MULTIPLICATION OF STRONGLY SUMMABLE SERIES

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1. Introduction. Given a series $\sum_{n=0}^{\infty} a_n$ we define A_n, $A_n^{(k)}$, $E_n^{(k)}$ $(k > -1)$, by the relations

$$A_n^{(k)} = \sum_{\nu=1}^{n} \binom{k+n-\nu}{n-\nu} a_\nu, \quad E_n^{(k)} = \binom{k+n}{n}, \quad A_n = A_n^{(0)}.$$

The series $\sum a_n$ is said to be summable (C, k), where $k > -1$, to the sum s if

$$a_n^{(k)} = A_n^{(k)}/E_n^{(k)} \to s \quad \text{as} \quad n \to \infty;$$

to be summable $(C, -1)$ to s if it converges to s and $na_n = o(1)$; to be absolutely summable (C, k), or summable $| C, k |$, to s if it is summable (C, k) to s and

$$\sum_{n=1}^{\infty} | a_n^{(k)} - a_{n-1}^{(k)} | < \infty;$$

and to be strongly Cesàro summable to s with order $k > 0$ and index p, or summable $[C; k, p]$ to s, if

$$\sum_{n=0}^{N} | a_n^{(k-1)} - s |^p = o(N).$$

Hyslop [1] has shown that necessary and sufficient conditions for $\sum a_n$ to be summable $[C; k, p]$, where $k > 0, p \geqslant 1$, to the sum s are that it be summable (C, k) to the sum s and that

$$\sum_{n=1}^{N} n^p | a_n^{(k)} - a_{n-1}^{(k)} |^p = o(N).$$

These conditions suggest that summability $[C; 0, p]$ be defined as convergence together with the condition

$$\sum_{n=0}^{N} n^p | a_n |^p = o(N),$$

and on the basis of this definition Hyslop [1] has proved the inclusion theorem that summability $[C; k, p]$ $(k \geqslant 0, p \geqslant 1)$ of a series implies summability $[C; k+\delta, q]$ of the series to the same sum for any $\delta > 0$ and $q \leqslant p$. He has also noted that, for $k \geqslant 0$, summability $| C, k |$ of a series implies its summability $[C; k, 1]$ to the same sum.

It will be shown that, with the above natural definition for summability $[C; 0, p]$, certain known results involving multiplication of $[C; k, 1]$-summable series with $k > 0$ cannot be extended to include $k = 0$.

2. The following results were proved by Winn [2]; his $[C, k]$ is the $[C; k, p]$ as defined above with $p = 1$.

THEOREM 2. *If $\sum u_n$ is summable $(C, k-1)$, where $k > 0$, then it is summable $[C, k]$ to the same sum.*

This result also holds for $k = 0$.

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THEOREM 5. *If Σu_n is summable $[C, k]$ to s, and if Σv_n is summable (C, l) to t, where $k > 0$ and $l \geqslant 0$, then $\Sigma w_n \equiv \Sigma(u_0 v_n + u_1 v_{n-1} + \ldots + u_n v_0)$ is summable $(C, k+l)$ to st.*

THEOREM 6. *Σu_n is summable $[C, k]$ to s and Σv_n is summable $[C, l]$ to t, where $k > 0$ and $l > 0$, then Σw_n is summable $[C, k+l]$ to st.*

3. The Case $k = l = 0$ of Theorem 6. If Σu_n and Σv_n are summable $[C, 0]$ then, by Hyslop's inclusion theorem, each of these series is summable $[C, \frac{1}{2}\delta]$ for any $\delta > 0$ and so, by Winn's Theorem 6, Σw_n is summable $[C, \delta]$. That Σw_n need not be summable $[C, 0]$ is shown by the following

Counterexample 1. Let

$$u_0 = u_1 = 0, \quad u_n = \frac{(-1)^n}{n \log n} \ (n \geqslant 2), \quad v_0 = v_1 = v_2 = 0, \quad v_n = \frac{(-1)^n}{n \log\log n} \ (n \geqslant 3).$$

Then Σu_n and Σv_n are each summable $(C, -1)$, and so also summable $[C, 0]$, but

$$\begin{aligned}
\sum_{n=0}^{N} n\,|w_n| &= \sum_{n=5}^{N} n \left| \sum_{r=2}^{n-3} \frac{(-1)^n}{r(\log r)(n-r)\log\log(n-r)} \right| \\
&= \sum_{r=2}^{N-3} \frac{1}{r \log r} \sum_{n=r+3}^{N} \frac{n}{n-r} \frac{1}{\log\log(n-r)} \\
&> \sum_{r=2}^{N-3} \frac{1}{r \log r} \frac{N}{N-r} \sum_{n=r+3}^{N} \frac{1}{\log\log(n-r)} \\
&> cN \sum_{r=2}^{N-3} \frac{1}{r \log r \log\log(N-r)}, \quad \text{where } c > 0, \\
&> \frac{cN}{\log\log N} \sum_{r=2}^{N-3} \frac{1}{r \log r} \\
&\sim cN \quad \text{as } N \to \infty.
\end{aligned}$$

Hence Σw_n is not summable $[C, 0]$.

