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The algebraic-geometric AKNS potentials

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Abstract. We characterize the algebraic-geometric potentials for the Schrödinger and AKNS operators using the Weyl m-functions and the Floquet exponent for these operators. The characterization is this: among random ergodic Schrödinger operators, the alebraic-geometric potentials are those for which (i) the spectrum is a union of finitely many intervals (or *bands*); (ii) the Lyapounov exponent vanishes on the spectrum.

1. *Introduction*

Our goal in this paper is to give a simple and rather surprising characterization of the finite-band potentials for the Schrödinger and AKNS (after Ablowitz, Kaup, Newell, and Segur ([**2**]; see also [**35**]) operators. Our characterization is based on properties of the Weyl m-functions $m_\pm$ of these operators, together with certain ideas from the theory of algebraic curves and from ergodic theory and dynamical systems.

Although we are primarily interested in the AKNS operator, we begin the discussion by considering the one-dimensional Schrödinger equation

$$L\phi = \left(-\frac{d^2}{dt^2} + y(t)\right)\phi = \lambda\phi$$

with a real bounded measurable potential $y(t)$. In recent years, L has been intensively studied from two points of view. The first is algebraic-geometric; one considers certain quasi-periodic potentials $y(t)$ for which L has finite-band spectrum. Such a potential y can be calculated by evaluating a meromorphic function (abelian function), defined on the Jacobi variety $J(C)$ of a certain hyperelliptic curve C, along a rectilinear winding in $J(C)$ [**8**], [**23**], [**26**]. Thus $y(t)$ can be expressed in terms of the Θ-function of C.

The second point of view is ergodic-theoretic. One begins with a family $Y = \{y\}$ of potentials which form a stationary ergodic process with ergodic measure ν (see § 3 for our definitions of these terms). One introduces a Floquet exponent $w(\lambda) = \beta(\lambda) + i\alpha(\lambda)$ [**20**], [**14**], where $\beta(\lambda)$ is the Lyapounov exponent and $\alpha(\lambda)$ the rotation number-integrated density of states [**20**], [**3**]. We delay definitions to § 3, and note only that $\beta(\lambda)$ determines the exponential growth rate of vectors $\begin{pmatrix}\phi(t)\\ \phi'(t)\end{pmatrix}$, where $L\phi = \lambda\phi$.

One of our objectives is to combine these two view-points in order to characterize the Θ-potentials $y(t)$. Namely let (Y, ν) be a stationary ergodic process, and suppose the following hypotheses are satisfied:

(H1) The spectrum Σ of $L = L_y$ (which is independent of y for ν-a.a. y) consists of finitely many intervals (or *bands*);

(H2) $\beta(\lambda) = 0$ for all $\lambda \in \Sigma$.

Then Y (assumed to equal the support of ν) is a real torus, ν is Haar measure on Y, and each $y \in Y$ is a Θ-potential. Thus the innocuous-looking hypothesis (H2) is very strong indeed.

A second objective is to give an analogous characterization of the finite-band potentials for the AKNS operator

$$K\vec{u} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left[\frac{d}{dt} - y(t) \right] \vec{u} = \lambda \vec{u} \qquad \vec{u} \in \mathbb{C}^2,$$

where

$$y = \begin{pmatrix} a & b+e \\ -b+e & -a \end{pmatrix} \in \mathrm{sl}(2, \mathbb{R}).$$

This is the 'sl$(2, \mathbb{R})$-form' of the AKNS operator; if $b \equiv 0$, the change of variable

$$\vec{v} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \vec{u}$$

takes K to the original 'su$(1, 1)$-form' found in the original paper [**2**]. We again start from (H1) and (H2). If $b = 0$, we find that a and e can be computed by evaluating meromorphic functions A and E, defined on the *generalized* Jacobian $J(C_0)$ corresponding to a two-point singularization C_0 of a hyperelliptic curve C, along a rectilinear winding in $J(C_0)$. It is then easy to show that, starting with such a and e, one can introduce an arbitrary function b in such a way that (H1) and (H2) remain satisfied.

We are going to give a detailed derivation of the formulae for the finite-band AKNS potentials (both periodic and non-periodic). One reason we do so is the elementary nature of our derivation. But the main reason, which also forms the third objective of this paper, is to show how the Weyl m-functions and the Floquet exponent can be used to go simply and naturally from hypotheses (H1), (H2) to precise formulae for the potentials $y(t)$. In particular, the usual Floquet theory for periodic systems is rendered completely unnecessary, and the source of the curve C is made very clear. We feel that the m-functions provide a better point of departure for discussing these algebraic-geometric potentials than, for example, the Green's function, or the Baker function.

In the above considerations, we use a preliminary spectral theoretic result which is of independent interest; our fourth and final objective is to prove it. Consider e.g. the AKNS operator

$$K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left[\frac{d}{dt} - y(t) \right].$$

Assume that $y(t)$ is both 'positive and negative Poisson recurrent'; see § 2. Consider the half-line boundary value problems

$$K^{\pm}\vec{u}=\lambda\vec{u}, \qquad \vec{u}=\begin{pmatrix}u_1\\u_2\end{pmatrix}\in\mathbb{C}^2;$$
$$u_1(0)=0, \qquad \vec{u}\in L^2((0,\pm\infty),\mathbb{C}^2).$$

Our result is that the essential spectra $\Sigma^{\pm}$ or $K^{\pm}$ are equal, and that these sets coincide with the complement of the set of λ in $\mathbb{C}$ for which the differential system

$$\vec{u}'=y(t)+\begin{pmatrix}0 & -\lambda\\ \lambda & 0\end{pmatrix}\vec{u}, \qquad {}'=\frac{d}{dt},$$

has an *exponential dichotomy* (i.e. the solution space has a hyperbolic splitting; see e.g. [**6**], [**31**], [**33**]). A corollary of this result is that, if $I\subset\mathbb{R}\backslash\Sigma^{\pm}$ is an interval, then at most one of the two operators $K^{\pm}$ has an eigenvalue in I, and there can be at most one such eigenvalue. This refines a result of Hartman [**16**].

An outline of the paper is as follows. In § 2, we introduce the Weyl functions and prove the spectral-theoretic result we need. In § 3, hypotheses (H1) and (H2) are introduced. We adapt and extend a fundamental argument of Kotani [**22**] to show how the Weyl functions $m_{\pm}$ fit together to form a single meromorphic function on a curve C. The description of $y(t)$ then proceeds along well-known lines in the Schrödinger case (e.g. [**26**]). In § 4, we give a self-contained, detailed, and elementary derivation of explicit expressions for the finite-band AKNS potentials. The Weyl functions and the generalized Jacobian are the fundamental tools.

The potentials we obtain form a set of initial conditions for which the 'non-focussing' non-linear Schrödinger equation (NLS) can be explicitly solved. We will not consider this topic, but instead refer to Previato [**30**] for the construction of these solutions of NLS. We wish to note that Previato obtains, with different methods, a description of the *periodic* finite-band AKNS potentials which is equivalent to ours (see also Ablowitz & Ma [**1**]).

2. *A spectral-theoretic result*

In this preliminary section, we introduce the Weyl m-functions, and discuss the result we need concerning the half-line operators $L^{\pm}$, $K^{\pm}$.

We will consider *Poisson recurrent* potentials $y(t)$. We give the definition first in the Schrödinger case. Let

$$\mathscr{L}=\left\{y:\mathbb{R}\to\mathbb{R}\;\middle|\;\sup_t\int_t^{t+1}|y(s)|ds<\infty\right\},$$

with the distribution topology defined as follows: $y_n\to y$ in $\mathscr{L}$ iff $\int_{\mathbb{R}} y_\alpha\phi\,ds\to\int_{\mathbb{R}} y\phi\,ds$ for all $\phi\in C_0^\infty(\mathbb{R})$. Note that the *translation flow*

$$\tau: B\times\mathbb{R}\to B\colon \tau(y,t)(s)=y(t+s)$$

is continuous for each bounded subset $B\subset\mathscr{L}$ (i.e., $\sup_t\int_t^{t+1}|y(s)|\,ds\le M<\infty$ for each $y\in B$). We will often write $\tau_t(y)$ for $\tau(y,t)$.

Suppose now that $y\in\mathscr{L}$ has the following properties:

(P1) $\lim_{\varepsilon\to 0}\sup_t\int_t^{t+\varepsilon}|y(s)|\,ds=0$;

(P2) there exist sequences $t_n\to\infty$, $s_n\to-\infty$ such that the translates $\tau(y,t_n)$ and $\tau(y,s_n)$ satisfy $\lim_{n\to\infty}\tau(y,t_n)=\lim_{n\to\infty}\tau(y,s_n)=y$.

