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NON-MINIMAL TREE ACTIONS AND THE EXISTENCE OF NON-UNIFORM TREE LATTICES

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A *uniform tree* is a tree that covers a finite connected graph. Let X be any locally finite tree. Then $G = \mathrm{Aut}(X)$ is a locally compact group. We show that if X is uniform, and if the restriction of G to the unique minimal G-invariant subtree $X_0 \subseteq X$ is not discrete then G contains *non-uniform lattices*; that is, discrete subgroups Γ for which $\Gamma\backslash G$ is not compact, yet carries a finite G-invariant measure. This proves a conjecture of Bass and Lubotzky for the existence of non-uniform lattices on uniform trees.

0. Introduction

Let X be a locally finite tree and let $G = \mathrm{Aut}(X)$. Then G is naturally a locally compact group ([**3, 4**]). For a discrete subgroup $\Gamma \leqslant G$, the vertex stabilizers Γ_x, $x \in VX$, are finite groups [**3**]. Let $V(\Gamma\backslash X)$ be the vertex set of the quotient graph $\Gamma\backslash X$. As in [**3**] and [**4**] we call Γ an X*-lattice*, or a *tree lattice* if

$$\mathrm{Vol}(\Gamma\backslash\backslash X) = \sum_{x \in V(\Gamma\backslash X)} \frac{1}{|\Gamma_x|}$$

is finite, and a *uniform* X*-lattice* if $\Gamma\backslash X$ is a finite graph, *non-uniform* otherwise.

Following [**3**] we call X *uniform* if X is the universal cover of a finite connected graph. We call X *rigid* if $G = \mathrm{Aut}(X)$ is discrete, and X is *minimal* if G acts minimally on X, that is, there is no proper G-invariant subtree [**4**]. If X is uniform then there is always a unique minimal G-invariant subtree $X_0 \subseteq X$ ([**4,** (5.7), (5.11), (9.7)]). We call X *virtually rigid* if X_0 is rigid.

The following results of Bass and Tits [**5**] and Bass and Lubotzky [**4**] indicate that uniform trees with discrete groups of automorphisms cannot give rise to non-uniform lattices.

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PROPOSITION 0.1. ([**5**, (5.5)].) *Let X be a locally finite tree. If X is uniform and rigid then all X-lattices are uniform.*

PROPOSITION 0.2. ([**4**, (3.7)].) *Let X be a locally finite tree. If X is uniform and virtually rigid then all X-lattices are uniform.*

In analogy with Borel's classical theorem establishing the co-existence of uniform and non-uniform lattices in connected non-compact semisimple Lie groups, Bass and Lubotzky conjectured that under some natural assumptions $G = \mathrm{Aut}(X)$ contains *both* uniform and non-uniform lattices ([**4**, Chapter 7,8]). In particular, they conjectured that when G contains uniform lattices, the only obstruction to the existence of non-uniform lattices is virtual rigidity of X ([**4**, Chapter 7,8]). Here we present a proof of this conjecture. We use a theorem of Bass and Kulkarni [**3**] which states that $G = \mathrm{Aut}(X)$ contains a uniform lattice if and only if X is uniform. Our main theorem is the following.

THEOREM 0.3. *If X is uniform and not virtually rigid then G contains a non-uniform X-lattice.*

In [**6**], the author proved Theorem 0.3 for minimal actions assuming also the (necessary) Bass-Tits criterion for non-discreteness of G ([**5**, (5.5)]), which is equivalent to non-rigidity of X. That is, in [**6**] the author proved:

THEOREM 0.4. ([**6**].) *Let X be a uniform tree, and let $G = \mathrm{Aut}(X)$. If G is not discrete and acts minimally on X, then there is a non-uniform X-lattice $\Gamma \leqslant G$.*

Here we no longer assume that G acts minimally. Suppose that X is a uniform tree and let $X_0 \subseteq X$ be the unique minimal G-invariant subtree of X, also a uniform tree. Let $G_0 = \mathrm{Aut}(X)|_{X_0}$. In ([**8**]) we showed that $G_0 = \mathrm{Aut}(X_0)$. If X is not virtually rigid, that is X_0 is not rigid, then by Theorem 0.4 G_0 contains a non-uniform X_0-lattice Γ_0. Thus our task is to show that Γ_0 extends to a non-uniform X-lattice $\Gamma \leqslant G = \mathrm{Aut}(X)$. This is achieved by Theorems 3.1 and 3.4 in Section 3.