It can, however, be proved that if Σu_n and Σv_n are summable $[C, 0]$ then Σw_n is summable $(C, 0)$. For,

$$\begin{aligned}
\sum_{n=0}^{N} n w_n &= \sum_{n=0}^{N} n \sum_{r=0}^{n} u_r v_{n-r} \\
&= \sum_{r=0}^{N} u_r \sum_{n=r}^{N} (n - r + r)\, v_{n-r} \\
&= \sum_{r=0}^{N} u_r \left\{ r V_{N-r} + \sum_{s=0}^{N-r} s v_s \right\} \\
&= \sum_{r=0}^{N} r u_r V_{N-r} + \sum_{s=0}^{N} s v_s U_{N-s} \\
&= O\left\{ \sum_{r=0}^{N} r\,|u_r| + \sum_{s=0}^{N} s\,|v_s| \right\} \\
&= o(N).
\end{aligned}$$

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Since Σw_n is summable $[C, \delta]$ for any $\delta > 0$, it is summable (C, δ) and hence summable $(C, 1)$. From the identity

$$(N+1) W_N^{(0)} = W_N^{(1)} + \sum_{n=0}^{N} n w_n,$$

we have that $W_n^{(0)}$ tends to the same limit as $W_n^{(1)}/E_n^{(1)}$ as $N \to \infty$. Hence Σw_n converges.

4. Counterexample 2. Let $u_0 = u_1 = 0$, $u_n = (-1)^n/(n \log n)$ $(n \geqslant 2)$, $v_0 = t$, $v_1 = v_2 = 0$, $v_3 = -3/\log\log 3$, $v_4 = 4/\log\log 4$,

$$v_n = (-1)^n \left\{ \frac{n}{\log\log n} - \frac{n-2}{\log\log(n-2)} \right\}, \quad \text{for } n \geqslant 5.$$

Then $V_0 = V_1 = V_2 = t$, $V_3 = t - (3/\log\log 3)$,

$$V_n = t + (-1)^n \left\{ \frac{n}{\log\log n} - \frac{n-1}{\log\log(n-1)} \right\}, \quad \text{for } n \geqslant 4.$$

Now $V_n \to t$ as $n \to \infty$, so that Σv_n is convergent, and hence summable $[C, 1]$ to t. As before, Σu_n is summable $[C, 0]$; let its sum be s. Then

$$W_n = \sum_{p=0}^{n} \sum_{q=0}^{p} u_q v_{p-q} = \sum_{r=0}^{n} u_r V_{n-r},$$

Hence, since

$$(V_r - t) u_{n-r} = (-1)^n |V_r - t| \cdot |u_{n-r}|,$$

$$\begin{aligned}
\sum_{n=0}^{N} |W_n - st| &= \sum_{n=0}^{N} \left| \sum_{r=0}^{n} (V_{n-r} - t) u_r + t \left\{ \sum_{r=0}^{n} u_r - s \right\} \right| \\
&\geqslant \sum_{n=0}^{N} \left| \sum_{r=0}^{n} (V_r - t) u_{n-r} \right| - t \sum_{n=0}^{N} \left| \sum_{r=0}^{n} u_r - s \right| \\
&= \sum_{n=0}^{N} \sum_{r=0}^{n} |V_r - t|\, |u_{n-r}| + o(N) \\
&= \sum_{m=0}^{N} |u_m| \sum_{r=0}^{N-m} |V_r - t| + o(N) \\
&= \sum_{m=2}^{N-3} \frac{N-m}{m \log m \log\log(N-m)} + o(N), \quad \text{if } N \geqslant 5, \\
&= S_1 + S_2 - S_3 + o(N),
\end{aligned}$$

where

$$\begin{aligned}
S_1 &= N \sum_{m=2}^{N/2} \frac{1}{m \log m \log\log(N-m)} \\
&\sim \frac{N}{\log\log N} \sum_{m=2}^{N/2} \frac{1}{m \log m} \\
&\sim N \quad \text{as } N \to \infty,
\end{aligned}$$

$$S_2 = N \sum_{m=\frac{1}{2}N+1}^{N-3} \frac{1}{m \log m \log\log(N-m)} > 0,$$

and

$$S_3 = \sum_{m=2}^{N-3} \frac{1}{\log m \log\log(N-m)}$$

$$= O\left\{\sum_{m=2}^{N-3} \frac{1}{\log m}\right\} = o(N).$$

Therefore $\sum_{n=0}^{N} |W_n - st| \neq o(N)$, and Σw_n is not summable $[C, 1]$ to st. By Theorem 6, Σw_n is summable $[C, 1+\delta]$ to st, so that, by the inclusion theorem, it cannot be summable $[C, 1]$ to any sum other than st. Hence Σw_n is not summable $[C, 1]$ and so is not summable $(C, 0)$. This proves that Winn's Theorem 6 cannot be extended to the case $k = 0$, $l = 1$ and his Theorem 5 cannot be extended to include $k = l = 0$.