If $y \in \mathscr{L}$ satisfies (P1) and (P2), then we say that y is *positive and negative Poisson recurrent*, or simply *Poisson recurrent.*

Property (P1) has the consequence that the *hull* Y of y is compact. Here Y is defined to be the closure $\mathrm{cl}\{\tau_t(y) | t \in \mathbb{R}\}$ of the set of translates of y.

For the AKNS operator

$$K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left[\frac{d}{dt} - y(t) \right]$$

with $y : \mathbb{R} \to \mathrm{sl}(2, \mathbb{R})$, we define $\mathscr{L}$ as above, except that now $|\cdot|$ is some norm on $\mathrm{sl}(2, \mathbb{R})$. We state properties (P1) and (P2) just as above, and we say that $y \in \mathscr{L}$ is Poisson recurrent if (P1) and (P2) hold. The hull Y of y is also defined just as above, and is compact.

Returning to the Schrödinger equation, fix a potential $y_0 \in \mathscr{L}$ which satisfies (P1), and let Y be the hull of y_0. Consider the family of operators

$$L_y = \frac{-d^2}{dt^2} + y(t) \qquad (y \in Y),$$

and the corresponding collection of ordinary differential equations

$$(1)_y \qquad \vec{u}' = \begin{pmatrix} \phi \\ \phi' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\lambda + y(t) & 0 \end{pmatrix} \vec{u} \qquad (y \in Y, \lambda \in \mathbb{C}).$$

When necessary, we will write $(1)_{y,\lambda}$ to indicate the λ-dependence. Let $\Phi_y(t)$ be the fundamental matrix solution of $(1)_y$ satisfying $\Phi_y(0) = I$. It is easy to show, using Gronwall's inequality [**5**] and (P1), that $(y, t, \lambda) \to \Phi_y^\lambda(t) : Y \times \mathbb{R} \times \mathbb{C} \to \mathrm{SL}(2, \mathbb{C})$ is continuous (for the first and last time, we have indicated the λ-dependence of Φ_y).

For fixed $\lambda \in \mathbb{C}$, the mappings Φ_y induce a flow $\hat{\tau}: Y \times \mathbb{C}^2 \times \mathbb{R} \to Y \times \mathbb{C}^2$, defined as follows:

$$\hat{\tau}((y, \vec{u}), t) = (\tau_t(y), \Phi_y(t)\vec{u}),$$

where τ_t refers to the translation flow on Y. We write $\hat{\tau}_t(y, \vec{u}) = \hat{\tau}((y, \vec{u}), t)$.

2.1. *Definition* [**6**], [**31**]. Fix $\lambda \in \mathbb{C}$. Equations $(1)_y$ are said to have *exponential dichotomy* (ED) if there are positive constants $M > 0$, $\alpha > 0$ and vector sub-bundles $V^\pm \subset Y \times \mathbb{C}^2$ with the following properties:

(i) $V^\pm$ are invariant; that is, $(y, \tau) \in V \Rightarrow \hat{\tau}_t(y, u) \in V^\pm$, $(y \in Y, u \in \mathbb{C}^2)$;

(ii) $(y, \vec{u}) \in V^+ \Rightarrow \|\Phi_y(t)\vec{u}\| \leq Me^{-\alpha t}, (t \geq 0)$;

(iii) $(y, \vec{u}) \in V^- \Rightarrow \|\Phi_y(t)\vec{u}\| \leq Me^{\alpha t}, (t \leq 0)$;

(iv) $Y \times \mathbb{C}^2 = V^+ \oplus V^-$.

If equations $(1)_y$ have ED, then $\dim V^+ = \dim V^- = 1$ (because $\det \Phi_y \equiv 1$).

Now we recall a result of [**18**]. That result is stated in slightly less generality than needed here, but the proof carries over simply to the present case [**19**].

2.2. THEOREM. *Suppose* $y_0 \in \mathscr{L}$ *satisfies* (P1), *and let* Y *be the hull of* y_0. *Then the spectrum of* $L_0 = (-d^2/dt^2) + y_0(t)$ *as a self-adjoint operator on* $L^2(\mathbb{R})$ *equals* $\{\lambda \in \mathbb{C} |$ *equations* $(1)_{y,\lambda}$ *do* <u>*not*</u> *have* ED$\}$.

Since L_0 is self-adjoint on $L^2(\mathbb{R})$, theorem 2.2 implies that, if $\mathrm{Im}\, \lambda \neq 0$, then equations $(1)_{y,\lambda}$ have ED.

Now we introduce the Weyl m-functions, using theorem 2.2. Let $\mathbb{P}^1(\mathbb{C})$ be the usual space of complex lines in $\mathbb{C}^2$, coordinatized as follows: if $l \in \mathbb{P}^1(\mathbb{C})$ contains the vector $\begin{pmatrix} 1 \\ m \end{pmatrix}$, then m is the coordinate of l; if $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in l$, then $m = \infty$. Fix $y \in Y$. For $\lambda \in \mathbb{C}$, $\operatorname{Im} \lambda \equiv 0$, let $m_+(y, \lambda)$ be the m-coordinate of $V^+(y) \overset{\text{def}}{=} V^+ \cap (\{y\} \times \mathbb{C}^2)$, where V^+ is the bundle of theorem 2.2. Similarly, let $m_-(y, \lambda)$ be the m-coordinate of $V^-(y)$.

We pause here to note that $\lambda \to m_\pm(y, \lambda)$ are the classical Weyl m-functions for the operator $L_y = (-d^2/dt^2) + y(t)$ [**34**]. In fact the classical Weyl functions $\tilde{m}_\pm(\lambda)$ are defined as follows. For $\operatorname{Im} \lambda \neq 0$, let $\phi_\pm(t)$ be non-zero solutions of $L_y \phi = \lambda \phi$ which are in $L^2(0, \pm\infty)$. Then $\phi_\pm$ are unique up to constant multiple, because L_y is in the limit-point case at $t = \pm\infty$; see e.g. [**5**]. Define $\tilde{m}_\pm(\lambda) = \phi'_\pm(0)/\phi_\pm(0)$. To see that $m_\pm(y, \lambda) = \tilde{m}_\pm(\lambda)$, note that, if $\begin{pmatrix} \psi_\pm(t) \\ \psi'_\pm(t) \end{pmatrix}$ is a non-zero solution of $(1)_{y,\lambda}$ with $\psi'_\pm(0) = m_\pm(y, \lambda)\psi_\pm(0)$, then by (2.2),

$$|\psi_+^2(t)| + |\psi_+'^2(t)| \leq M^2 e^{-2\alpha t} \qquad (t \geq 0),$$

and

$$|\psi_-^2(t)| + |\psi_-'^2(t)| \leq M^2 e^{2\alpha t} \qquad (t \leq 0).$$

Hence $\psi_\pm = \text{const} \cdot \phi_\pm$.

The preceding discussion also applies to the AKNS operator. Fix $y_0 \in \mathscr{L}$ which satisfies (P1), let $Y = \operatorname{Hull}(y_0)$, and consider the operators

$$K_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left[\frac{d}{dt} - y(t) \right] \qquad (y \in Y),$$

together with the associated differential equations

$$(2)_y \qquad \vec{u}' = \left[y(t) + \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} \right] \vec{u} \qquad (y \in Y, \lambda \in \mathbb{C}).$$

We will write $(2)_{y,\lambda}$ when necessary. Theorem 2.2 is true as stated with

$$K_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left[\frac{d}{dt} - y_0(t) \right]$$

replacing L_0 and $L^2(\mathbb{R}, \mathbb{C}^2)$ replacing $L^2(\mathbb{R})$ [**15**], [**19**]. The Weyl m-functions are defined just as above.

We need two facts concerning $m_\pm(y, \lambda)$. For the Schrödinger operator, they are part of the classical spectral theory (e.g. [**5**]). For the AKNS operator, they are proved, for example, in [**14**], [**15**].

2.3. PROPOSITION. (a) *For* $\operatorname{Im} \lambda \neq 0$, *one has* $\operatorname{sgn}(\operatorname{Im} \lambda \cdot \operatorname{Im} m_\pm(y, \lambda)) = \pm 1$.

(b) *The functions* $\lambda \to m_\pm(y, \lambda)$ *are holomorphic in* $\operatorname{Im} \lambda \neq 0$ $(y \in Y)$.

It follows immediately from theorem 2.2 that $m_\pm$ are jointly continuous in $(y, \lambda) \in Y \times \{\lambda \mid \operatorname{Im} \lambda \neq 0\}$.

