BULL. AUSTRAL. MATH. SOC. 46B28
VOL. 70 (2004) [469–473]

A COUNTEREXAMPLE USING 4-LINEAR FORMS

DAVID PÉREZ-GARCÍA

We prove that, for $n \geqslant 4$ and arbitrary infinite dimensional Banach spaces $X_1, \dots, X_n$, there exists an extendible n-linear form $T : X_1 \times \cdots \times X_n \longrightarrow \mathbb{K}$ that is not integral.

1. INTRODUCTION AND NOTATION

The fact that one cannot expect a general Hahn-Banach theorem for multilinear forms or homogeneous polynomials has been known for a long time. One can see it as follows [**7**]. Since every bilinear form on ℓ_∞ is weakly sequentially continuous, one can take, for instance, the inner product on a real Hilbert space, which is not weakly sequentially continuous (see [**13**]) and, therefore, cannot be extended to ℓ_∞.

However, which partial results can one expect? The first line of work in this direction was the search of the superspaces of a given space to which every multilinear form (polynomial) can be extended. The work in this direction goes back to the work of Arens in 1951 [**1, 2**], where he extended the product of a Banach algebra A to its bidual A^{**}. This work was continued by Aron and Berner in [**3**], where they gave a general procedure to extend every multilinear form to the product of the biduals. Lindström and Ryan gave in [**12**] a method to obtain extensions to ultrapowers, generalising in some sense the previous work. A nice review about *Aron-Berner extensions* and related topics can be found in [**4**].

Another profitable line of research has been the characterisation of the extendible multilinear forms (polynomials), that is, the forms that can be extended to *every* superspace of a given space. In this direction, Carando and Zalduendo [**6**] showed that every integral homogeneous polynomial is extendible. In [**11**], Kirwan and Ryan characterised the extendible 2- homogeneous polynomials from $\mathcal{L}_1$, $\mathcal{L}_\infty$ and L_2 spaces. This work was completed by Carando [**5**] and Castillo, García and Jaramillo [**7**], where they showed (independently) that

THEOREM 1.1. *For a Banach space with cotype 2, the extendible bilinear forms (2-homogeneous polynomials) are exactly the integral ones.*

Received 21st June, 2004
The author was partially supported by DGICYT grant BMF2001-1284.

In [**7**], they also contributed to the topic by showing the relation of the problem with the local complementability defined in [**10**], and characterising the spaces $\mathcal{L}_\infty$ as the spaces in which every multilinear form is extendible in a linear and continuous way.

However, with the exception of this last result, very little is known apart from the bilinear case. In this paper (Section 2) we shall show that Theorem 1.1 is a bilinear (maybe trilinear) result, in the sense that every possible generalisation to the 4-linear case fails.

The notation and terminology used in the paper will be the standard in Banach space theory, as for instance in [**9**], which is also our main source for unexplained notation. All the operators that appear in the paper will be continuous. The basis field $\mathbb{K}$ can be both $\mathbb{R}$ or $\mathbb{C}$ and $(e_i)_{i=1}^n$ will denote the canonical basis in $\mathbb{K}^n$. We shall denote by ε and π, respectively, the injective and projective tensor norms and we refer the reader to [**8**] for more about tensor norms. A multilinear form is said to be integral if it is continuous for the injective norm.

If $1 \leqslant p \leqslant \infty$ and $\lambda > 1$, a Banach space X is said to be an $\mathcal{L}_{p,\lambda}$ space if, for every finite dimensional subspace $E \subset X$ there exists another finite dimensional subspace F, with $E \subset F \subset X$ and such that there exists an isomorphism $v : F \longrightarrow \ell_p^{\dim F}$ with $\|v\|\,\|v^{-1}\| < \lambda$.