Theorems 0.3 and 0.4 together give a complete proof of the Bass-Lubotzky conjecture for the existence of non-uniform lattices on uniform trees ([**4**, Chapter 7,8]). Together with [**2**], and with [**9**] and [**10**] where we address the Bass–Lubotzky existence question in the case that X is not uniform, we have answered the Bass-Lubotzky conjectures in full. We refer the reader to [**7**] for an overview of the Bass–Lubotzky conjectures and their proofs.

1. Tree lattices, edge-indexed graphs, volumes and coverings

Let Γ be a group acting without inversions on a tree X. The fundamental theorem of Bass and Serre ([**1**, **12**]) states that Γ is encoded (up to isomorphism) in a 'quotient graph of groups' $\mathbb{A} = \Gamma \backslash\backslash X$ ([**1**, **12**]). Conversely a graph of groups $\mathbb{A}$ gives rise to a group $\Gamma = \pi_1(\mathbb{A}, a)$, $a \in V\mathbb{A}$, acting on a tree $X = \widetilde{(\mathbb{A}, a)}$ without inversions, and the vertex stabilizers Γ_x, $x \in VX$, are (conjugate to) the vertex groups of $\mathbb{A}$ ([**1, 12**]).

Now assume that X is locally finite, and that Γ acts on X with quotient graph of groups $\mathbb{A} = \Gamma\backslash\backslash X$. Then $\mathbb{A}$ naturally gives rise to an 'edge-indexed' graph (A, i), defined as follows. The graph A is the underlying graph of $\mathbb{A}$ with vertex set VA, edge set EA, initial and terminal functions $\partial_0, \partial_1 : EA \longmapsto VA$ which pick out the endpoints of an edge and with fixed point free involution $- : EA \longmapsto EA$ which reverses the orientation. The indexing $i : EA \longmapsto \mathbb{Z}_{>0}$ of (A, i) is defined to be the group theoretic index

$$i(e) = \left[\mathcal{A}_{\partial_0 e} : \alpha_e(\mathcal{A}_e)\right],$$

where

$$(\mathcal{A}_a)_{a \in VA} \text{ and } (\mathcal{A}_e = \mathcal{A}_{\overline{e}})_{e \in EA}$$

are the vertex and edge groups of $\mathbb{A}$, and $\alpha_e : \mathcal{A}_e \hookrightarrow \mathcal{A}_{\partial_0 e}$ are the boundary monomorphisms of $\mathbb{A}$. We write $(A, i) = I(\mathbb{A})$ when $i(e) = \left[\mathcal{A}_{\partial_0 e} : \alpha_e(\mathcal{A}_e)\right]$ for data

$$\{\mathcal{A}_a,\ \mathcal{A}_e = \mathcal{A}_{\overline{e}},\ \alpha_e : \mathcal{A}_e \hookrightarrow \mathcal{A}_{\partial_0 e}\}$$

from $\mathbb{A}$. Conversely, an *edge-indexed* graph (A, i) is defined to be a graph A and an assignment $i : EA \longmapsto \mathbb{Z}_{>0}$ of a positive integer to each oriented edge. Then (A, i) determines a universal covering tree $X = \widetilde{(A, i)}$ up to isomorphism ([**3, 4**]), and every edge-indexed graph arises from a tree action [**4**]. Here we assume $i(e)$ is finite for each $e \in EA$. Under this assumption the universal covering tree $X = \widetilde{(A, i)}$ is locally finite ([**3, 4**]).

Given an edge-indexed graph (A, i), a graph of groups $\mathbb{A}$ such that $I(\mathbb{A}) = (A, i)$, is called a *grouping* of (A, i). We call $\mathbb{A}$ a *finite grouping* if the vertex groups $\mathcal{A}_a$ are finite and a *faithful grouping* if $\mathbb{A}$ is a faithful graph of groups, that is if $\pi_1(\mathbb{A}, a)$, $a \in VA$ acts faithfully on $X = \widetilde{(\mathbb{A}, a)}$ [**3**]. If $\mathbb{A}$ is not faithful, then a faithful quotient of $\mathbb{A}$ always exists ([**1**]).