2.4. *Geometric discussion.* Fix $y \in Y$, and consider the function $m_+(\lambda) \equiv m_+(y, \lambda)$ (what we will say is true for both the Schrödinger and AKNS operators). Let $I \subset \mathbb{R}$ be an open interval such that m_+ is meromorphic on $I \cup \{\lambda \mid \operatorname{Im} \lambda \neq 0\} \equiv D$. We view

m_+ as a mapping from D into $\mathbb{P}^1(\mathbb{C})$. Let $\mathbb{P}^1(\mathbb{R})$ be the usual one-dimensional projective space; we embed $\mathbb{P}^1(\mathbb{R})$ in $\mathbb{P}^1(\mathbb{C})$ by identifying a real line in $\mathbb{R}^2 \subset \mathbb{C}^2$ with the complex line which it generates. Then $\mathbb{P}^1(\mathbb{R}) \cong \{m \mid m \in \mathbb{R} \text{ or } m = \infty\}$. Since $\operatorname{Im} \lambda \cdot \operatorname{Im} m_+(\lambda) > 0$ for $\operatorname{Im} \lambda \neq 0$, we must have $m_+(\lambda) \in \mathbb{P}^1(\mathbb{R})$ for $\lambda \in I$.

The point we wish to emphasize is this: as λ increases in I, $m_+(\lambda)$ rotates without rest in the direction of increasing m on the circle $\mathbb{P}^1(\mathbb{R})$. This is simply a restatement of the facts that: (i) $dm_+/d\lambda > 0$ if $m_+ \neq \infty$; (ii) $d(1/m_+)/d\lambda < 0$ if $m_+ \neq 0$; these statements in turn follow easily from the inequality $\operatorname{Im} m_+ \cdot \operatorname{Im} \lambda > 0$ ($\operatorname{Im} \lambda \neq 0$).

Similarly we see that, if $m_-(\lambda) \equiv m_-(y, \lambda)$ is meromorphic on D, then $m_-(\lambda)$ rotates without rest on $\mathbb{P}^1(\mathbb{R})$ in the direction of decreasing m, as λ increases in I. Thus m_+ and m_- rotate in opposite directions on $\mathbb{P}^1(\mathbb{R})$ as λ increases in I.

Fix once again an element $y_0 \in \mathscr{L}$ satisfying (P1). We consider the half-line boundary value problems ($y \in Y$)

$$
\begin{array}{ll}
L_y^+ \phi = \lambda \phi & L_y^- \phi = \lambda \phi \\
\phi(0) = 0,\ \phi \in L^2(0, \infty) & \phi(0) = 0,\ \phi \in L^2(-\infty, 0) \\
K_y^+ \vec{u} = \lambda \vec{u} & K_y^- \vec{u} = \lambda \vec{u} \\
u_1(0) = 0,\ \vec{u} \in L^2((0, \infty), \mathbb{C}^2) & u_1(0) = 0,\ \vec{u} \in L^2((-\infty, 0), \mathbb{C}^2)
\end{array}
$$

The boundary conditions are used to define self-adjoint operators on $L^2(0, \pm\infty)$; we use the symbols $L_y^\pm$, $K_y^\pm$ to refer to these operators. Thus, e.g. L_y^+ is the closure of the operator $(-d^2/dt^2) + y(t)$ on $C_0^\infty(\mathbb{R}^+) \subset L^2(0, \infty)$.

From now on we consider only the operators $K_y^\pm$. It will be clear that all statements made (in particular our main results, 2.5 and 2.6) hold also for $L_y^\pm$.

Let us write $R^\pm(y)$ for the *essential resolvent* of $K_y^\pm$; that is, $R^\pm(y) = \{\lambda \in \mathbb{C} \mid \lambda$ is not in the essential spectrum of $K_y^\pm\}$. Thus $R^\pm(y)$ consists of isolated point eigenvalues and points in the resolvent of $K_y^\pm$.

We now state the main results of this section.

2.5. THEOREM. *Let $y_0 \in \mathscr{L}$ satisfy* (P1) *and* (P2); *thus y_0 is positive and negative Poisson recurrent. Let $Y = \operatorname{Hull}(y_0) \subset \mathscr{L}$. Then $R^+(y_0) = R^-(y_0) = \{\lambda \in \mathbb{C} \mid$ equations $(2)_{y,\lambda}$ have exponential dichotomy$\}$.*

2.6. COROLLARY. *Let $I \subset R^+(y_0) = R^-(y_0)$ be an interval. Let $K_0^\pm \equiv K_{y_0}^\pm$. Then K_0^+ (K_0^-) has at most one eigenvalue in I. If K_0^+ (K_0^-) has an eigenvalue in I, then K_0^- (K_0^+) has no eigenvalue in I.*

Corollary 2.6 is a strengthened version (for potentials y_0 satisfying (P1) and (P2)) of the main result of Hartman [**16**].

Proof of 2.5. First we recall some elementary facts about the spectral theory of $K_y^\pm$ for fixed $y \in Y$; they are proved e.g. in [**14**], [**15**]. There are monotone increasing, right-continuous *spectral functions* $\rho_y^\pm(t)$, unique up to an additive constant, whose points of increase are exactly the spectra of $K_y^\pm$. Moreover

$$\pm \frac{\operatorname{Im} m_\pm(y, \lambda)}{\operatorname{Im} \lambda} = \int_{-\infty}^{\infty} \frac{d\rho_y^\pm(t)}{|t - \lambda|^2} \qquad (\operatorname{Im} \lambda > 0).$$

An open interval I is contained in $R^{+}(y)$ if and only if $\lambda \to m_{\pm}(y, \lambda)$ extends meromorphically to $I \cup \{\lambda | \operatorname{Im} \lambda \neq 0\}$. The poles of the meromorphic extension are exactly the (isolated) eigenvalues of $K_y^{\pm}$ in I.

Turning now to 2.5, let $I \subset R^{+}(y_0)$ be an open interval. We show that equations $(2)_{y,\lambda}$ have ED for each $\lambda \in I$.

Let $y \in Y$. By (P2), there exists a sequence $t_n \to \infty$ such that $\tau(y_0, t_n) \to y$. By lemma 6.7 of [**18**], $I \subset R^{+}(y)$.

It is convenient to recall part of the proof of [**18**, lemma 6.7]. Let $T > 0$, and consider the boundary value problem

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left[\frac{d}{dt} - y_0(t) \right] \vec{u} = \lambda \vec{u}$$
$$u_1(0) = u_1(T) = 0.$$

There is a corresponding self-adjoint operator $K_{y_0}^{T}$ on $L^2([0, T], \mathbb{C}^2)$; the spectrum of this operator consists of discrete simple eigenvalues. Let $J \subset I$ be an open interval containing k eigenvalues of $K_{y_0}^{+}$. Then an argument of Hartman [**16**] shows that J contains no more than $k+1$ eigenvalues of $K_{y_0}^{T}$. As a simple corollary of this statement, one has the following: let J contain k eigenvalues of $K_{y_0}^{+}$; then J contains no more than $k+3$ eigenvalues of $K_{\tau(y_0, T)}^{+}$ (and $J \subset R^{+}(\tau(y_0, T))$).

Continuing the proof of 2.5, let $y \in Y$, $t_n \to \infty$, and $\tau(y_0, t_n) \to y$. Let $\rho_n^{+} = \rho_{\tau(y_0, t_n)}^{+}$. Then $\rho_n^{+} \to \rho_y^{+}$ in the sense that $\int_{-\infty}^{\infty} f \, d\rho_n^{+} \to \int_{-\infty}^{\infty} f \, d\rho_y^{+}$ for each $f \in C_0^{\infty}(\mathbb{R})$. Let $J \subset I$ be an interval containing k eigenvalues of $K_{y_0}^{+}$. By the preceding paragraph, $K_{\tau(y_0, t_n)}^{+}$ has $\leq k+3$ eigenvalues in J for each n, and by the convergence of ρ_n^{+} to ρ_y^{+} we see that ρ_y^{+} has $\leq k+3$ points of discontinuity on J (and $J \subset R^{+}(y)$). This holds for *all* $y \in Y$.

Now we use Hartman's argument again: the problem

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left[\frac{d}{dt} - \tau_{-T}(y_0)(t) \right] \vec{u} = \lambda \vec{u}, \tag{3}$$
$$u_1(0) = u_1(T) = 0,$$

has no more than $k+4$ eigenvalues in J $(T > 0)$. Consider the problem

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left[\frac{d}{dt} - y_0(t) \right] \vec{u} = \lambda \vec{u}, \tag{4}$$
$$u_1(-T) = u_1(0) = 0 \qquad (T > 0).$$

The number of eigenvalues in J of this problem is equal to the number of eigenvalues in J of (3), hence is $\leq k+4$.

Now let $T_n \to \infty$ be a sequence such that $\tau(y_0, -T_n) \to y_0$. Let ρ_n be the spectral function of the problem (4) with T_n in place of T. Thus ρ_n is piecewise constant with jump $(\int_{-T_n}^{0} \|\vec{u}_s(t)\|^2 \, dt)^{-1}$ at λ_s, where λ_s is the sth eigenvalue of (4) and $\vec{u}_s$ is the corresponding normalized eigenfunction. Then one can show (see e.g. [**15**, § 3] for the necessary arguments) that $\int_{-\infty}^{\infty} f \, d\rho_n \to \int_{-\infty}^{\infty} f \, d\rho_{y_0}^{-}$ for each $f \in C_0^{\infty}(\mathbb{R})$. Hence $\rho_{y_0}^{-}$ has at most $k+4$ jumps on J. Hence $m_{-}(y_0, \lambda)$ extends meromorphically through J, and therefore through I as well.

