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ON THE GROWTH OF ENTIRE FUNCTIONS BOUNDED ON LARGE SETS

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There have been many indications of a relationship between the rate of growth of an entire function and the "size" of the set, $E(c)$, where the modulus of the function is larger than the constant, c. Theorems of this type include the classical theorem of Wiman on functions of bounded minimum modulus, the Phragmén-Lindelöf Theorem, the Denjoy-Carleman-Ahlfors Theorem, and its many subsequent improvements. These theorems can all be understood as quantitative versions of the statement that if f is an entire function such that, for some $c > 0$, the set $E(c)$ is "small", then the maximum modulus function $M(R, f)$ is forced to grow rapidly with R. The object of this paper is to prove the following theorem, which reinforces the notion just expressed:

THEOREM 1. *Let f be an entire function, let $c > 0$ be fixed, and let Ω be a component of $E(c) = \{z : |f(z)| > c\}$. Put $A(R)$ = area* of $[\Omega \cap \{|z| < R\}]$.

(1) *If* $\limsup_{R\to\infty} A(R)R^{-2} = 0$, *then*

$$\liminf_{R\to\infty} [\log \log M(R, f)]A(R)R^{-2} > 0.$$

(2) *If* $\liminf_{R\to\infty} A(R)R^{-2} = 0$, *then*

$$\limsup_{R\to\infty} [\log \log M(R, f)]A(R)R^{-2} > 0.$$

This theorem is sharp in the sense that there exist many examples of entire functions satisfying the hypothesis of the theorem for which $[\log \log M(R, f)]\cdot A(R)R^{-2}$ is bounded above.

Part (1) of Theorem 1 was announced in [**4**] together with the following:

If $A(R)$ is bounded, then $$\int^{\infty} R[\log \log M(R, f)]^{-1}\, dR < \infty.$$

This latter result had been conjectured by W. K. Hayman [**6**] in response to a problem posed by P. Erdös. We have been informed by letter that the above result with $A(R)$ bounded had already been proved by G. Camera, so we omit its proof and concentrate on the case where $A(R)$ is unbounded.

Theorem 1 gives no information in the case where $A(R)R^{-2}$ is bounded away from zero, but some information on the growth of f is still possible in that case. We refer the reader to Theorem 7.2 of [**3**] and Theorem 1 of [**5**] which together imply:

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THEOREM 2. *Let f be an entire function.*

(A) *If in the above notation,* $\limsup_{R\to\infty} A(R)R^{-2} = p$ *where* $0 \leqq p < 1$, *then*

$$\liminf_{R\to\infty} \frac{\log\log M(R,f)}{\log R} \geqq \lambda(p),$$

where $\lambda(0) = +\infty$, $\lambda(p) = (2p)^{-1}$ *if* $0 < p \leqq 1/2$, *and* $\lambda(p) = 2(1-p)$ *if* $1/2 \leqq p < 1$.

(B) *For* $c > 0$, *let* $h_c = \sup\{h(\Omega) : \Omega$ *is a component of* $E(c)\}$ *where* $h(\Omega)$ *denotes the Hardy number of* Ω *(see* [**3**] *for the definition of the Hardy number). Then* h_c *is non-decreasing with* c. *If* $\lambda = \lim_{c\to\infty} h_c$, *then*

$$\liminf_{R\to\infty} \frac{\log\log M(R,f)}{\log R} \geqq \lambda.$$

We now introduce some notation which will be needed. Let Ω be an unbounded region in the complex plane. For t such that $\Omega \cap \{|z| = t\} \neq \emptyset$, we define

$$t\alpha(t) = \text{the length of the longest arc in } \Omega \cap \{|z| = t\}$$

and

$$\chi(t) = \begin{cases} 0 & \text{if } \{|z| = t\} \subset \Omega \\ 1 & \text{if } \{|z| = t\} \not\subset \Omega. \end{cases}$$

In the proof of Theorem 1 we shall use the following estimate for harmonic measure due to Tsuji [**9**, Theorem 2]:

LEMMA. *Let* $\Omega(R)$ *be a component of* $\Omega \cap \{|z| < R\}$ *and let* $\omega_R(z)$ *be the harmonic measure of* $\{|z| = R\}$ *with respect to* $\Omega(R)$. *Then if* $0 < K < 1$ *and* $|z| < KR/2$ *and* $z \in \Omega(R)$, *we have*

$$\omega_R(z) \leqq \frac{9}{\sqrt{1-K}} \exp\left\{-\pi \int_{2|z|}^{KR} \frac{\chi(t)}{t\alpha(t)}\, dt\right\}.$$

Proof of Theorem 1. Let f, Ω, and $A(R)$ be as described in Theorem 1. For $t > 0$ let $t\theta(t)$ = total length of $\Omega \cap \{|z| = t\}$. Then

$$A(R) = \int_0^R t\theta(t)dt \geqq \int_0^R t\alpha(t)dt$$
$$\geqq 2\pi \int_{[0,R]\cap\{\alpha(t)=2\pi\}} t\, dt.$$

Let $m(R)$ = Lebesgue measure of $[0, R] \cap \{\alpha(t) = 2\pi\}$. Then, since $g(t) = t$ is an increasing function of t, we get

$$A(R) \geqq 2\pi \int_0^{m(R)} t\, dt = \pi[m(R)]^2.$$

Thus if $A(R)R^{-2} \to 0$ as $R \to \infty$, we also have $m(R)R^{-1} \to 0$ as $R \to \infty$. Hence if r_0 is fixed, $[R - r_0 - m(R)] \geqq 0$ for large R, and so

