = |G : G_\alpha| = 3^3$.

The candidate in (b) is not possible because $N = \mathrm{Z}_3^3$ and $\mathrm{PSU}(4, 2) \nleq \mathrm{GL}(3, 3)$. For the candidates in (a), since $G \nleq \mathrm{GL}(1, p)$, we only need to consider the last case where $G = \mathrm{PSL}(n, q)$ and $p^d = (q^n - 1)/(q - 1)$. Inspecting the smallest linear degree of $\mathrm{PSL}(n, q)$ given in [17, page 188], we conclude that $X = 2^3 : \mathrm{PSL}(3, 2) = \mathrm{AGL}(3, 2)$, as in statement (iii).

Finally, assume that $N = T_1 \times \cdots \times T_d$ is nonabelian, with $d \geq 2$. Since N is a socle of X, we have $\mathbf{C}_X(N) = 1$, and hence G acts faithfully on $\{T_1, T_2, \ldots, T_d\}$, and hence $G \leq \mathrm{S}_d$. Suppose that N has a normal subgroup M which is regular on Ω. Then $N = M : N_\omega$, where N_ω is the stabilizer of a point $\omega \in \Omega$. It follows that $|N_\omega| \leq |M|$, see [22], and hence $|M| = |T|^m$ divides $|G|$ with $m \geq d/2$. Since 4 divides $|T|$, we have that $4^m = 2^{2m}$ divides $|G|$, and hence $2^{2m} \mid d!$, which is not possible by Lemma 3.1. Thus N has no normal subgroup which is regular on Ω, and X is in the product action. Hence there exists $n \geq 5$ such that n^d divides $|\Omega|$. Let p be a prime divisor of n. Since $G \leq \mathrm{S}_d$, we conclude that p^d divides $d!$, which is not possible by Lemma 3.1. Therefore, $d = 1$, and X is an almost simple group, as in statement (ii). □

Let $\Gamma = (V, E)$ be a connected G-vertex-transitive cubic graph. We will prove Conjecture 1.2 for the following families of simple groups.

Hypothesis 3.4. Let G be a simple group satisfying one of the following items:

(a) G is a sporadic simple group and $G \neq \mathrm{M}_{11}, \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{J}_2, \mathrm{Suz}$;

(b) $G = \mathrm{A}_n$, with $n \notin \{8, 11, 23, 47\} \cup \{2^m - 1 | m \geq 3\}$;

(c) G is simple group of Lie type of odd characteristic;

(d) $G = \mathrm{PSL}(2, q)$, $\mathrm{PSL}(3, q)$, $\mathrm{PSU}(3, q)$, $\mathrm{PSp}(4, q)$, $\mathrm{E}_8(q)$, $\mathrm{F}_4(q)$, ${}^2\mathrm{F}_4(q)'$, $\mathrm{G}_3(q)$, $\mathrm{Sz}(q)$, where $q = 2^e$.

We have been unable to prove Conjecture 1.2 for the simple groups which do not satisfy the hypothesis. Verifying Conjecture 1.2 for those simple groups would be a crucial step in solving the conjecture.

Before stating and proving the main theorem, we introduce a few definitions. For a G-vertex-transitive graph $\Gamma = (V, E)$ and a normal subgroup N of G, let V_N be the set of N-orbits on V, and let Γ_N denote the *normal quotient* induced by N which is the graph with vertex set V_N and two orbits $\alpha^N, \beta^N \in V_N$ with $\alpha, \beta \in V$ adjacent in Γ_N if and only if α', β' are adjacent in Γ for some $\alpha' \in \alpha^N$ and $\beta' \in \beta^N$.

The following theorem shows that for each of these groups, all connected G-vertex-transitive cubic graphs are GI-graphs.

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THEOREM 3.5. *Let G be one of the simple groups listed in (a)–(d) of Hypothesis 3.4. Let $\Gamma = (V, E)$ be a connected G-vertex-transitive cubic graph. Then Γ is a GI-graph of G, and G is a normal subgroup of* $\mathrm{Aut}\Gamma$. *In particular, if G is a simple group of Lie type of odd characteristic, then Γ is a GI-graph of G.*

PROOF. Let $X = \mathrm{Aut}\Gamma$. Suppose that G is not normal in X.

Let $M \lhd X$ be maximal subject to the condition that M is intransitive on V. Let $\overline{X} = X/M$ and $\overline{G} = MG/M$. Then $\overline{G} \leq \overline{X} \leq \mathrm{Aut}\Gamma_M$, and $\overline{X}$ is quasiprimitive on V_M. Since $\overline{G} \leq \mathrm{Aut}\Gamma_M$ is nonabelian simple and Γ_M is connected, the valency of Γ_M is larger than 2. Since the valency of Γ_M is less than or equal to the valency of Γ which is 3, Γ_M is cubic. It follows from Lemma 3.3 that the quasiprimitive group $\overline{X}$ is almost simple because a group in the cases (i) and (iii) of Lemma 3.3 has no orbital graph of valency 1, 2 or 3.

Since both Γ and Γ_M are cubic, it implies that M is semiregular on V, and so the order $|M|$ divides the number of vertices $|V|$. Since G is transitive on V, the order $|G|$ is divisible by $|M|$. Therefore, if G does not centralize M, then there exists a prime power p^d dividing $|M|$ such that $G \leq \mathrm{GL}(d, p)$ by Lemma 3.2.

Let $Y = MG$. Then $M \lhd Y$, and $M \cap G \lhd G$. Since G is simple, it implies that $M \cap G = 1$. Since G is transitive on V, we have $Y = GY_\alpha$, where $\alpha \in V$. Thus $Y = M : G$, and then $|M|\,|G| = |Y| = |GY_\alpha| = |G|\,|Y_\alpha|/|G \cap Y_\alpha|$. Therefore, $|M| = |Y_\alpha|/|G_\alpha|$; in particular, $|M|$ divides the order $|X_\alpha|$. We proceed with our analysis using two cases.