We now know that both $m_{\pm}(y_0, \lambda)$ extend meromorphically through I. Hence so does the *spectral matrix*

$$M(y_0, \lambda) = \frac{1}{m_+ - m_-} \begin{pmatrix} 1 & \frac{1}{2}(m_+ + m_-) \\ \frac{1}{2}(m_+ + m_-) & m_+ m_- \end{pmatrix} (y_0, \lambda).$$

It follows [**15**, § 3] that the *full*-line operator K_y has at most isolated eigenvalues on I. However ([**18**, 6.9]) K_y has no isolated eigenvalues. Hence by [**18**, 3.1], $I \subset \mathrm{ED}$, and we conclude that $R^+(y_0) \subset \mathrm{ED}$.

Along entirely similar lines one proves that $R^-(y_0) \subset \mathrm{ED}$. It is straightforward to show that $\mathrm{ED} \subset R^+(y_0) \cap R^-(y_0)$. Hence 2.5 is proved. □

Proof of 2.6. First of all, by a perturbation theorem [**6**], [**32**], the bundles $V^{\pm}$ of definition 2.1 vary continuously in λ. It follows that $m_+(y, \lambda) \neq m_-(y, \lambda)$ for all $\lambda \in I$, $y \in Y$. Now, according to the geometric discussion in 2.4, the functions $\lambda \to m_+(y, \lambda)$ and $\lambda \to m_-(y, \lambda)$ rotate in opposite directions on $\mathbb{P}^1(\mathbb{R})$ as λ increases in I $(y \in Y)$. Hence, in I, only one of the functions m_+, m_- can take on the value ∞, and ∞ can be attained at most once. This proves 2.6, since $m_+(y_0, \lambda) = \infty$ $(m_-(y_0, \lambda) = \infty) \Leftrightarrow \lambda \in I$ is an isolated eigenvalue of $K^+_{y_0}(K^-_{y_0})$. □

2.7. *Remark.* We will use the statement in the proof of 2.6 later, namely that at most one of $m_{\pm}(y, \lambda)$ can take on the value ∞, and ∞ is attained for at most one $\lambda \in I$.

3. *Properties of the Weyl functions*

Our purpose in this section is to study the implications for the Weyl functions $m_{\pm}(y, \lambda)$ of hypotheses (H1) and (H2) in the introduction. Extending an argument of Kotani [**22**], we show that, for each $y \in Y$, the functions $m_{\pm}$ glue together to form a meromorphic function M_y on a fixed hyperelliptic Riemann surface C which is completely determined by the spectrum of L_y. We can then describe with precision the potentials $y(t)$ satisfying (H1) and (H2). For the Schrödinger equation, they are the usual Θ-potentials [**26**], [**8**], [**23**]; fo the AKNS operator, they are described in § 4 using a generalized Jacobian.

We first define what we mean by 'stationary ergodic process'. We start with a subset Y of $\mathscr{L}$, where $\mathscr{L}$ is as defined in § 2. We require that

$$\lim_{\varepsilon \to 0^+} \sup_t \int_t^{t+\varepsilon} |y(s)| \, ds = 0$$

uniformly in $y \in Y$ (this implies that Y is a compact metrizable subset of $\mathscr{L}$), and that Y is invariant under translation: $\tau_t(Y) \subset Y$, where $(\tau_t y)(s) = y(t+s)$ $(y \in \mathscr{L})$. We further fix an *ergodic measure* ν on Y: thus μ is invariant (i.e. $\nu(\tau_t(B)) = \nu(B)$ for each Borel $B \subset Y$ and each $t \in \mathbb{R}$), and satisfies the following indecomposability condition: if B is a Borel subset of Y such that $\nu(B \Delta \tau_t(B)) = 0$ for all $t \in \mathbb{R}$, then either $\nu(B) = 0$ or $\nu(B) = 1$. See e.g. [**25**].

3.1. *Definition.* A *stationary ergodic process* is (in this paper) a triple $(Y, \{\tau_t\}_{t \in \mathbb{R}}, \nu)$ as described above, with the additional property that $\mathrm{Supp}\, \nu = Y$; i.e. if $W \subset Y$ is open, then $\nu(W) \neq 0$.

Let us now see to what extent conditions (P1) and (P2) of § 2 hold for the elements y of a stationary ergodic process. First of all, (P1) holds uniformly in $y \in Y$. As for (P2),

3.2. LEMMA. *For ν-a.a. $y \in Y$, the function $y(t)$ is both positive and negative Poisson recurrent, and moreover the positive and negative semi-orbits $\{\tau_t(y) | t > 0\}$ and $\{\tau_t(y) | t < 0\}$ are dense in Y.*

Proof. Note first that the second statement implies the first statement.

To prove the second statement, let $\{W_n\}_{n=1}^{\infty}$ be a countable base for the topology of Y. It follows from the Birkhoff ergodic theorem (e.g. [**25**]) that, if χ_n is the characteristic function of W_n, then

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \chi_n(\tau_s(y))\, ds = \int_Y \chi_n(y)\, d\nu(y) > 0$$

for ν-a.a. y. The second conclusion of 3.2 follows immediately.

We recall two facts concerning the operators L_y, K_y $(y \in Y)$. The first is a special case of the Oseledec theorem [**27**].

3.3. THEOREM. *Fix $\lambda \in \mathbb{C}$, and let $\Phi_y(t)$ be the fundamental matrix solution of equation $(1)_{y,\lambda}$ (Schrödinger) or $(2)_{y,\lambda}$ (AKNS) satisfying $\Phi_y(0) = I$. Then*

$$\beta(\lambda) \stackrel{\text{def}}{=} \lim_{t\to\infty} \frac{1}{t} \ln \|\phi_y(t)\|$$

exists and is independent of y for ν-a.a. $y \in Y$.

The number $\beta(\lambda)$ is called the *Lyapounov exponent* of equations $(1)_{y,\lambda}$ resp. $(2)_{y,\lambda}$. The transparent proof of the Oseledec theorem given in [**21**] applies in our case.

The second fact concerns the spectrum of the operators L_y, K_y $(y \in Y)$.

3.4. PROPOSITION. *The spectrum Σ_y of the full-line operator L_y, K_y is independent of y for ν-a.a. $y \in Y$. We write Σ for this common spectrum.*

For the Schrödinger operator, 3.4 is proved by Pastur [**29**] and Ishii [**17**]. Their methods can be applied to the AKNS operator. Alternatively, the proof in [**18**] for the AKNS operator with bounded uniformly continuous y extends to the present case [**19**].

We now restate (H1) and (H2) with precision:

(H1) Σ is a finite union of intervals; we write

$$\Sigma = [\lambda_0, \lambda_1] \cup \cdots \cup [\lambda_{2g}, \infty)$$

in the Schrödinger case and

$$\Sigma = (-\infty, \lambda_1] \cup \cdots \cup [\lambda_{2g+2}, \infty)$$

in the AKNS case;

(H2) $\beta(\lambda) = 0$ for Lebesgue-a.a. $\lambda \in \Sigma$.

It follows from 2.1, 2.2 and 3.2 that $\beta(\lambda) > 0$ for all $\lambda \in \mathbb{C} \backslash \Sigma$. Thus (H2) says that β attains the value zero 'where it is allowed to do so'.

We now prove two fundamental lemmas (3.5 and 3.6).

3.5. LEMMA. *Fix a stationary ergodic process as in* 3.1, *and let* $\{L_y: y \in Y\}$ (*resp.* $\{K_y: y \in Y\}$) *be the corresponding families of full-line Schrödinger* (*resp.* AKNS) *operators. Let* $I \subset \mathbb{R}$ *be an open interval, and suppose* $\beta(\lambda) = 0$ *for Lebesgue-a.a.* $\lambda \in I$. *Then for* ν*-a.a.* y, *the functions* $\lambda \to m_{\pm}(y, \lambda)$ *extend holomorphically from the upper half-plane* H^+ *through* I *to* $\mathcal{D} = I \cup \{\lambda \mid \operatorname{Im} \lambda \neq 0\}$. *Write* $h_{\pm}(y, \lambda)$ *for the extension of* $m_{\pm}(y, \lambda)$ *to* $\mathcal{D}$. *Then*

$$\operatorname{Im} h_+(y, \lambda) > 0 > \operatorname{Im} h_-(y, \lambda)$$

on I. *Moreover*

$$h_+(y, \lambda) = \overline{h_-(y, \lambda)} \qquad (\lambda \in I;\ \nu\text{-a.a. } y \in Y).$$

Proof. In the Schrödinger case, this lemma is proved by Kotani [**22**] by means of a beautiful calculation. Thus we need only give the proof in the AKNS case, which we do by adapting Kotani's argument.