2. The result

The ideas needed to get the example can be summarised as follows. In Lemma 2.1 we see that $\ell_2^n \bigotimes_\varepsilon \ell_2^n$ can be seen as a subspace of $\ell_\infty^N \bigotimes_\pi \ell_\infty^N$. This is essentially Grothendieck's Theorem. In Lemma 2.2 we use this, the fact that the diagonal of $\ell_2^n \bigotimes_\varepsilon \ell_2^n$ is isometric to ℓ_∞^n, and the injectivity of ℓ_∞^n to obtain the counterexample in ℓ_2^n. Then, using Dvoretzky's Theorem, we extend the counterexample to every Banach space in Theorem 2.3.

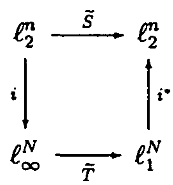
Lemma 2.1. *If $i : \ell_2^n \hookrightarrow X_n \subset \ell_\infty^N$ is an isomorphic inclusion with $\|i\| \leqslant 2$ and $\|i^{-1}\| \leqslant 1$, we have that $I_2 = i \otimes i : \ell_2^n \bigotimes_\varepsilon \ell_2^n \longrightarrow X_n \bigotimes X_n \subset \ell_\infty^N \bigotimes_\pi \ell_\infty^N$ is an isomorphic inclusion with $\|I_2\| \leqslant 4K_G$ and $\|I_2^{-1}\| \leqslant 1$, where K_G is Grothendieck's constant and we are considering the norm in $X_n \bigotimes X_n$ inherited from $\ell_\infty^N \bigotimes_\pi \ell_\infty^N$.*

Proof: Clearly I_2^{-1} factorises as

$$I_2^{-1} : X_n \bigotimes X_n \xrightarrow{\text{id}} X_n \bigotimes_\varepsilon X_n \xrightarrow{i^{-1}\otimes i^{-1}} \ell_2^n \bigotimes_\varepsilon \ell_2^n,$$

which gives us that $\|I_2^{-1}\| \leqslant \|\text{id}\|\,\|i^{-1}\|^2 \leqslant 1$.

To see that $\|I_2\| \leqslant 4K_G$ we take a bilinear form $T : \ell_\infty^N \times \ell_\infty^N \longrightarrow \mathbb{K}$ and consider the operator $S = T(i,i) : \ell_2^n \times \ell_2^n \longrightarrow \mathbb{K}$. It is enough to see that the norm of S as a linear form $S : \ell_2^n \bigotimes_\varepsilon \ell_2^n \longrightarrow \mathbb{K}$ is bounded by $4K_G$. So, we consider the linear operator $\widetilde{S} : \ell_2^n \longrightarrow \ell_2^n$ associated to S and we have the following diagram:



By Grothendieck's Theorem i^* is $1-$summing and $\pi_1(i^*) \leqslant K_G\|i\| \leqslant 2K_G$ and by [**9**, Corollary 5.8] $i^*\widetilde{T}$ is integral with $\iota(i^*\widetilde{T}) \leqslant 2K_G$. Our result follows trivially. □

LEMMA 2.2. *If $i : \ell_2^n \hookrightarrow X_n \subset \ell_\infty^N$ is an isomorphic inclusion with $\|i\| \leqslant 2$ and $\|i^{-1}\| \leqslant 1$, then the operator*

$$I_4 = i \otimes i \otimes i \otimes i : \ell_2^n \bigotimes_\varepsilon \ell_2^n \bigotimes_\varepsilon \ell_2^n \bigotimes_\varepsilon \ell_2^n \longrightarrow \ell_\infty^N \bigotimes_\pi \ell_\infty^N \bigotimes_\pi \ell_\infty^N \bigotimes_\pi \ell_\infty^N,$$

verifies that $\|I_4\| \geqslant (1/128K_G^2)\sqrt{n}$.

