ON DISCRIMINANTS OF BINARY QUADRATIC FORMS WITH A SINGLE CLASS IN EACH GENUS

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1. Introduction. Consider the classes of positive, primitive binary quadratic forms $ax^2 + bxy + cy^2$ of discriminant $-\Delta = d = b^2 - 4ac < 0$. Dickson (**2**, p. 89) lists 101 values of Δ such that $-\Delta$ is a discriminant having a single class in each genus. The largest value given is 7392, and Swift (**7**) has shown that there are no more up to 10^7. Sixty-five of these values are divisible by 4. For these values, $\Delta/4$ is called an idoneal number; its properties were investigated by Euler.

We write as usual

$$L_k(s) = \sum_1^\infty \chi(n)n^{-s}, \qquad \Re(s) > 0,$$

where throughout $\chi(n)$ is a real non-principal character modulo k; also $\zeta(s)$ is the Riemann zeta function defined for $R(s) > 1$ by

$$\zeta(s) = \sum_1^\infty n^{-s}.$$

We prove the two theorems:

THEOREM I. *If $\Delta > 10^{60}$, there is at most one fundamental discriminant $-\Delta$ with a single class in each genus.*

THEOREM II. *If $L_k(53/54) \geqslant 0$ for $k > 10^{14}$, there are for $\Delta > 10^{14}$ no fundamental discriminants $-\Delta$ with a single class in each genus.*

Chowla (**1**) proved that as d approaches $-\infty$, the number of classes in each genus tends to ∞, so that after some indeterminate point, there are no discriminants with a single class in each genus. This also follows from the well-known inequality of Siegel (**6**) which states that $L_k(1) > k^{-\epsilon}$, $k > k_0(\epsilon)$.

If $h(d)$ is the class number, then for fundamental discriminants $d < -4$,

$$h(d) = \frac{\sqrt{\Delta}}{\pi}\sum_1^\infty \left(\frac{d}{n}\right)\frac{1}{n} = \frac{\sqrt{\Delta}}{\pi} L_\Delta(1),$$

since the Kronecker symbol is a real non-principal character modulo Δ. The number of genera into which these classes are divided is either 2^{t-1} or 2^t, where t is the number of distinct prime factors of d.

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2. Some lemmas.

LEMMA 1. $|\zeta(\frac{1}{2} + it)| \leqslant 2(|t| + 1)$.

Since **(8**, p. 14**)**

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{x-[x]}{x^{s+1}}\,dx, \qquad \Re(s) > 0,$$

$$|\zeta(\tfrac{1}{2} + it)| \leqslant \left|\frac{\frac{1}{2} + it}{-\frac{1}{2} + it}\right| + (\tfrac{1}{2} + |t|)\int_1^\infty x^{-3/2}dx = 1 + 2(\tfrac{1}{2} + |t|).$$

LEMMA 2. $|L_k(\frac{1}{2} + it)| \leqslant (2|t| + 1)\sqrt{k}\log k$.

Let

$$S(x) = \sum_{n \leqslant x} \chi(n).$$

Then, for $\Re(s) > 0$,

$$L_k(s) = \sum_1^\infty \frac{S(n) - S(n-1)}{n^s} = \sum_1^\infty S(n)\{n^{-s} - (n+1)^{-s}\}$$

$$= \sum_1^\infty S(n)\, s\int_n^{n+1} \frac{dx}{x^{s+1}} = s\int_1^\infty \frac{S(x)}{x^{s+1}}\,dx.$$

But $|S(x)| \leqslant \sqrt{k}\log k$ **(5**, Satz 494**)**, hence,

$$|L_k(\tfrac{1}{2} + it)| \leqslant (\tfrac{1}{2} + |t|)\sqrt{k}\log k\int_1^\infty x^{-3/2}\,dx = 2\sqrt{k}(\tfrac{1}{2} + |t|)\log k.$$