First we recall a result from the theory of functions holomorphic with positive imaginary part in H^+.

Result (see e.g. [**10**]). Let $I \subset \mathbb{R}$ be an open interval, let $f: H^+ \to H^+$ be holomorphic, and suppose that $\lim_{\varepsilon \to 0^+} \operatorname{Re} f(\lambda + i\varepsilon) = 0$ for Lebesgue-a.a. $\lambda \in I$. Then f extends holomorphically through I, and any extension $\tilde{f}$ satisfies $\operatorname{Im} \tilde{f} > 0$ on I (and of course $\operatorname{Re} \tilde{f} \equiv 0$ on I).

Note that, by the Schwarz reflection principle, one can find an extension $\tilde{f}$ defined on all of $\mathcal{D} = I \cup \{\lambda \mid \operatorname{Im} \lambda \neq 0\}$.

Next we introduce the Floquet exponent $w = w(\lambda)$ for our family $\{K_y \mid y \in Y\}$ of AKNS operators. See [**14**], [**15**] for a detailed discussion. These papers are inspired by [**20**], which discusses the Floquet exponent for the Schrödinger operator; see also [**3**]. Moser [**24**] uses the Floquet exponent in his discussion of the Θ-potentials for the Schrödinger equation.

The properties of w which we need are the following. First, $-w$ is holomorphic in H^+ and has positive *real* part there. Second, w restricted to a bounded subset of H^+ is uniformly bounded. Third, for $\lambda \in H^+$, $-\operatorname{Re} w(\lambda) = \beta(\lambda) =$ Lyapounov exponent. Fourthly,

$$\frac{dw}{d\lambda} = \int_Y \frac{1 + m_- m_+}{m_- - m_+}(y, \lambda)\, d\nu(y) \qquad (\lambda \in H^+).$$

Motivated by Kotani's computations, we now show that, if $\lambda \in H^+$, then for ν-a.a. y:

$$-4\left(\frac{\operatorname{Re} w}{\operatorname{Im} \lambda} + \operatorname{Im} \frac{dw}{d\lambda}\right) = \int_Y \left[\frac{1 + |m_+|^2}{\operatorname{Im} m_+} - \frac{1 + |m_-|^2}{\operatorname{Im} m_-}\right] \times \left[\frac{(\operatorname{Re} m_- - \operatorname{Re} m_+)^2 + (\operatorname{Im} m_- + \operatorname{Im} m_+)^2}{|m_- - m_+|^2}\right] d\nu.$$

To prove this equality, we start with a formula for $\operatorname{Re} w(\lambda)$: for ν-a.a. $y \in Y$,

$$\operatorname{Re} w(\lambda) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \operatorname{Re}\,[a(t) + (\lambda + b(t) + e(t)) m_+(\tau_t(y), \lambda)]\, dt. \tag{5}$$

We sketch the proof of this formula. First of all, $-\operatorname{Re} w(\lambda) = \beta(\lambda)$, so we must show that $-\beta(\lambda)$ equals the expression on the right-hand side for ν-a.a. $y \in Y$. Let

$V^{\pm} = V^{\pm}(\lambda)$ be the one-dimensional subbundles of $Y \times \mathbb{C}^2$ defined by exponential dichotomy (2.1). Let $\vec{u}(t)$ be a non-zero solution of $(2)_{y,\lambda}$ with $\vec{u}(0) \in V^+ \cap \{y\} \times \mathbb{C}^2$. Using the Oseledec theorem [**27**] and 2.1, one shows that, for ν-a.a. y,

$$-\beta(\lambda) = \lim_{t\to\infty} \frac{1}{t} \ln \|\vec{u}(t)\|.$$

Now $u_2(t) = m_+(\tau_t(y), \lambda) u_1(t)$, and $|m_+|$ is uniformly bounded. Thus (5) follows from the equation $(\ln u_1)' = a + (\lambda + b + e) m_+$.

Now fix $y \in Y$ for a moment, and consider the corresponding Riccati equation for $m = u_2/u_1$:

$$m' = (-\lambda - b + e) - 2am - (\lambda + b + e)m^2,$$

where we set $m = m(t) \equiv m_+(\tau_t(y), \lambda)$ for fixed $\lambda \in H^+$. Taking imaginary parts, we get

$$\begin{aligned}
\operatorname{Im} m' &= -\operatorname{Im}\lambda - 2a \operatorname{Im} m - \operatorname{Im}[(\lambda+b)m^2] - e \operatorname{Im} m^2 \\
&= -\operatorname{Im}\lambda - 2a \operatorname{Im} m - (\operatorname{Re}\lambda + b)\operatorname{Im} m^2 - \operatorname{Im}\lambda \operatorname{Im} m^2 - e \operatorname{Im} m^2 \\
&= -\operatorname{Im}\lambda - 2a \operatorname{Im} m - 2(\operatorname{Re}\lambda + b)\operatorname{Re} m \operatorname{Im} m \\
&\quad - \operatorname{Im}\lambda[(\operatorname{Re} m)^2 - (\operatorname{Im} m)^2] - 2e \operatorname{Re} m \cdot \operatorname{Im} m.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\operatorname{Im} m'}{\operatorname{Im} m} &= -2a - 2(\operatorname{Re}\lambda + b) \operatorname{Re} m + \operatorname{Im}\lambda \operatorname{Im} m - 2e \operatorname{Re} m - \operatorname{Im}\lambda \left(\frac{1 + (\operatorname{Re} m)^2}{\operatorname{Im} m}\right) \\
&= -2a - 2(\operatorname{Re}\lambda + b + e) \operatorname{Re} m + 2 \operatorname{Im}\lambda \operatorname{Im} m \\
&\quad - \operatorname{Im}\lambda \left(\frac{1 + (\operatorname{Im} m)^2 + (\operatorname{Re} m)^2}{\operatorname{Im} m}\right).
\end{aligned}$$

Rewriting, we get

$$\frac{\operatorname{Im} m'}{\operatorname{Im} m} + 2 \operatorname{Re}[a + (\lambda + b + e)m] = \frac{-\operatorname{Im}\lambda}{\operatorname{Im} m}(1 + |m|^2).$$

Using (5) and $\operatorname{Im} m > 0$, we have for ν-a.a. $y \in Y$:

$$-2 \operatorname{Re} w(\lambda) = \operatorname{Im}\lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{1 + |m_+(\tau_t(y), \lambda)|^2}{\operatorname{Im} m_+(\tau_t(y), \lambda)} dt.$$

By the Birkhoff ergodic theorem,

$$-2 \operatorname{Re} w(\lambda) = \operatorname{Im}\lambda \int_Y \frac{1 + |m_+(y, \lambda)|^2}{\operatorname{Im} m_+(y, \lambda)} d\nu(y). \tag{6}$$

In a similar way, one proves

$$2 \operatorname{Re} w(\lambda) = \operatorname{Im}\lambda \int_Y \frac{1 + |m_-(y, \lambda)|^2}{\operatorname{Im} m_-(y, \lambda)} d\nu(y). \tag{7}$$

Now using the fourth property of $w(\lambda)$, and suppressing the arguments (y, λ), we have

$$\begin{aligned}
\operatorname{Im} \frac{dw}{d\lambda} &= \operatorname{Im} \int_Y \frac{1 + m_- m_+}{m_- - m_+} d\nu = \int_Y \operatorname{Im} \frac{(1 + m_- m_+)(\bar{m}_- - \bar{m}_+)}{|m_- - m_+|^2} d\nu \\
&= \int_Y \frac{(1 + |m_-|^2) \operatorname{Im} m_+ - (1 + |m_+|^2) \operatorname{Im} m_-}{|m_- - m_+|^2} d\nu.
\end{aligned}$$

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Combining this formula with (6) and (7), we get

$$-4\left(\frac{\operatorname{Re} w}{\operatorname{Im}\lambda}+\operatorname{Im}\frac{dw}{d\lambda}\right)=\int_Y\left\{\frac{1+|m_+|^2}{\operatorname{Im} m_+}-\frac{1+|m_-|^2}{\operatorname{Im} m_-}\right.$$
$$\left.-4\left[\frac{(1+|m_-|^2)\operatorname{Im} m_+-(1+|m_+|^2)\operatorname{Im} m_-}{|m_--m_+|^2}\right]\right\}d\nu$$
$$=\int_Y[(1+|m_+|^2)\operatorname{Im} m_--(1+|m_-|^2)\operatorname{Im} m_+]$$
$$\times\left[\frac{1}{\operatorname{Im} m_-\operatorname{Im} m_+}+\frac{4}{|m_--m_+|^2}\right]d\nu$$
$$=\int_Y\left[\frac{1+|m_+|^2}{\operatorname{Im} m_+}-\frac{1+|m_-|^2}{\operatorname{Im} m_-}\right]\left[\frac{|m_--m_+|^2+4\operatorname{Im} m_+\operatorname{Im} m_-}{|m_--m_+|^2}\right]d\nu$$
$$=\int_Y\left[\frac{1+|m_+|^2}{\operatorname{Im} m_+}-\frac{1+|m_-|^2}{\operatorname{Im} m_-}\right]$$
$$\times\left[\frac{(\operatorname{Re} m_--\operatorname{Re} m_+)^2+(\operatorname{Im} m_-+\operatorname{Im} m_+)^2}{|m_--m_+|^2}\right]d\nu.$$

This is the formula which we wanted to prove.