Further, if Σv_n were summable $[C, 1-\delta]$ for some $\delta > 0$, then, since Σu_n is summable $[C, \frac{1}{2}\delta]$, Σw_n would be summable $[C, 1-\frac{1}{2}\delta]$ and hence summable $[C, 1]$. Since this is not the case, it follows that Σv_n is not summable $[C, 1-\delta]$, although it is summable $(C, 0)$ and $[C, 1]$. This shows that, when $k = 1$, Winn's Theorem 2 is in a sense " best possible ".

5. A multiplication theorem. Following the method of proof of Theorem 6, we have the

THEOREM. *If Σu_n is summable $[C, k]$, where $k > 0$, to s, and Σv_n is summable $|C, 0|$ to t, then Σw_n is summable $[C, k]$ to st.*

LEMMA (Winn [**2**]). *If $\sum_{n=1}^{N} |\alpha_n| = o(N)$, then $\sum_{n=1}^{N} n^p |\alpha_n| = o(N^{p+1})$ for $p > -1$.*

Proof of the Theorem. We have that Σv_n is summable $[C, 0]$. Suppose that $s = 0$. Equating coefficients of x^n in the identity

$$(1-x)^{-k} \sum_{n=0}^{\infty} u_n x^n \sum_{n=0}^{\infty} v_n x^n = \sum_{n=0}^{\infty} U_n^{(k-1)} x^n \sum_{n=0}^{\infty} v_n x^n,$$

we get
$$W_n^{(k-1)} = \sum_{r=0}^{n} v_r U_{n-r}^{(k-1)}.$$

Then
$$w_n^{(k-1)} = W_n^{(k-1)}/E_n^{(k-1)}$$

$$= \frac{1}{E_n^{(k-1)}} \sum_{r=0}^{n} v_r E_{n-r}^{(k-1)} u_{n-r}^{(k-1)},$$

and

$$\sum_{n=0}^{N} |w_n^{(k-1)}| \leqslant \sum_{n=0}^{N} \sum_{r=0}^{n} \frac{|v_r| E_{n-r}^{(k-1)} |u_{n-r}^{(k-1)}|}{E_n^{(k-1)}}$$

$$\leqslant \sum_{m=0}^{N} \sum_{r=0}^{N} \frac{E_m^{(k-1)} |v_r| |u_m^{(k-1)}|}{E_{m+r}^{(k-1)}}$$

$$= |v_0 u_0^{(k-1)}| + S_1 + S_2 + S_3,$$

where
$$S_1 = |v_0| \sum_{m=1}^{N} |u_m^{(k-1)}| = o(N)$$

$$S_2 = |u_0^{(k-1)}| \sum_{r=1}^{N} \frac{|v_r|}{E_r^{(k-1)}}$$

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$$= O\left\{\sum_{r=1}^{N} r^{1-k} |v_r|\right\} = o(N),$$

and

$$S_3 = \sum_{m=1}^{N} \sum_{r=1}^{N} \frac{E_m^{(k-1)} |v_r| \, |u_m^{(k-1)}|}{E_{m+r}^{(k-1)}}$$

$$< K \sum_{m=1}^{N} \sum_{r=1}^{N} \frac{m^{k-1} |v_r| \, |u_m^{(k-1)}|}{(m+r)^{k-1}}.$$

To show that $S_3 = o(N)$ it suffices to prove the result for any particular $k_0 > 0$; it will then follow for all $k \geqslant k_0$, since $\{m/(m+r)\}^{k-1}$ is a decreasing function of k. If $k \leqslant 1$, we have, since Σv_n is $|C, 0|$,

$$S_3 > K(2n)^{1-k} \sum_{m=1}^{N} m^{k-1} |u_m^{(k-1)}| \sum_{r=1}^{N} |v_r|$$

$$= O\left\{N^{1-k} \sum_{m=1}^{N} m^{k-1} |u_m^{(k-1)}|\right\}$$

$$= o(N^{1-k}N^k) = o(N),$$

by the Lemma. Σw_n is then summable $[C, k]$ to 0.

If $s \neq 0$, put $u_0' = u_0 - s$, $u_n' = u_n (n > 0)$, so that $\Sigma u_n'$ is summable $[C, k]$ to 0. Then $\Sigma w_n'$ is summable $[C, k]$ to 0. But $\Sigma w_n = \Sigma w_n' + s\Sigma v_n$ and Σv_n is summable $[C, k]$ to t. Hence Σw_n is summable $[C, k]$ to st.

Whether this result remains true when $k = 0$ is at present unsettled.

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