LEMMA 1.1. ([**3, 4**].) *Let (A, i) be an edge-indexed graph and let $\mathbb{A}$ be a finite faithful grouping of (A, i). Then for $a \in VA$, $\Gamma = \pi_1(\mathbb{A}, a)$ is a discrete subgroup of $G = \mathrm{Aut}\,(X)$, where $X = \widetilde{(A, i)}$.*

For an edge e in (A, i), define:

$$\Delta(e) = \frac{i(\overline{e})}{i(e)}.$$

If $\gamma = (e_1, \ldots, e_n)$ is a path, set $\Delta(\gamma) = \Delta(e_1) \ldots \Delta(e_n)$. An indexed graph (A, i) is then called *unimodular* if $\Delta(\gamma) = 1$ for all closed paths γ in A. This is equivalent to unimodularity of $G = \mathrm{Aut}\,(X)$ where $X = \widetilde{(A, i)}$ [**3**].

Assume now that (A,i) is unimodular. Pick a base point $a_0 \in VA$, and define, for $a \in VA$,

$$N_{a_0}(a) = \frac{\Delta a}{\Delta a_0} \quad \big(= \Delta(\gamma) \text{ for any path } \gamma \text{ from } a_0 \text{ to } a\big) \in \mathbb{Q}_{>0}.$$

For $e \in EA$, set

$$N_{a_0}(e) = \frac{N_{a_0}(\partial_0(e))}{i(e)}.$$

Following ([**4**, (2.6)]), we say that (A,i) has *bounded denominators* if $\{N_{a_0}(e) \mid e \in EA\}$ has bounded denominators, that is, if for some integer $D > 0$, $D \cdot N_{a_0}$ takes only integer values on edges. This condition is automatic if A is finite, and since

$$N_{a_1} = \frac{\Delta a_0}{\Delta a_1} N_{a_0},$$

this condition is independent of $a_0 \in VA$. As in [**3**] the functions $N : A \longrightarrow \mathbb{Q}^{\times}_{>0}$ as above are called *vertex orderings* of (A,i). We call N *integral* if for all $e \in EA$, we have $N(\partial_0(e))/i(e) \in \mathbb{Z}$ and hence $N(a) \in \mathbb{Z}$ for $a \in VA$.

Theorem 1.2. ([**3**, (2.4)].) *The following conditions on an edge-indexed graph (A,i) are equivalent.*

(a) *(A,i) admits a finite (faithful) grouping.*
(b) *(A,i) is unimodular and has bounded denominators.*
(c) *(A,i) admits an integral vertex ordering.*

We define the *volume* of an indexed graph (A,i) at a basepoint $a_0 \in VA$:

$$\mathrm{Vol}_{a_0}(A,i) = \sum_{a \in VA} \frac{1}{((\Delta a)/(\Delta a_0))} = \sum_{a \in VA} \left(\frac{\Delta a_0}{\Delta a}\right).$$

Then

$$\mathrm{Vol}_{a_1}(A,i) = \frac{\Delta a_0}{\Delta a_1} \mathrm{Vol}_{a_0}(A,i),$$

as in ([**4**, Chapter 2]). We write $\mathrm{Vol}(A,i) < \infty$ if $\mathrm{Vol}_a(A,i) < \infty$ for some, and hence every $a \in VA$.

If $\mathbb{A}$ is a finite grouping of (A,i), then we have ([**4**, (2.6.15)]):

$$\mathrm{Vol}(\mathbb{A}) = \frac{1}{|\mathcal{A}_a|} \mathrm{Vol}_a(A,i),$$

which is automatically finite if $\mathrm{Vol}(A,i) < \infty$.

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We now describe a method for constructing X-lattices which follows naturally from the fundamental theory of Bass and Serre, and was first suggested in [**3**]. We begin with an edge-indexed graph (A, i). Then (A, i) determines its universal covering tree $X = \widetilde{(A, i)}$ up to isomorphism ([**4**, Chapter 2]). If (A, i) is unimodular and has bounded denominators, then by Theorem 1.2 we can find a finite (faithful) grouping $\mathbb{A}$ of (A, i). By Lemma (1.1), $\Gamma = \pi_1(\mathbb{A}, a_0)$, $a_0 \in VA$, is a discrete subgroup of $G = \mathrm{Aut}(X)$. If further (A, i) has finite volume, then $\mathbb{A} \cong \Gamma\backslash\backslash X$ has finite volume $\mathrm{Vol}(\mathbb{A}) = \mathrm{Vol}(\Gamma\backslash\backslash X)$. It follows that $\Gamma = \pi_1(\mathbb{A}, a_0)$ is an X-lattice, uniform if $A = \Gamma\backslash X$ is a finite graph, non-uniform otherwise.