Next recall that $\beta(\lambda)=0$ for Lebesgue-a.a. $\lambda\in I$. It is easily seen that, for almost all (in fact *all*) $\lambda\in\mathbb{R}$, one has $\lim_{\varepsilon\to0^+}-\operatorname{Re} w(\lambda+i\varepsilon)=\beta(\lambda)$; see e.g. [**15**]. Using the boundedness property of w, we see that the Schwarz reflection principle applies, and thus w extends holomorphically to $\mathcal{D}=I\cup\{\lambda|\operatorname{Im}\lambda\neq0\}$. In particular $\operatorname{Re} w(\lambda)=0$ for all $\lambda\in I$.

We use these facts in the following way. Let $\lambda\in I$. Then

$$\operatorname{Im}\frac{dw}{d\lambda}(\lambda)=-\lim_{\varepsilon\to0}\frac{\operatorname{Re} w(\lambda+i\varepsilon)}{\varepsilon}\Rightarrow\lim_{\varepsilon\to0}\left(\operatorname{Im}\frac{dw}{d\lambda}(\lambda+i\varepsilon)+\frac{\operatorname{Re} w(\lambda+i\varepsilon)}{\varepsilon}\right)=0.$$

By Fubini's theorem and a standard result on boundary values of holomorphic functions $f: H^+\to H^+$ [**10**], the limits $\lim_{\varepsilon\to0^+}m_\pm(y,\lambda+i\varepsilon)\overset{\text{def}}{=}\hat m_\pm(y,\lambda)$ exist Lebesgue-a.e., for ν-a.a. $y\in Y$. Hence by Fatou's lemma and our formula for $-4((\operatorname{Re} w/\operatorname{Im}\lambda)+\operatorname{Im}(dw/d\lambda))$, we get:

$$\operatorname{Re}\hat m_-(y,\lambda)=\operatorname{Re}\hat m_+(y,\lambda),\tag{8}$$

$$\operatorname{Im}\hat m_-(y,\lambda)=-\operatorname{Im}\hat m_+(y,\lambda),\tag{9}$$

Lebesgue-a.e. in I, for ν-a.a. $y\in Y$.

By (8), (9), and the result alluded to earlier, we see that, for ν-a.a. $y\in Y$, $m_+(y,\lambda)-m_-(y,\lambda)$ extends holomorphically through I to $\mathcal{D}$, with positive imaginary part on I (and on $\mathcal{D}$, in fact). The same is true of $((m_-m_+)/(m_--m_+))(y,\lambda)$. It is now an exercise to show that, for ν-a.a. y, $m_-(y,\lambda)$ and $m_+(y,\lambda)$ both extend holomorphically to $\mathcal{D}$ (hint: consider the quadratic polynomial with roots m_+ and $-m_-$). If $h_\pm(y,\lambda)$ are the extensions, then (8) and (9) imply the last two statements of 3.5. □

In our second lemma we show that the extensions $h_\pm$ of lemma 3.5 satisfy $h_+(y,\lambda)=m_-(y,\lambda)$, $h_-(y,\lambda)=m_+(y,\lambda)$ ($\operatorname{Im}\lambda<0$). This amounts to saying that the potential

y is reflectionless. This statement should be true (with an appropriate definition of 'reflectionless' [7]) for any stationary ergodic set of potentials.

3.6. LEMMA. *Let $I \subset \mathbb{R}$ be as in lemma 3.5. Then for all $y \in Y$, $m_\pm(y, \lambda)$ admits a holomorphic extension $h_\pm(y, \lambda)$ to $\mathscr{D} = I \cup \{\lambda \mid \operatorname{Im} \lambda \neq 0\}$, and $h_\pm(y, \lambda) = m_\mp(y, \lambda)$ $(\operatorname{Im} \lambda < 0)$.*

Proof. We consider only $m_+(y, \lambda)$.

Let $Y_1 = \{y \in Y \mid$ the conclusions of 3.5 hold$\}$, and let $Y_2 = \{y \in Y \mid$ the positive and negative semi-orbits of y are dense in $Y\}$. Then both Y_1 and Y_2 have ν-measure 1. Let $\bar{y}$ be an element of $Y_1 \cap Y_2$, and write $h(\lambda) = h_+(\bar{y}, \lambda)$. Here h_+ is defined by 3.5.

For $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda \neq 0$, consider the corresponding Riccati equation for m:

$$m' + m^2 = -\lambda + y(t) \qquad \text{(Schrödinger)}, \tag{10}$$

$$m' = (-\lambda - b + e) - 2am - (\lambda + b + e)m^2 \qquad \text{(AKNS)}. \tag{11}$$

Suppose $\operatorname{Im} \lambda < 0$. Then for each each real intial condition $m(0) \in \mathbb{R}$, the solution $m(t)$ of (10) (resp. (11)) satisfies $\operatorname{Im} m(t) > 0$ for all $t > 0$. Moreover, if $n = 1/m$, and $n(0) \in \mathbb{R}$, then $\operatorname{Im} n(t) < 0$ for all $t > 0$.

Geometrically, these statements mean the following. View $\{m \mid \operatorname{Im} m > 0\}$ as a disc B in $\mathbb{P}^1(\mathbb{C}) \cong$ Riemann sphere, with boundary $\partial B = \{\infty\} \cup \{m \mid m \in \mathbb{R}\}$. We define a flow $\{\tau_t\}_{t \in \mathbb{R}}$ on $Y \times \mathbb{P}^1(\mathbb{C})$ in the natural way: $\tilde{\tau}_t(y, m_0) = (\tau_t(y), m(t))$, where $m(t)$ satisfies (10) (resp. (11)) with $m(0) = m_0$. (Alternatively, $\tilde{\tau}$ is induced from the flow $\hat{\tau}$ of § 2 by considering the action of $\hat{\tau}$ on complex lines in $\mathbb{C}^2$.) Then $\tilde{\tau}_t(B \cup \partial B) \subset B$ for all $t > 0$; i.e. the closed disc $\bar{B}$ is mapped entirely into its interior for all $t > 0$ (if $\operatorname{Im} \lambda < 0$).

Consider next the sections $S^\pm(\lambda) = \{(y, m_\pm(y, \lambda)) \mid y \in Y\}$ of the sphere bundle $Y \times \mathbb{P}^1(\mathbb{C})$. For fixed $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda \neq 0$, these sections are invariant under the flow $\tilde{\tau}$; this follows from 2.1(i). Moreover, $S^\pm(\lambda)$ are the *only* continuous invariant sections of $Y \times \mathbb{P}^1(\mathbb{C})$. To see this, let S_0 be a third such section. Using the Birkhoff ergodic theorem, we see that $\{y \in Y \mid S_0 \cap \{y\} \times \mathbb{P}^1(\mathbb{C})$ equals neither $m_+(y, \lambda)$ nor $m_-(y, \lambda)\}$ has ν-measure 1. By 3.2, we can find such a y with dense positive semi-orbit $\{\tau_t(y) \mid t > 0\}$. Choose non-zero vectors $\vec{u}_+, \vec{u}_0, \vec{u}_-$ in the complex lines $S^+(\lambda)$, S_0, $S^-(\lambda) \cap \{y\} \times \mathbb{P}^1(\mathbb{C})$ respectively; then these vectorse are pairwise linearly independent. Let $\Phi_y(t)$ be the fundamental matrix solution of $(1)_{y,\lambda}$ resp. $(2)_{y,\lambda}$ such that $\Phi_y(0) = I$. Using 2.1(i) and (ii), we find $\lim_{t \to \infty} \|\Phi_y(t)\vec{u}_-\| = \infty$, and therefore $\liminf_{t \to \infty} \|\Phi_y(t)\vec{u}_0\| = 0$ (because $\det \Phi_y(t) = 1$). However $\lim_{t \to \infty} \|\Phi_y(t)\vec{u}_+\| = 0$, and thus we obtain the contradiction $\liminf_{t \to \infty} \det \Phi_y(t) = 0$.