Case 1. Suppose that X is arc-transitive on Γ. Then the quotient graph Γ_M is $\overline{X}$-arc-transitive. By Tutte's theorem [26], the stabiliser $X_\alpha = \mathrm{Z}_3, \mathrm{S}_3, \mathrm{D}_{12}, \mathrm{S}_4$ or $\mathrm{S}_4 \times \mathrm{S}_2$. In particular, the order $|X_\alpha|$ divides $2^4 3$, and so does the order $|M|$, since $|M| \mid |X_\alpha|$. Since $\overline{X} = X/M$ is almost simple, the socle $T := \mathsf{soc}(\overline{X})$ is the only insoluble composition factor of X.

Let $\Omega = [X : G]$. Since G is transitive on V, then $X = GX_\alpha$, and thus X_α acts transitively on Ω. Then $|\Omega|$ divides $|X_\alpha|$, and so $|\Omega| \mid 2^4 3$. Since G is simple and G is not normal in X by our assumption, G is core-free in X, and X is a transitive permutation group on Ω. Let $\mathcal{B}$ be an X-invariant partition of Ω such that X is primitive on $\mathcal{B}$. Then $|\mathcal{B}|$ divides 48, and G fixes a point in $\mathcal{B}$. Inspecting the primitive permutation groups of degree dividing 48 listed in [6, Appendix B], we conclude that either $X^{\mathcal{B}} = \mathrm{AGL}(3, 2)$ and $G = \mathrm{PSL}(2, 7)$, or $X^{\mathcal{B}}$ and G lie in the following table:

$\mathsf{soc}(X^{\mathcal{B}})$	A_6	A_8	A_{12}	A_{24}	A_{48}	$\mathrm{PSU}(3,3)$	M_{11}	M_{12}	M_{24}
G	A_5	A_7	A_{11}	A_{23}	A_{47}	$\mathrm{PSL}(2,7)$	$\mathrm{PSL}(2,11)$	M_{11}	M_{23}

We first consider the latter case. Since G satisfies Hypothesis 3.4, then $G = \mathrm{A}_5$, $\mathrm{PSL}(2, 7)$ or $\mathrm{PSL}(2, 11)$. Since the only insoluble composition factor of X is $T = \mathsf{soc}(\overline{X})$, we conclude that $T = \mathsf{soc}(X^{\mathcal{B}})$, and $\overline{X} = X^{\mathcal{B}}$. Thus

$$(G, T) = (\mathrm{A}_5, \mathrm{A}_6), \quad (\mathrm{PSL}(2,7), \mathrm{PSU}(3,3)) \quad \text{or} \quad (\mathrm{PSL}(2,11), \mathrm{M}_{11}).$$

Since $T \geq G$ and G is transitive, $T = GT_\alpha$. For the first candidate $(G, T) = (\mathrm{A}_5, \mathrm{A}_6)$, noticing that A_6 does not have a subgroup D_{12} and $\mathrm{A}_6 \neq \mathrm{A}_5\mathrm{D}_6$, then $T_\alpha = \mathrm{S}_4$, which is

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not possible since $|A_6 : S_4| = 15$ is odd. If $(G, T) = (\mathrm{PSL}(2, 7), \mathrm{PSU}(3, 3))$, then T_α has order divisible by $|T : G| = 36$, which is not possible. Thus $G = \mathrm{PSL}(2, 11)$. However, by [11], M_{11} does not have a factorization with one factor isomorphic to X_α (which is Z_3, S_3, D_{12}, S_4 or $S_4 \times S_2$).

The former case therefore occurs, with $X^{\mathcal{B}} = \mathrm{AGL}(3, 2)$ and $G = \mathrm{PSL}(2, 7)$. Since $\overline{X} = X/M$ is almost simple, it follows that $X = MG = M : G$. Suppose that G does not centralize M. By Lemma 3.2, there exists a prime power p^d dividing $|G|$ such that $G \le \mathrm{GL}(d, p)$. The only possibility is $p^d = 2^3$. However, in this case, the number of vertices of the quotient graph Γ_M divides an odd number $|G|/2^3 = 21$, which is a contradiction since Γ_M is cubic. Therefore, G is normal in X, and by Theorem 2.2, Γ is a GI-graph.

Case 2. Assume that Γ is not arc-transitive. Then the stabilizer X_α is a 2-group, where $\alpha \in V$, and the index $|X : G|$ is a power of 2.

Suppose that X is quasiprimitive on V. By Lemma 3.3, X is almost simple. Since $|X : G|$ is a power of 2, it follows from [15] that $(\mathrm{soc}(X), G) = (A_{2^m}, A_{2^m-1})$, where $m \ge 3$, which contradicts Hypothesis 3.4.

The group X is thus not quasiprimitive on V, and so $M \ne 1$. Suppose that G does not centralize M. By Lemma 3.2, there exists an integer d such that 2^d divides $|G|$ and $G \le \mathrm{GL}(d, 2)$. Let G_2 be a Sylow 2-subgroup. Then $2^d \mid |G_2|$, and $d \le \log_2 |G|_2$. On the other hand, d is equal to or larger than the linear degree of the simple group G. An inspection of the list of simple groups and their linear degrees in [17, Sections 5.3–5.4] shows that this is not possible. Then G centralizes M, and $X = M : G \cong G \times M$, and so G is normal in X. Since $|M| = |X : G|$ is a 2-power, G is a unique nonabelian simple subgroup of X, and by Theorem 2.2, Γ is a GI-graph. □

Acknowledgement

The authors are grateful for the referee's constructive suggestions.

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