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$$[R - r_0 - m(R)]^2 \leqq \left(\int_{r_0}^{R} \chi(t)dt\right)^2.$$

An application of the Schwarz Inequality yields

$$[R - r_0 - m(R)]^2 \leqq \left(\int_{r_0}^{R} t\theta(t)dt\right)\left(\int_{r_0}^{R} \frac{\chi(t)}{t\theta(t)}dt\right)$$
$$\leqq A(R)\int_{r_0}^{R} \frac{\chi(t)}{t\alpha(t)}dt.$$

Therefore, for large R,

$$(1) \quad \int_{r_0}^{R} \frac{\chi(t)}{t\alpha(t)}dt \geqq \left[1 - \frac{r_0}{R} - \frac{m(R)}{R}\right]^2 \frac{R^2}{A(R)}.$$

On the boundary of $\Omega \cap \{|z| < R\}$, the inequality

$$\log\left(\frac{|f(z)|}{c}\right) \leqq \omega_R(z)\log M\left(R, \frac{f}{c}\right)$$

is satisfied. Therefore, by virtue of the Maximum Principle, this same inequality must be satisfied throughout $\Omega \cap \{|z| < R\}$. Hence if $z \in \Omega$ is fixed, $0 < K < 1$ and $R > 2|z|/K$, the lemma implies that

$$0 < \log\left(\frac{|f(z)|}{c}\right) \leqq \log M\left(R, \frac{f}{c}\right)\frac{9}{\sqrt{1-K}}\exp\left\{-\pi\int_{2|z|}^{KR} \frac{\chi(t)}{t\alpha(t)}dt\right\}$$

and so

$$(2) \quad \log\log M\left(R, \frac{f}{c}\right) \geqq \log\left(\frac{\sqrt{1-K}}{9}\log\frac{|f(z)|}{c}\right) + \pi\int_{2|z|}^{KR} \frac{\chi(t)}{t\alpha(t)}dt.$$

If we combine inequalities (1) and (2), we may conclude that for all large R,

$$\log\log M\left(R, \frac{f}{c}\right) \geqq \log\left(\frac{\sqrt{1-K}}{9}\log\frac{|f(z)|}{c}\right) + \pi\left[1 - \frac{2|z|}{KR} - \frac{m(KR)}{KR}\right]\frac{(KR)^2}{A(KR)}.$$

Therefore there exists a constant $B = B(f) > 0$ so that for all large R,

$$\log\log M(R, f) \geqq \frac{BR^2}{A(R)}.$$

For the proof of part (2) of Theorem 1, notice that if there exists a sequence $\{R_n\}$ tending to ∞ such that $A(R_n)R_n^{-2} \to 0$, we still have $m(R_n)R_n^{-1} \to 0$ as $n \to \infty$. And this implies that all of the preceding inequalities remain valid with R replaced by R_n so that we are able to conclude that, for all large n,

$$\log\log M(R_n, f) \geqq \frac{BR_n^{\,2}}{A(R_n)}.$$

This completes the proof of Theorem 1.

Theorem 1 does not always give the right order of growth since the Schwarz Inequality can introduce some error and, in many cases, the estimate for harmonic measure given in the lemma is not a very close estimate. For example, if $f(z) = \exp[\exp(z^2)]$, then $A(R)$ grows like a constant times $\log R$. Thus in this case, $R^2/A(R)$ increases as $R^2/\log R$ whereas $\log \log M(R, f) = R^2$.

But there are many examples where Theorem 1 is sharp. An easy example is given by $f(z) = \exp[\exp(z)]$, where $A(R)$ is approximately πR, and hence $A(R)R^{-2} \log \log M(R, f)$ is bounded above as well as being bounded away from zero.

Further examples indicating the sharpness of Theorem 1 can be given using the work of W. Al-Katifi [**2**]. She refined the earlier work of P. B. Kennedy [**7**] for constructing an entire function bounded off a given region D, and then estimating the growth of the function. Her method can be used to construct, for each λ between 0 and 2, an entire function, f_λ, having the following properties:

(i) For large $c > 0$, the set $\{|f_\lambda| > c\}$ is contained in and is only slightly smaller than $\{re^{i\theta} : r > 1 \text{ and } 0 < \theta < r^{-\lambda}\}$.

(ii) $\limsup_{R\to\infty} [\log \log M(R, f_\lambda)]R^{-\lambda} < \infty$.

If in the notation of Theorem 1, $A_\lambda(R)$ denotes the area function defined relative to $\{|f_\lambda| > c\}$, an easy calculation shows that $A_\lambda(R)$ grows no faster than a constant times $R^{2-\lambda}$. Thus Theorem 1 allows one to conclude that

$$\liminf_{R\to\infty} [\log \log M(R, f_\lambda)]R^{-\lambda} > 0.$$

This inequality, together with the inequality of (ii) above, indicates that Theorem 1 gives just the right order of growth for the functions f_λ.

We conclude by remarking that the inequalities in both part (1) and part (2) of Theorem 1 would remain valid if $A(R)$ were replaced by $[A(R) - A(R/2)]$. This would give significantly better estimates than those of Theorem 1 in the case where $A(R)$ is bounded, since then $[A(R) - A(R/2)]$ tends to zero as $R \to \infty$.

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