A *covering* $p : (B, j) \longrightarrow (A, i)$ of edge-indexed graphs ([**4**, (2.5)]), is a graph morphism $p : B \longrightarrow A$ such that for all $e \in EA$, $\partial_0(e) = a$, and $b \in p^{-1}(a)$, we have

$$i(e) = \sum_{f \in p_{(b)}^{-1}(e)} j(f),$$

where $p_{(b)} : E_0^B(b) \longrightarrow E_0^A(a)$ is the local map on the star $E_0(v)$ of a vertex v, that is, the set of edges with initial vertex v. If $b \in VB$, $p(b) = a \in VA$, then we can identify

$$\widetilde{(A, i, a)} = X = \widetilde{(B, j, b)}$$

so that the diagram of natural projections

$$\begin{tikzcd} & X \arrow[dl, "p_B"'] \arrow[dr, "p_A"] & \\ B \arrow[rr, "p"] & & A \end{tikzcd}$$

commutes. Let $G_{(B,j)} = \{g \in G \mid g \circ p_B = p_B\}$ and $G_{(A,i)} = \{g \in G \mid g \circ p_A = p_A\}$ be the groups of deck transformations of (B, j) and (A, i) respectively. If $p : (B, j) \longrightarrow (A, i)$ is a covering of edge-indexed graphs, then we have $G_{(B,j)} \leqslant G_{(A,i)}$ ([**4**, Chapter 2]). If $\mathbb{A}$ is a grouping of (A, i) and $\mathbb{B}$ is a grouping of (B, j) then by ([**4**, Chapter 2]) we have

$$\pi_1(\mathbb{A}, a) \leqslant G_{(A,i)} \text{ and } \pi_1(\mathbb{B}, b) \leqslant G_{(B,j)}.$$

2. Existence and structure of unique minimal subtree and its quotient

Let X be a locally finite tree, and let $G = \mathrm{Aut}(X)$. We recall that X is minimal if there is no proper G-invariant subtree. The following gives an existence theorem for minimal invariant subtrees of X.

Proposition 2.1. ([**4**, (5.5), (5.11), (9.7)].) *Let X be a tree and let $G = \mathrm{Aut}(X)$. If X is a uniform tree then there is a unique minimal G-invariant subtree $X_G \leqslant X$. Moreover the (hyperbolic) length function $l(G) \neq 0$, and if Γ is any X-lattice, $l(\Gamma) \neq 0$ and $X_G = X_\Gamma$.*

In this section we describe minimality of a group action $H \leqslant G = \mathrm{Aut}(X)$ in terms of its edge-indexed quotient graph, $(A, i) = I(H\backslash\backslash X)$, as in [**4**] and [**8**].

Let (A, i) be any edge-indexed graph. We say that (A, i) is *minimal* if (A, i) is the edge-indexed quotient of a minimal tree action. A vertex $a \in VA$ is called a *terminal vertex* of (A, i) if $\deg_{(A,i)}(a) = 1$, where

$$\deg_{(A,i)}(a) = \sum_{e \in E_0(a)} i(e),$$

and $E_0(a) = \{e \in EA \mid \partial_0 e = a\}$. A terminal vertex in (A, i) is then a geometrically terminal vertex in the graph A, that is, there is a unique edge e with $\partial_0 e = a$. The following gives a geometric characterisation of a minimal edge-indexed graph.

Proposition 2.2. ([**8**]) *Let Γ be a group acting without inversions on a tree X with quotient graph of groups $\mathbb{A} = \Gamma\backslash\backslash X$ and edge-indexed quotient graph $(A, i) = I(\mathbb{A})$.*

(1) *If (A, i) is minimal then (A, i) has no terminal vertices.*
(2) *If (A, i) is finite and has no terminal vertices then (A, i) is minimal.*

Let (T, i) be an edge-indexed graph. As in ([**4, 11**]) say that (T, i) is a *dominant rooted edge-indexed tree* if T is a tree and there is a vertex $a \in VT$ such that for all $e \in ET$

$$d(\partial_0 e, a) > d(\partial_1 e, a) \implies i(e) = 1.$$

We call such a vertex $a \in VT$ a *dominant root* of (T, i) and we write (T, i, a) when (T, i) is a dominant edge-indexed tree rooted at $a \in VT$.

