CORRESPONDENCE

To the Editor of the *Mathematical Gazette*

DEAR SIR,

When teaching a difficult but fundamental idea one of the essential things, as the experienced teacher knows well, is to slow down the process without losing interest. It is also better to start with the familar and lead towards the unfamilar. I have just tried out a method of teaching the integral of $\frac{1}{x}$ that was designed to do just this.

We investigate the existence of a function y such that $\frac{dy}{dx} = \frac{1}{x}$. Since we can differentiate x^δ for $\delta \neq 0$ and get $\frac{\delta}{x^{1-\delta}}$ we try to approach this by a limiting process.

$$\int \frac{dx}{x^{1-\delta}} = \frac{x^\delta}{\delta} + C$$

In order to give this answer a chance to be finite as $\delta \to 0$, we put $C = \frac{-1}{\delta}$, giving $\frac{x^\delta - 1}{\delta}$. We now graph this for $\delta = 2$, $\delta = 1$, $\delta = \frac{1}{2}$, $\delta = -\frac{1}{2}$ and any other values between $\frac{1}{2}$, $-\frac{1}{2}$ we like to add.

These curves have a common tangent at $x = 1$.

It is useful to consider the curve for which $\delta = \frac{-1}{p}$ where p is a positive integer, since it has a horizontal asymptote $y = p$.

We now have a sequence of graphs with a gap, which we can fill graphically by inserting a curve which, for $x > 1$, is below the lowest of those for which δ is >0 and above the highest of those for which δ is <0.

We then see that we have a function that tends to infinity more slowly than any positive power of x, and has unit slope at (1, 0). We next try to analyse this result.

Put

$$y = \underset{\delta \to 0}{Lt} \frac{x^\delta - 1}{\delta}.$$

In order to deal with the exponent δ, we take logs, first writing $x^\delta = 1 + z$. Then $\delta \,.\, \log x = \log (1 + z)$. Substituting this value of δ in the denominator of y, we have

$$y = \underset{\delta \to 0}{Lt} \left\{ \log x \quad \frac{z}{\log (1 + z)} \right\}$$

Provided $z/\log (1 + z) \to$ a limit L as $\delta \to 0$, $y = L \log x$; and since $z/\log (1 + z)$ is clearly a function of the base of logarithms as well as a function of z, we suppose we can choose the base (e) so that the limit $L = 1$. Then $y = \log_e (x)$. This process lacks rigour, since, apart from this doubtful last step, we have assumed the existence of a limit from

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a perusal of a graph and that

$$\frac{d}{dx}\left\{\underset{\delta\to 0}{Lt}\left(\frac{x^\delta - 1}{\delta}\right)\right\} = \underset{\delta\to 0}{Lt}\left\{\frac{d}{dx}\left(\frac{x^\delta - 1}{\delta}\right)\right\}.$$

We therefore begin all over again to find an argument that is more readily justifiable. Personally, I next try to differentiate $\log x$ from first principles and find inevitably that this process involves the same limit as held us up before. I then go straight to $\int_1^x \frac{dt}{t}$ and use this as a new definition for $\log_e x$. It gives the same function as the one we interpolated on the set of graphs with which we began.

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P.S.

A mistake in the textbook I am using suggests a useful introduction to "conditions for validity."

$$\text{"Show that } \tan^{-1}\frac{1+x}{1-x} = \frac{\pi}{4} + \tan^{-1} x\text{"}$$

We differentiate $\tan^{-1}\left(\frac{1+x}{1-x}\right) - \tan^{-1} x$ and find the derivative zero. We conclude that the difference is constant, and find it by putting $x = 0$. The result is false for $x > 1$.

We are led to the condition "Provided y is a (continuous and) differentiable function of x, and $\frac{dy}{dx} = 0$, then y is constant."

$$\left(\text{Tan}^{-1}\frac{1+x}{1-x} \text{ is discontinuous at } x = 1.\right)$$

To the Editor of the *Mathematical Gazette*

DEAR SIR,

May I suggest a new approach to one corner of our nomenclature? For "anti-log" there are neither apologists nor alternatives. Yet our verb for the inverse operation to taking logs seems to carry the implication that one is either opposed to logs (anti-capital punishment) or compensating for them (anti-missile missile).

Similarly generation of students after generation have to be cautioned that $\sin^{-1} A$ does not mean $1/\sin A$ [just as $\sin^2 A$ does not mean $\sin(\sin A)$.] Here again a perfectly legitimate inverse function is handicapped by having incompletely unambiguous symbolism.

Should we replace anti-log by gol

$\sin^{-1}$	by nis
$\cos^{-1}$	by soc
$\tan^{-1}$	by nat,

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