PROOF: We consider a generic vector

$$x = \sum_{i,j=1}^n \lambda_{i,j} e_i \otimes e_i \otimes e_j \otimes e_j \in \ell_2^n \bigotimes_\varepsilon \ell_2^n \bigotimes_\varepsilon \ell_2^n \bigotimes_\varepsilon \ell_2^n.$$

As the map $d : e_i \mapsto e_i \otimes e_i$ is an isometric inclusion of ℓ_∞^n in $\ell_2^n \bigotimes_\varepsilon \ell_2^n$, we have

$$\|x\| = \Big\|\sum_{i,j=1}^n \lambda_{ij} e_i \otimes e_j\Big\|_{\ell_\infty^n \bigotimes_\varepsilon \ell_\infty^n}.$$

Moreover, using Lemma 2.1, we have that $I_2 d : \ell_\infty^n \hookrightarrow Y_n \subset \ell_\infty^N \bigotimes_\pi \ell_\infty^N$ is an isomorphic inclusion with $\|I_2 d\| \leqslant 4K_G$ and $\|(I_2 d)^{-1}\| \leqslant 1$. But ℓ_∞^n is a 1-injective space, which implies that Y_n is $4K_G$-complemented in $\ell_\infty^N \bigotimes_\pi \ell_\infty^N$. As π preserves complemented subspaces,

$$\begin{aligned} \|I_4(x)\| &\geqslant (1/(4K_G)^2)\Big\|(I_2 d) \otimes (I_2 d)\Big(\sum_{i,j=1}^n \lambda_{ij} e_i \otimes e_j\Big)\Big\|_{Y_n \bigotimes_\pi Y_n} \\ &\geqslant (1/(4K_G)^2)\Big\|\sum_{i,j=1}^n \lambda_{ij} e_i \otimes e_j\Big\|_{\ell_\infty^n \bigotimes_\pi \ell_\infty^n}. \end{aligned}$$

By, for example, [**14**, Proposition 7], there exist scalars λ_{ij} such that

$$\Big\|\sum_{i,j=1}^n \lambda_{ij} e_i \otimes e_j\Big\|_{\ell_\infty^n \bigotimes_\varepsilon \ell_\infty^n} \leqslant 1,$$

$$\Big\|\sum_{i,j=1}^n \lambda_{ij} e_i \otimes e_j\Big\|_{\ell_\infty^n \bigotimes_\pi \ell_\infty^n} \geqslant \frac{1}{8}\sqrt{n}.$$

https://doi.org/10.1017/S0004972700034717 Published online by Cambridge University Press

Hence, we can conclude that

$$\|I_4\| \geqslant \frac{1}{128K_G^2}\sqrt{n}.$$

∎

THEOREM 2.3. *Let X be an infinite dimensional Banach space and $n \geqslant 4$. There exists an extendible n-linear form $T : X \times \cdots \times X \longrightarrow \mathbb{K}$ that is not integral.*

PROOF: It is equivalent to show that, if $j : X \hookrightarrow \ell_\infty(B_{X^*})$ is the canonical isometric inclusion, the linear operator

$$J_n = j \otimes \cdots \otimes j : X \bigotimes_\varepsilon \cdots \bigotimes_\varepsilon X \longrightarrow \ell_\infty(B_{X^*}) \bigotimes_\pi \cdots \bigotimes_\pi \ell_\infty(B_{X^*})$$

is not bounded.

For this, it is enough to show that J_4 is not bounded. Now, by Dvoretzky's Theorem [**9**, Theorem 19.1], for each $n \in \mathbb{N}$ we can find a subspace Z_n of dimension n in X and an isomorphism $k : \ell_2^n \hookrightarrow Z_n$ with $\|k\| \leqslant \sqrt{2}$ and $\|k^{-1}\| \leqslant 1$. Moreover, as $\ell_\infty(B_{X^*})$ is an $\mathcal{L}_{\infty,\lambda}$ space for every $\lambda > 1$, we can find $N \in \mathbb{N}$, an N-dimensional subspace $W_N \subset \ell_\infty(B_{X^*})$ such that $jk(\ell_2^n) \subset W_N$ and an isomorphism $\widetilde{h} : W_N \hookrightarrow \ell_\infty^N$ with $\|\widetilde{h}\| \leqslant \sqrt{2}$ and $\|\widetilde{h}^{-1}\| \leqslant 1$. We define h to be the extension of $\widetilde{h}$ to $\ell_\infty(B_{X^*})$ with the same norm.