We define for complex $s \neq 1$,

$$F(s) = \zeta(s)\, L_k(s).$$

For $\Re(s) > 1$, we write

$$\zeta(s)L_k(s) = \sum_1^\infty a_n n^{-s}.$$

Then $a_1 = 1$, $a_n \geqslant 0$, and $a_n \geqslant 1$ if $n = r^2$ **(3**, p. 428**)**. Let

$$G(x) = \sum_1^\infty a_n e^{-nx}, \qquad x > 0.$$

By a theorem of Mellin **(5**, Satz 231**)**,

$$e^{-x} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)x^{-s}ds, \qquad x > 0.$$

Therefore

$$G(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)x^{-s}F(s)\,ds.$$

This integral can be evaluated by applying Cauchy's Theorem to the rectangle with vertices $2 \pm Ti$, $\frac{1}{2} \pm Ti$, $T > 0$. On the horizontal paths, the integral has the order **(5**, Satz 229, Satz 407**)**

$$O\left(\frac{T^{5/2}}{e^{\frac{1}{2}\pi T}\sqrt{x}}\right).$$

Letting $T \to \infty$, we obtain, because of the singularity at $s = 1$,

$$G(x) = \frac{L_k(1)}{x} + \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(s) x^{-s} F(s)\, ds. \tag{1}$$

LEMMA 3.

$$|\Gamma(\tfrac{1}{2} + it)| = \frac{\sqrt{\pi}}{\sqrt{\cosh \pi t}}.$$

This follows from $\Gamma(s)\,\Gamma(1 - s) = \pi/\sin s\pi$.

From Lemma 3,

$$|\Gamma(\tfrac{1}{2} + it)| \leqslant \sqrt{2\pi}\, e^{-\frac{1}{2}\pi|t|}. \tag{2}$$

LEMMA 4. *If* $L_k(53/54) \geqslant 0$ *for* $k > 10^{14}$, *then*

$$L_k(1) > \frac{1}{54} k^{1/27}.$$

From (1), (2), Lemmas 1 and 2, we obtain

$$\left|G(x) - \frac{L_k(1)}{x}\right| \leqslant \frac{1}{2\pi\sqrt{x}}\int_{-\infty}^{\infty} \sqrt{2\pi}\, e^{-\frac{1}{2}\pi|t|} 2\sqrt{k}(2|t|^2 + 3|t| + 1)\log k\, dt$$

$$= \frac{2\sqrt{2k}\log k}{\sqrt{\pi x}}\int_0^{\infty}(2t^2 + 3t + 1)\, e^{-\frac{1}{2}\pi t}\, dt$$

$$= \frac{2\sqrt{2k}\log k}{\sqrt{\pi x}}\left(\frac{32}{\pi^3} + \frac{12}{\pi^2} + \frac{2}{\pi}\right) < \frac{5\sqrt{k}\log k}{\sqrt{x}},$$

and

$$\left|G\left(\frac{x}{k}\right) - \frac{kL_k(1)}{x}\right| < \frac{5k\log k}{\sqrt{x}}. \tag{3}$$

Next for $\Re(s) > 1$,

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1}\, dx = \left(\frac{n}{k}\right)^s \int_0^{\infty} e^{-nx/k} x^{s-1}\, dx,$$

$$k^s \Gamma(s)\, F(s) = \int_0^{\infty} x^{s-1} G\left(\frac{x}{k}\right) dx.$$

Therefore

$$k^s\Gamma(s)F(s) - \frac{kL_k(1)t^{s-1}}{s-1} = \int_0^t x^{s-1}\left\{G\left(\frac{x}{k}\right) - \frac{kL_k(1)}{x}\right\}dx + \int_t^{\infty} x^{s-1} G\left(\frac{x}{k}\right) dx \tag{4}$$

$$= I_1 + I_2.$$

From now on suppose $53/54 \leqslant s < 1$. Then (4) still holds on noting (3).