Theorem 2.3. ([8]) *Let (A, i) be a finite edge-indexed graph. Then*

(1) *(A, i) contains a unique minimal connected subgraph (A_0, i_0).*
(2) *(A, i) has the form*

$$(A, i) = (A_0, i_0) \amalg \coprod_{a_j \in \Delta} (T_j, i_j, a_j),$$

where (T_j, i_j, a_j) are finite dominant-rooted edge-indexed trees with root vertices $a_j \in \Delta$, $j = 1 \ldots n$, $\Delta \subseteq VA_0$ and (A_0, i_0) has no terminal vertices.

Note that for (A, i) as in Theorem 2.3 the covering tree $X = \widetilde{(A, i)}$ has the form

$$X = X_0 \amalg \coprod_{x_{j,k} \in p^{-1}(a_j),\ a_j \in \Delta \subseteq VA_0} \left(T_{j,k}, \widetilde{i_{j,k}}, x_{j,k}\right),$$

where $j = 1 \ldots n$, $k \geqslant 1$, $X_0 = \widetilde{(A_0, i_0)}$, p is the covering map and $p\left(\widetilde{T_{j,k}}\right) = T_j$ ([8]).

3. Existence of Non-uniform Lattices

Let X be a locally finite tree. Let $H \leqslant G = \mathrm{Aut}(X)$ and let $G_H = \{g \in G \mid p \circ g = p\}$ be the deck transformation group of H, where $p : X \longrightarrow H\backslash X$ is the quotient morphism. Let $(A, i) = I(H\backslash\backslash X)$. Then $G_{(A,i)} = G_H$. Let (A_0, i_0) be the unique minimal subgraph of (A, i) as in Theorem 2.3. Let $X_0 \subseteq X$ be the unique minimal subtree of X. Then by [8], $X_0 = \widetilde{(A_0, i_0)}$ and H acts minimally on X_0. Our main theorem is the following.

Theorem 3.1. *Let X be a uniform tree and let $H \leqslant G = \mathrm{Aut}(X)$. Let $X_0 \subseteq X$ be the unique minimal G-invariant subtree of X. Let $G_0 = \mathrm{Aut}(X)|_{X_0} = \mathrm{Aut}(X_0)$, $(A, i) = I(H\backslash\backslash X)$, and let (A_0, i_0) be the unique minimal subgraph of (A, i). Assume that G_0 is not discrete (X_0 is not rigid). Then*

(i) *There is a non-uniform X_0-lattice $\Gamma_0 \leqslant G_{(A_0, i_0)} \leqslant G_0$.*

(ii) *Γ_0 extends to a non-uniform X-lattice $\Gamma \leqslant G_{(A,i)} \leqslant G = \mathrm{Aut}(X)$.*

The author proved (i) of Theorem 3.1 in [6] where the assumptions on X and G were restated as combinatorial conditions on (A, i). It remains to prove (ii). We shall give a constructive proof of (ii) by constructing the appropriate (infinite) edge-indexed graph (B, j) and taking a finite faithful grouping $\mathbb{B}$ of (B, j) of finite volume so that $\pi_1(\mathbb{B}, b)$ is a lattice, for $b \in VB$. In order to do this we describe the combinatorial restatement of the assumptions of Theorem 3.1 used in [6].

By [3] we have the following equivalent conditions:

(1) X is a uniform tree.

(2) There is a uniform X-lattice $\Lambda \leqslant G_H = G_{(A,i)}$.

(3) (A, i) is unimodular and finite.

(4) $\overline{H}$ is unimodular and $H\backslash X$ is finite, where $\overline{H}$ denotes the closure of H.

Similarly we have the following equivalent conditions:

$(1)_0$ $X_0 \subseteq X$ is uniform.

$(2)_0$ There is a uniform X_0-lattice $\Lambda_0 \leqslant G_{(A_0, i_0)}$.

$(3)_0$ (A_0, i_0) is unimodular and finite.