If one replaces lemma 3.2 with Poincaré recurrence in the above argument, one can show that $S^\pm(\lambda)$ are the only ν-measurable invariant sections of $Y \times \mathbb{P}^1(\mathbb{C})$.

Using 2.3(i), we see that, if $\operatorname{Im} \lambda > 0$, then $S^+(\lambda)$ is the only continuous invariant section in $Y \times B \subset Y \times \mathbb{P}^1(\mathbb{C})$; if $\operatorname{Im} \lambda < 0$, then $S^-(\lambda)$ is the only continuous invariant section in $Y \times B$.

Now return to the fixed point $\bar{y} \in Y_1 \cap Y_2$. Let $\mathscr{D}_0 \subset \mathbb{C}$ be a domain containing $I \cup \{\lambda \mid \operatorname{Im} \lambda > 0\}$ on which $h(\lambda)$ is holomorphic and has positive imaginary part

(3.5). For $t>0$ and $\lambda\in\mathscr{D}_0$, define $g_t(\lambda)$ by $(\tau_t(\bar{y}), g_t(\lambda))=\tilde{\tau}_t(\bar{y}, h(\lambda))$. Then g_t maps $\mathscr{D}_0$ into $H^+\cong B$: if $\operatorname{Im}\lambda>0$ this follows from invariance of $S^+(\lambda)$, if $\lambda\in I$ it follows from $\operatorname{Im} h>0$, and if $\operatorname{Im}\lambda<0$ it follows from $\operatorname{Im} h>0$ and the first part of the present proof. Moreover $g_t:\mathscr{D}_0\to H^+$ is holomorphic because $\Phi_y(t)$ is *linear*. Hence $\{g_t \mid t>0\}$ is a normal family of holomorphic functions on $\mathscr{D}_0$.

Let $y\in Y$ be arbitrary, and let $t_n\to\infty$ be a sequence such that $\tau(\bar{y}, t_n)\to y$. We can choose a subsequence t_m of t_n such that q_{t_m} converges to a holomorphic function v_y, uniformly on compact subsets of $\mathscr{D}_0$. By invariance and continuity of $S^+(\lambda)$, we have $v_y(\lambda)=m_+(y,\lambda)$ for all $\operatorname{Im}\lambda>0$. By an elementary argument, we conclude that $g_{t_n}\to v_y$, uniformly on compact subsets of $\mathscr{D}_0$, for *every* sequence $t_n\to\infty$ such that $\tau(\bar{y}, t_n)\to y$.

Since $\{\tau(\bar{y}, t)\mid t>0\}$ is dense in Y, we can define a limiting function $v_y:\mathscr{D}_0\to H^+$ as above for every $y\in Y$. Since $v_y(\lambda)=m_+(y,\lambda)$ for $\operatorname{Im}\lambda>0$, and since $S^+(\lambda)$ is invariant, we conclude via a normal families argument that the section $V_0=\{(y, v_y(\lambda))\mid y\in Y\}$ is continuous and invariant for each $\lambda\in\mathscr{D}_0$.

For $\operatorname{Im}\lambda<0$, $\lambda\in\mathscr{D}_0$, we have $S^-(\lambda)=V_0$ by uniqueness. Thus we can conclude that $m_+(y,\lambda)$ extends holomorphically through I, and the extension $h_+(y,\lambda)$ equals $m_-(y,\lambda)$ for $\operatorname{Im}\lambda<0$ $(y\in Y)$.

Similarly one proves the statement concerning m_- and its extension h_- to $\mathscr{D}$. This completes the proof of 3.6. □

3.7. *Remark.* It follows from 3.6 that the resolvent of L_y(resp. K_y) equals $\mathbb{C}\backslash\Sigma$ for *all* $y\in Y$. This is proved as follows (we treat the AKNS case). First of all, $\mathbb{C}\backslash\Sigma=\mathscr{E}=\{\lambda\in\mathbb{C}\mid$ equations $(2)_{y,\lambda}$ have ED$\}$ (2.2). It follows that $\mathbb{C}\backslash\Sigma$ is in the resolvent of K_y for all $y\in Y$. On the other hand, if $\lambda\in\operatorname{int}\Sigma$, then it follows from (3.5 and) 3.6 that all solutions of $(2)_{y,\lambda}$ are bounded. This implies (**[9]**) that λ is in the spectrum of K_y. This proves our assertion.

Let C be the Riemann surface of

(i) $\kappa^2=-(\lambda-\lambda_0)\cdots(\lambda-\lambda_{2g})=-\prod_{i=1}^{2g}(\lambda-\lambda_i)$ in the Schrödinger case;

(ii) $\kappa^2=-(\lambda-\lambda_1)\cdots(\lambda-\lambda_{2g+2})=-\prod_{i=1}^{2g+2}(\lambda-\lambda_i)$ in the AKNS case.

Then C is a non-singular, hyperelliptic algebraic curve of genus g. We will show that the m-functions $m_\pm$ define, for fixed $y\in Y$, a meromorphic function on C.

We need some notation. Let S^2 be the λ-sphere. C is precisely $\operatorname{cls}\{(\lambda,\kappa)\mid\lambda\in\mathbb{C}\}\subset S^2\times S^2$ where κ runs over the square roots of the corresponding polynomial. Let $\pi:C\to S^2:(\lambda,\kappa)\to\lambda$ be the projection. In the Schrödinger case, $p_0=\pi^{-1}(\lambda_0),\ldots,p_{2g}=\pi^{-1}(\lambda_{2g})$ are ramification points; in the AKNS case, $p_i=\pi^{-1}(\lambda_i)$ $(1\le i\le 2g+2)$. In the Schrödinger case, $\pi^{-1}(\infty)$ is also a ramification point; in the AKNS case this is not true, and $\pi^{-1}(\infty)$ consists of two points ∞_+,∞_- which we distinguish by requiring that $\operatorname{sgn}\operatorname{Im}\kappa=\pm1$ in a neighbourhood of $\infty_\pm$. There are no other ramification points on C. We use p to denote points of C.

3.8. THEOREM. *Fix a stationary ergodic process Y, and suppose* (H1) *and* (H2) *are satisfied. Then for each $y\in Y$, the functions $m_\pm(y,\lambda)$ define a meromorphic function M_y on C. The map $M: Y\times C\to\mathbb{P}^1(\mathbb{C}):(y,p)\to M_y(p)$ is jointly continuous.*

Proof. We do the AKNS case, then remark on the (completely analogous) Schrödinger case.

It follows immediately from 3.6 that the functions $m_\pm(y, \lambda)$ form a function M_y which is meromorphic on C except possibly at the branch points $p_1, \ldots, p_{2g+2}$ and at ∞_+, ∞_-. Actually, there are two possibilities for M_y; we choose one by requiring that $\operatorname{Im} M_y(p) > 0$ if $\operatorname{Im} \kappa(p) > 0$. The other possibility is $M_y \circ \sigma$, where $\sigma:(\lambda, \kappa) \to (\lambda, -\kappa)$ is the involution on C which interchanges sheets.

Let $p \in \{p_1, \ldots, p_{2g+2}\}$ be a branch point, and let $D \subset C$ be a disc centred at p. Letting $\mathscr{E} = \{\lambda \in \mathbb{C} \mid \text{equations } (2)_{y,\lambda} \text{ have ED}\}$ as in 3.7, we can choose D so that $\pi(\bar{D}) \cap \mathbb{R} \cap \mathscr{E}$ is contained in an open interval I, which can be assumed to have the form $(\lambda_{2i-1}, \lambda_{2i})$.

Let $\bar{y} \in Y$, and let λ_0 be the endpoint in I of $\pi(\bar{D}) \cap I$. Since the bundles $V^\pm$ of 2.1 are continuous in λ ([**6**], [**32**]), we see that $(y, \lambda) \to m_\pm(y, \lambda)$ are jointly continuous on $Y \times D_0$, where now $D_0 \subset \mathbb{C}$ is a disc with centre λ_0 such that $D_0 \cap \mathbb{R} \subset I$. This fact together with the geometric discussion of 2.4 and the inequalities $m_+(y, \lambda) \neq m_-(y, \lambda)$ $(y \in Y, \lambda \in I)$ show that there is a neighbourhood B of $\bar{y}$ in Y such that, for each $y \in B$, the restriction of M_y to $D \backslash \{p\}$ omits the *same* set of infinitely many real values. By the Picard theorem [**4**], M_y extends meromorphically to p, and by the Montel theorem [**4**], $\{M_y \mid y \in B\}$ is a normal family. It follows easily that $(y, p) \to M(y, p)$ is continuous on $B \times D$, and hence on $Y \times D$.