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Now set $t = 1/k$. Then, for $k > 10^{14}$,

$$I_2 = \int_{k^{-1}}^{\infty} x^{s-1} G\left(\frac{x}{k}\right) dx = k^s \int_{k^{-2}}^{\infty} x^{s-1} G(x)\, dx > k^s \int_{k^{-2}}^{1} G(x)\, dx$$

$$\geqslant k^s \int_{k^{-2}}^{1} \sum_{r=1}^{\infty} e^{-r^2 x}\, dx \geqslant k^s \sum_{r=1}^{100} \frac{1}{r^2} (e^{-r^2/k^2} - e^{-r^2})$$

$$\geqslant k^s \left\{ \left[\frac{\pi^2}{6} - 10^{-2} \right] [1 - 10^{-24}] - \sum_{1}^{3} \frac{e^{-r^2}}{r^2} - 97 \frac{e^{-16}}{16} \right\}$$

$$\geqslant \frac{5}{4} k^s.$$

$$|I_1| < \int_0^t x^{s-1} \frac{5k \log k}{\sqrt{x}} dx = \frac{5kt^{s-\frac{1}{2}}}{s - \frac{1}{2}} \log k.$$

Hence

$$I_1 + I_2 > \frac{5}{4} k^s - \frac{5k^{(3/2)-s}}{s - \frac{1}{2}} \log k;$$

and it is easily seen that for $k > 10^{14}$

$$I_1 + I_2 > k^s.$$

To complete the proof of Lemma 4, take $s = 53/54$. Then $\zeta(s) < 0$ and $L_k(s) \geqslant 0$. Hence the first term of (4) is non-positive and so

$$\frac{kL_k(1)t^{s-1}}{1 - s} > k^s$$

or

(5) $$L_k(1) > (1 - s)k^{2(s-1)}.$$

This is the result, since (5) holds at 53/54.

Lemma 5. *If* $-d = \Delta > 10^{14}$ *then* $2^t < \Delta^{0\cdot 3}$, *and if* $-d = \Delta > 10^{60}$ *then* $2^t < \Delta^{0\cdot 2}$.

The smallest positive integer with r prime factors is the product of the first r primes. Let this product be P_r. Then the lemma follows easily by induction, since if

$$2^r < (P_r)^m,$$

$$2^{r+1} < 2(P_r)^m < (P_{r+1})^m, \qquad p_{r+1} > 2^{1/m},$$

and $r = 13$ is the smallest value of r such that $P_r > 10^{14}$, and $r = 37$ is the smallest value of r such that $P_r > 10^{60}$.

3. Proof of the theorems. We first prove Theorem II. From

$$h(d) = \frac{\sqrt{\Delta}}{\pi} L_\Delta(1),$$

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and Lemma 4, we have for $\Delta > 10^{14}$,

$$h(d) > \frac{\sqrt{\Delta}}{\pi} \frac{1}{54\,\Delta^{1/27}} = \frac{\Delta^{25/54}}{54\pi}.$$

By Lemma 5, the number of genera is less than $\Delta^{0\cdot 3}$ for $\Delta > 10^{14}$. Therefore the theorem is true whenever

$$\frac{\Delta^{25/54}}{54\pi} > \Delta^{0.3}$$

which holds for $\Delta > 10^{14}$.

We now prove Theorem I. We assume there are two such discriminants d_1, d_2 with $\Delta_1 > \Delta_2 > 10^{60}$ and show that this leads to a contradiction. The tests given by Swift **(7)** for a discriminant to have more than a single class in each genus show that if d has a single class in each genus, then d, $d/4$, or $d/16$ is a fundamental discriminant. From this Theorem 1 can be extended to all discriminants without difficulty but with tedium.

Landau **(4**, p. 281**)** proved

$$\text{(6)} \qquad \frac{h(d_1)}{\sqrt{\Delta_1}\log^2 \Delta_1} + \frac{h(d_2)}{\sqrt{\Delta_2}\log^2 \Delta_2} > \frac{1}{5 \log^5 (\Delta_1 \Delta_2)}.$$