The assumption that X_0 is not rigid (G_0 is not discrete) is equivalent (by [5]) to the assumption that (A_0, i_0) is '*non-discretely ramified*'. As in ([4, 5]) we say that an edge-indexed graph (A, i) is *non-discretely ramified* if:

there exists $e \in EA$ such that $i(e) \geqslant 3$, or $i(e) = 2$ and e is not separating,
or $i(e) = 2$, and $(A_1(e), i)$ is either a ramified tree, or an unramified graph,

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where

$$(A_1(e), i) = \Big\{ v \in VA \mid d\big(v, \partial_1(e)\big) > d\big(v, \partial_0(e)\big) \Big\}.$$

If (A, i) is minimal this simplifies to:

> there exists $e \in EA$ such that $i(e) \geqslant 3$, or $i(e) = 2$ and $\partial_0 e$ is not a geometrically terminal vertex.

Let (A, i) be a finite edge-indexed graph. We say that (A, i) is *virtually discretely ramified* if the unique minimal subgraph (A_0, i_0) is discretely ramified. We can now describe our combinatorial restatement of (i) of Theorem 3.1 proven in [**6**].

THEOREM 3.2. ([**6**]) *Let X_0 be a uniform tree, let $H_0 \leqslant G_0 = \mathrm{Aut}(X_0)$ and let $(A_0, i_0) = I(H_0\backslash\backslash X_0)$. If H_0 acts minimally on X_0 and is not discrete (X_0 is not rigid) then there is a non-uniform X_0-lattice $\Gamma_0 \leqslant G_{(A_0,i_0)} \leqslant G_0$. Equivalently, assume that (A_0, i_0) is finite, unimodular, minimal and non-discretely ramified. Then (A_0, i_0) has a covering $p_0 : (B_0, j_0) \longrightarrow (A_0, i_0)$ such that (B_0, j_0) is infinite, unimodular, has finite volume and bounded denominators.*

If instead X_0 is the unique minimal invariant subtree of a uniform tree X, we obtain:

COROLLARY 3.3. *Let X be a uniform tree and let $H \leqslant G = \mathrm{Aut}(X)$. Let $X_0 \subseteq X$ be the unique minimal G-invariant subtree of X, also a uniform tree. Let $G_0 = \mathrm{Aut}(X)|_{X_0} = \mathrm{Aut}(X_0)$, $(A, i) = I(H\backslash\backslash X)$, and let (A_0, i_0) be the unique minimal subgraph of (A, i). If X_0 is not rigid then there is a non-uniform X_0-lattice $\Gamma_0 \leqslant G_{(A_0,i_0)} \leqslant G_0$. Equivalently, assume that (A_0, i_0) is finite, unimodular, minimal and non-discretely ramified. Then (A_0, i_0) has a covering $p_0 : (B_0, j_0) \longrightarrow (A_0, i_0)$ such that (B_0, j_0) is infinite, unimodular, has finite volume and bounded denominators.*

It remains to show that Γ_0 extends to a non-uniform X-lattice $\Gamma \leqslant G_{(A,i)} \leqslant G$. We achieve this with the following theorem. Our strategy is to start with a minimal edge-indexed graph (B_0, j_0) that admits a non-uniform lattice, and extend this to a non-minimal edge-indexed graph (B, j) that also admits a non-uniform lattice.

THEOREM 3.4. *Let (A_0, i_0) be an edge-indexed graph that is finite, unimodular, minimal and non-discretely ramified. Let $p_0 : (B_0, j_0) \longrightarrow (A_0, i_0)$ be a covering such that (B_0, j_0) is infinite, unimodular, has finite volume and bounded denominators. Let (A, i) be obtained from (A_0, i_0) by attaching to vertices $a_k \in \Delta$, $k = 1 \ldots n$, $\Delta \subseteq VA_0$ finite dominant-rooted edge-indexed trees (T_k, i_k, a_k), $k = 1 \ldots n$. Let (B, j) be obtained from (B_0, j_0) by attaching to each $b_k^t \in p_0^{-1}(a_k)$ a copy of (T_k, i_k, a_k), $k = 1, \ldots, n$, $t > 0$, denoted $\widetilde{(T_k, i_k, a_k)}$. Then there is a covering $p : (B, j) \longrightarrow (A, i)$ such that (B, j) is infinite, unimodular, has finite volume and bounded denominators.*