If we now let $i = hjk : \ell_2^n \hookrightarrow Y_n \subset \ell_\infty^N$, then the hypothesis of Lemma 2.2 is satisfied. So we obtain $\|I_4\| \geqslant \left(1/128K_G^2\right)\sqrt{n}$.

But, if $K_4 = k \otimes k \otimes k \otimes k$ and $H_4 = h \otimes h \otimes h \otimes h$, then $I_4 = H_4 J_4 K_4$, and therefore

$$\frac{1}{128K_G^2}\sqrt{n} \leqslant \|I_4\| \leqslant \|h\|^4 \|J_4\| \|k\|^4 \leqslant 16 \|J_4\|$$

for every n. ∎

Standard arguments can lead us now to the symmetric version, that is

THEOREM 2.4. *Let X be an infininte dimensional Banach space and $n \geqslant 4$. There exists an extendible n-homogeneous polynomial $P : X \longrightarrow \mathbb{K}$ that is not integral.*

Similarly, it is straightforward to change the above reasoning to obtain Theorem 2.3 for different Banach spaces.

THEOREM 2.5. *Let $X_1, \ldots, X_n$ be arbitrary infinite dimensional Banach spaces with $n \geqslant 4$. There exists an extendible n-linear form $T : X_1 \times \cdots \times X_n \longrightarrow \mathbb{K}$ that is not integral.*

REFERENCES

[1] R. Arens, 'The adjoint of a bilinear operator', *Proc. Amer. Math. Soc.* **2** (1951), 839–848.

[2] R. Arens, 'Operations induced in function classes', *Monatsh. Math.* **55** (1951), 1–19.

[3] R. Aron and P.D. Berner, 'A Hahn-Banach extension theorem for analytic mappings', *Bull. Soc. Math. France* **106** (1978), 3–24.

[4] F. Cabello, R. García and I. Villanueva, 'Extension of multilinear operators on Banach spaces', *Extracta Math.* **15** (2000), 291–334.

[5] D. Carando, 'Extendibility of polynomials and analytic functions on ℓ_p', *Studia Math.* **145** (2001), 63–73.

[6] D. Carando and I. Zalduendo, 'A Hahn-Banach theorem for integral polynomials', *Proc. Amer. Math. Soc.* **127** (1999), 241–250.

[7] J.M.F. Castillo, R. García, and J.A. Jaramillo, 'Extensions of bilinear forms on Banach spaces', *Proc. Amer. Math. Soc.* **129** (2001), 3647–3656.

[8] A. Defant and K. Floret, *Tensor norms and operator ideals*, North Holland Math. Studies **176** (North-Holland Publishing Co., Amsterdam, 1993).

[9] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely summing operators* (Cambridge Univ. Press, Cambridge, 1995).

[10] N. Kalton, 'Locally complemented subspaces and $\mathcal{L}_p$-spaces for $0 < p < 1$', *Math. Nachr.* **115** (71–97).

[11] P. Kirwan and R. Ryan, 'Extendibility of homogeneous polynomials on Banach spaces', *Proc. Amer. Math. Soc.* **124** (1998), 1023–1029.

[12] M. Lindström and R. Ryan, 'Applications of ultraproducts to infinite dimensional holomorphy', *Math. Scand.* **71** (1992), 229–242.

[13] R. A. Ryan, 'Dunford-Pettis properties', *Bull. Acad. Polon. Sci. Ser. Sci. Math.* **27** (1979), 373–379.

[14] C. Schütt, 'Unconditionality in tensor products', *Israel J. Math.* **31** (1978), 209–216.

Área de Matemática Aplicada
Departamento de Matemáticas y Física Aplicadas y
Ciencias de la Naturaleza
Escuela Superior de Ciencias Experimentales y Tecnología
Universidad Rey Juan Carlos
Edificio Departamental II
28933 Móstoles (Madrid)
e-mail: dperezg@escet.urjc.es