By assumption

$$\text{(7)} \qquad h(d_1) \leqslant 2^{t_1} < \Delta_1^{\delta};\ h(d_2) \leqslant 2^{t_2} < \Delta_2^{\delta}, \qquad \delta < 1/5,$$

where the upper bound for δ follows from Lemma 4. From (6)

$$\frac{2}{\Delta_1^{\frac{1}{2}-\delta} \log^2 \Delta_1} > \frac{1}{5 \log^5 (\Delta_2^2)} = \frac{1}{160 \log^5 \Delta_2},$$

or

$$\text{(8)} \qquad \log \Delta_2 > \Delta_1^{(1-2\delta)/10}.$$

Next define

$$P(s) = \zeta(s)\, L_{k_1}(s)\, L_{k_2}(s)\, L_{k_1,k_2}(s),$$

where χ_1, χ_2, the characters in

$$L_{k_1}(s),\ L_{k_2}(s),$$

are real primitive non-principal characters modulo k_1 and k_2, $k_1 \neq k_2$. Also

$$L_{k_1,k_2}(s) = \sum_1^{\infty} \frac{\chi_1(n)\, \chi_2(n)}{n^s}.$$

Write for $\Re(s) > 1$

$$P(s) = \sum_1^{\infty} b_n n^{-s}.$$

Again $b_1 = 1$, $b_n \geqslant 0$, and $b_n \geqslant 1$ if $n = r^2$. Let

$$H(x) = \sum_1^{\infty} b_n e^{-nx}, \qquad x > 0.$$

As we obtained (1), we now obtain

$$(9) \qquad H(x) = \frac{L^*}{x} + \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(s)\, x^{-s} P(s)\, ds,$$

where

$$L^* = L_{k_1}(1)\, L_{k_2}(1)\, L_{k_1,k_2}(1).$$

From (9), (2), Lemmas 1 and 2, results,

$$\left|H(x) - \frac{L^*}{x}\right|$$
$$< \frac{1}{2\pi\sqrt{x}}\int_{-\infty}^{\infty} \sqrt{2\pi}\, e^{-\frac{1}{4}\pi|t|} 2(|t|+1)(2|t|+1)^3 k_1k_2 \log k_1 \log k_2 \log (k_1k_2)\, dt$$
$$= \frac{2\sqrt{2}\, k_1k_2 \log k_1 \log k_2 \log(k_1k_2)}{\sqrt{\pi x}} \int_0^{\infty} e^{-\frac{1}{4}\pi t}(8t^4 + 20t^3 + 18t^2 + 7t + 1)\, dt$$
$$= \frac{2\sqrt{2}\, k_1k_2 \log k_1 \log k_2 \log(k_1k_2)}{\sqrt{\pi x}}\left(\frac{6144}{\pi^5} + \frac{1920}{\pi^4} + \frac{288}{\pi^3} + \frac{28}{\pi^2} + \frac{2}{\pi}\right)$$
$$< \frac{100\, k_1k_2 \log k_1 \log k_2 \log (k_1k_2)}{\sqrt{x}}.$$

Therefore

$$(10) \qquad \left|H\left(\frac{x}{k_1k_2}\right) - \frac{L^* k_1k_2}{x}\right| < \frac{100(k_1k_2)^{3/2} \log k_1 \log k_2 \log(k_1k_2)}{\sqrt{x}}.$$

As we obtained (4), we now obtain, for $\Re(s) > 1$,

$$(11) \quad (k_1k_2)^s\Gamma(s)\, P(s) - \frac{k_1k_2L^* q^{s-1}}{s-1}$$
$$= \int_0^q x^{s-1}\left\{H\left(\frac{x}{k_1k_2}\right) - \frac{k_1k_2L^*}{x}\right\} dx + \int_q^{\infty} x^{s-1} H\left(\frac{x}{k_1k_2}\right) dx$$
$$= J_1 + J_2.$$

Suppose now $53/54 \leqslant s < 1$. Then (11) still holds by (10). Put $q = (k_1k_2)^{-2}$. As before, we obtain

$$J_2 > \frac{5}{4}(k_1k_2)^s,$$

and

$$|J_1| < 100(k_1k_2)^{3/2} \log k_1 \log k_2 \log(k_1k_2)\, \frac{q^{s-\frac{1}{2}}}{s-\frac{1}{2}}.$$

Hence for $s \geqslant 53/54$

$$J_1 + J_2 > (k_1k_2)^s, \qquad k_1, k_2 > 10^{60}.$$

Therefore from (11) follows

LEMMA 6. *If* $P(s_0) \leqslant 0$, $53/54 \leqslant s_0 < 1$, *then*

$$L_{k_1}(1)\, L_{k_2}(1)\, L_{k_1,k_2}(1) > (1 - s_0)(k_1 k_2)^{3(s_0-1)}$$

for $k_1, k_2 > 10^{60}$.