PROOF: The existence of a covering $p_0 : (B_0, j_0) \longrightarrow (A_0, i_0)$ is guaranteed by Theorem 3.2, and p_0 extends to $p : (B, j) \longrightarrow (A, i)$ in such a way that

$$p \mid_{(B_0, j_0)} = p_0 \quad \text{and} \quad p\Big(\widetilde{T_k, i_k, a_k}\Big) = (T_k, i_k, a_k).$$

Moreover (B, j) is automatically infinite. Since we are attaching finite trees (T_k, i_k, a_k), $k = 1, \ldots, n$, to (A_0, i_0) at single vertices, (A, i) is unimodular, and since (B_0, j_0) is unimodular, it follows that (B, j) is unimodular. Let

$$V_k = \mathrm{Vol}_{v_k}(T_k, i_k, a_k), \qquad k = 1, \ldots, n.$$

Let $V = max\{V_1, V_2, \ldots, V_n\}$. Choose $b_0 \in VB_0$ and let $V_0 = \mathrm{Vol}_{b_0}(B_0, j_0)$. Then

$$\begin{aligned}
\mathrm{Vol}_{b_0}(B, j) &= \sum_{v \in VB} \frac{1}{(\Delta v / \Delta b_0)} \\
&= \sum_{v \in p^{-1}(v_1)} \frac{V_1}{(\Delta v / \Delta b_0)} + \sum_{v \in p^{-1}(v_2)} \frac{V_2}{(\Delta v / \Delta b_0)} + \cdots + \sum_{v \in p^{-1}(v_n)} \frac{V_n}{(\Delta v / \Delta b_0)} \\
&\leqslant \sum_{v \in p^{-1}(v_1)} \frac{V}{(\Delta v / \Delta b_0)} + \sum_{v \in p^{-1}(v_2)} \frac{V}{(\Delta v / \Delta b_0)} + \cdots + \sum_{v \in p^{-1}(v_n)} \frac{V}{(\Delta v / \Delta b_0)} \\
&= V \sum_{v \in VB_0} \frac{1}{(\Delta v / \Delta b_0)} \\
&= VV_0 \\
&< \infty
\end{aligned}$$

Hence (B, j) has finite volume. Let $b_0 \in VB_0$. Then

$$\left\{ \frac{\Delta x}{\Delta b_0} \mid x \in VB_0 \right\} \subset \mathbb{Q}$$

has bounded denominators, since (B_0, j_0) has bounded denominators. Consider

$$\left\{ \frac{\Delta y}{\Delta b_0} \mid y \in p_B^{-1}(T_k) \right\} = \left\{ \frac{\Delta y}{\Delta v_k} \frac{\Delta v_k}{\Delta b_0} \mid y \in p_B^{-1}(T_k) \right\}.$$

Then the denominator of $(\Delta v_k)/(\Delta b_0)$ is bounded, since $v_k \in VB_0$, and the denominator of $(\Delta y)/(\Delta v_k)$ can increase only by a bounded amount for $y \in p_B^{-1}(T_k)$, since T_k is finite for each $k = 1, \ldots, n$. It follows that (B, j) has bounded denominators. ∎

COROLLARY 3.5. *In the setting of Theorem 3.4, there is a non-uniform lattice* $\Gamma \leqslant G_{(B,j)} \leqslant G_{(A,i)}$.

PROOF: Since (B, j) is unimodular and has bounded denominators, by Theorem 1.2 (B, j) admits a finite faithful grouping $\mathbb{B}$. Let $b \in VB$, $p(b) = a \in VA$, and set

$$\Gamma = \pi_1(\mathbb{B}, b) \text{ and } X = \widetilde{(A, i, a)} = \widetilde{(B, j, b)}.$$

Then $\Gamma \leqslant G_{(B,j)} \leqslant G_{(A,i)} \leqslant G = \mathrm{Aut}\,(X)$. By Lemma (1.1), Γ is a discrete subgroup of G. Since $\mathrm{Vol}(B,j) < \infty$,

$$\mathrm{Vol}\,(\mathbb{B}) = \mathrm{Vol}\,(\Gamma\backslash\backslash X) < \infty.$$

Thus Γ is an X-lattice, non-uniform since (B,j) is infinite. ∎

The subgroup $\Gamma \leqslant G$ is the non-uniform lattice, conjectured to exist in ([**4,** Chapter 7, 8]) and our proofs of Theorems 0.3 and 3.1 are complete.

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