From (7),

$$L_{\Delta_1}(1) < \frac{\pi}{\Delta_1^{\frac{1}{2}-\delta}}\,;\ L_{\Delta_2}(1) < \frac{\pi}{\Delta_2^{\frac{1}{2}-\delta}}. \tag{12}$$

But

$$\frac{\pi}{\Delta_1^{\frac{1}{2}-\delta}} < \frac{1}{54\Delta_1^{1/27}}, \qquad \Delta_1 > 10^{60},$$

and therefore by Lemma 4,

$$L_{\Delta_1}(53/54) < 0,$$

which means that

$$L_{\Delta_1}(s_0) = 0, \qquad 53/54 < s_0 < 1,$$

and that $P(s_0) = 0$. Furthermore

$$L_{\Delta_1}(1) = (1 - s_0)\, L'_{\Delta_1}(v), \qquad s_0 < v < 1. \tag{13}$$

Let $53/54 \leqslant s < 1$ and $S(x) = \sum_1^x \chi(n)$. Then

$$L'_k(s) = -\sum_1^\infty \frac{\chi(n)\log n}{n^s} = -\sum_{x=1}^\infty S(x)\left[\frac{\log x}{x^s} - \frac{\log(x+1)}{(x+1)^s}\right],$$

so that

$$\begin{aligned} |L'_k(s)| &\leqslant \sum_{x=1}^k x\left|\frac{\log x}{x^s} - \frac{\log(x+1)}{(x+1)^s}\right| + \sum_{k+1}^\infty \sqrt{k}\log k\left|\frac{\log x}{x^s} - \frac{\log(x+1)}{(x+1)^s}\right| \\ &\leqslant \sum_{x=1}^k x\left|\frac{1 - s\log(x+c_x)}{(x+c_x)^{s+1}}\right| + \frac{\log^2 k}{k^{s-\frac{1}{2}}}, \qquad 0 < c_x < 1, \\ &\leqslant 1 + 1 + \sum_{x=3}^k x\frac{s\log x}{x^{s+1}} + \frac{\log^2 k}{k^{s-\frac{1}{2}}} \\ &\leqslant 2 + 54\log k[k^{1/54} - 2^{1/54}] + 10^{-24}\log^2 k, \qquad k > 10^{60}. \\ &< 55\, k^{1/54}\log k. \end{aligned}$$

Also

$$L_{\Delta_1}(1) = \frac{\pi}{\sqrt{\Delta_1}}h(d_1) \geqslant \frac{\pi}{\sqrt{\Delta_1}}.$$

Therefore from (13), we obtain

$$1 - s_0 > \frac{\pi}{55\Delta_1^{14/27}\log\Delta_1}. \tag{14}$$

By (8),

$$\Delta_2 > \exp \Delta_1^{3/50} > \Delta_1^5, \qquad \Delta_1 > 10^{60},$$

or

$$\Delta_1 < \Delta_2^{1/5}. \tag{15}$$

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As is well known (**4**, p. 281),

$$L_{\Delta_1,\Delta_2}(1) < 3\log(\Delta_1\Delta_2).$$

Applying this, (12), (14), (15) to Lemma 6, gives

$$L_{\Delta_2}(1) > \frac{(\Delta_2^{6/5})^{-1/18}}{165(\Delta_2^{1/5})^{\delta+1/54}\log(\Delta_2^{1/5})\log(\Delta_2^{6/5})}$$

$$> \frac{1}{40\Delta_2^{(2/27)+\delta/5}\log^2\Delta_2} > \frac{1}{40\Delta_2^{0\cdot 2}} > \frac{\pi}{\Delta_2^{\frac{1}{2}-\delta}},$$

which contradicts (12).

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