In a neighbourhood of $p = \infty_+$ or ∞_- the argument is even simpler. Choose a disc $D \subset C$ centred at p such that $\pi(\bar{D}) \cap \mathbb{R} \subset (-\infty, \lambda_1) \cup (\lambda_{2g+2}, \infty)$. Then, by 3.6, the restriction of M_y to $D \backslash \{p\}$ omits *every* real value. Hence we can apply the theorems of Picard and Montel just as above to show that M is continuous on $Y \times D$. □

3.9. *Remark.* In the AKNS case, it follows from 3.8 and [**14**], [**15**] that $\lim_{p \to \infty_\pm} M_y(p) = \pm i$. In the Schrödinger case, one has $\lim_{p \to \infty} M_y(p) = \infty$.

3.10. *Remark.* Consider the AKNS operators K_y. It follows from 2.5, 2.7, and 3.8 that each M_y has one and only one pole $P_i(y)$ in each circle $\pi^{-1}[\lambda_{2i-1}, \lambda_{2i}] \subset C$ $(1 \leq i \leq g+1)$. Moreover it is easy to see that each such pole is simple (use 2.3). It follows that each function $y \to P_i(y): Y \to C$ is continuous. Analogous statements hold in the Schrödinger case.

3.11. *Remark.* It now follows from well-known arguments (see especially [**8**]) that, in the Schrödinger case, Y is a real sub-torus T of a complex torus, τ is a Kronecker winding on T, ν is normalized Haar measure on T, and $y(t)$ is the evaluation of an Abelian function along an orbit of τ. We do not give details here; the uninitiated reader may wish to work them out for her/himself after reading § 4.

4. *The finite-band* AKNS *potentials*

In this section, we describe precisely those stationary ergodic AKNS potentials

$$y = \begin{pmatrix} a & b+e \\ -b+e & -a \end{pmatrix}$$

for which hypotheses (H1) and (H2) hold. We first assume $b \equiv 0$, then at the end of the section show how to introduce an (essentially arbitrary) b.

We divide the discussion into four steps. The first aims at formulae (22) and (23) for $e(t)$ and $a(t)$. The second shows how to linearize the motion of the pole divisor. The third shows that this linear motion takes place on the generalized Jacobian. The fourth involves 'going backward': if $e(t)$, $a(t)$ are given by (22), (23), and the Jacobian data, then (H1) and (H2) hold.

To begin, let $\sigma: C \to C: (\lambda, \kappa) \to (\lambda, -\kappa)$ be the sheet-interchange involution. We fix $y \in Y$ until further notice, and write $\mu_+(t, p) = M(\tau_t(y), p)$, $\mu_-(t, p) = M(\tau_t(y), \sigma(p))$. Thus for example if $\lambda = \pi(p)$ has positive imaginary part, and if $\operatorname{Im} \kappa(p) > 0$, then

$$\mu_\pm(t, p) = m_\pm(\tau_t(y), p). \tag{12}$$

For each $p \in C$ with $\operatorname{Im} \pi(p) \neq 0$, the functions $\mu_\pm$ are absolutely continuous and satisfy the Riccati equation

$$\frac{d\mu_\pm}{dt} = (-\lambda + e(t)) - 2a(t)\mu_\pm(t, p) - (\lambda + e(t))\mu_\pm(t, p)^2 \tag{13}$$

for (Lebesgue)-a.a. $t \in \mathbb{R}$. In fact there is a Borel set $B_0 \subset \mathbb{R}$ with null complement such that, for each $t \in B_0$, (13) holds for *all* $p \in C$ with $\operatorname{Im} \pi(p) \neq 0$. We will not prove this; it is an exercise using the Cauchy integral formula, Fubini's theorem, and elementary existence-uniqueness theory for ordinary differential equations.

Next expand $p \to \mu_+(t, p)$ in a Laurent series near ∞_+ (see 3.9):

$$\mu_+(t, p) = i + \sum_{n=1}^{\infty} \frac{\mu_n(t)}{\lambda^n}, \qquad \lambda = \pi(p). \tag{14}$$

Another exercise shows that $\mu_n(t)$ is absolutely continuous for each n and that for a.a. $t \in \mathbb{R}$, say $t \in B_+$, the derivatives $d\mu_n/dt$ can be evaluated by plugging (14) into (13). Similarly, if we expand $p \to \mu_-(t, p)$ near ∞_+:

$$\mu_-(t, p) = -i + \sum_{n=1}^{\infty} \frac{\tilde{\mu}_n(t)}{\lambda^n}, \tag{15}$$

then each $\tilde{\mu}_n$ is absolutely continuous and for a.a. $t \in \mathbb{R}$, say $t \in B_-$, the derivatives can be evaluated by plugging (15) into (13). Let $B = B_0 \cap B_+ \cap B_-$; then $\mathbb{R} \setminus B$ has null complement.

Now consider the function $p \to \mu_+(t, p) - \mu_-(t, p)$ for fixed $t \in \mathbb{R}$. Let $P_i(t) \equiv P_i(\tau_t(y))$ be the poles of $\mu_+(t, \lambda) = M(\tau_t(y), \lambda)$ (3.9). If no pole $P_i(t)$ equals a ramification point $p_j = \pi^{-1}(\lambda_j)$ $(1 \leq j \leq 2g+2)$, then $\mu_+ - \mu_-$ has simple zeros at p_j $(1 \leq j \leq 2g+2)$ and simple poles at each of the two elements of $\pi^{-1}\pi(P_i(t))$ $(1 \leq i \leq g+1$; see 3.10). Since $\mu_+ - \mu_- \to 2i$ as $p \to \infty_+$, we have

$$\mu_+ - \mu_- = 2\frac{\kappa(p)}{\prod_{i=1}^{g+1} (\lambda - P_i(t))}, \qquad \lambda = \pi(p). \tag{16}$$

If one of the $P_i(t)$'s equals a ramification point p_*, then (16) still holds. To see this it suffices to show that p_* is a simple pole of $\mu_+ - \mu_-$. But this follows from 2.3, 2.4, and 3.8. Indeed, as $\lambda \to \pi(p_*)$ along the resolvent interval adjacent to λ_*, one has either $m_+(\tau_t(y), \lambda) \to -\infty$ and $m_-(\tau_t(y), \lambda) \to +\infty$, or $m_+(\tau_t(y), \lambda) \to \infty$ and $m_-(\tau_t(y), \lambda) \to -\infty$. Hence $\lim_{p \to p_*} (\mu_+ - \mu_-)(p) = \infty$, and the simplicity of the pole follows from 2.3.

Let us also consider the function $p \to \mu_+(t, p) + \mu_-(t, p)$. It commutes with σ, hence defines a rational function on the λ-sphere which is, moreover, 0 at $\lambda = \infty$. Its pole divisor is a subset of $d(t) = (\pi(P_1(t)), \dots, \pi(P_{g+1}(t)))$, and equals it if no $P_i(t)$ coincides with a ramification point. Therefore

$$\mu_+ + \mu_- = \frac{2Q(\lambda)}{\prod_{i=1}^{g+1} (\lambda - \pi(P_i(t)))}, \tag{17}$$

where $Q(\lambda) = Q(t, \lambda)$ is a polynomial of degree $\le g$.

Write $H(\lambda) = H(t, \lambda) = \prod_{i=1}^{g+1} (\lambda - \pi(P_i(t)))$. Then

$$\mu_+ = \frac{\kappa(p) + Q(\lambda)}{H(\lambda)}, \tag{18}$$

$$\mu_- = \frac{Q(\lambda) - \kappa(p)}{H(\lambda)}, \qquad \lambda = \pi(p). \tag{19}$$

The polynomial Q can be determined. Suppose first that no pole $P_1(t), \dots, P_{g+1}(t)$ is a ramification point. Then from (19), we must have $\kappa_i \equiv \kappa(P_i(t) = Q(\pi(P_i(t))$ $(1 \le i \le g+1)$. Then the coefficients of Q are uniquely determined by the linear system with van der Monde coefficient matrix

$$\begin{bmatrix} 1 & \pi(P_1) & \cdots & \pi(P_1)^g \\ & & \vdots & \\ 1 & \pi(P_{g+1}) & \cdots & \pi(P_{g+1})^g \end{bmatrix} \begin{pmatrix} q_0 \\ \vdots \\ q_g \end{pmatrix} = \begin{pmatrix} \kappa_1 \\ \vdots \\ \kappa_{g+1} \end{pmatrix}. \tag{20}$$

This matrix is non-singular, since by 3.10 the $\pi(P_i)$ are all distinct. In particular, the highest-order coefficient q_g of Q satisfies

$$q_g = \sum_{j=1}^{g+1} \frac{\kappa_j}{\prod_{r \ne j} (P_j - P_r)}, \tag{21}$$

where here and below we write P_r instead of $\pi(P_r)$ when no confusion can arise.

The polynomial Q is determined by (20) even when some of the P_i's are ramification points. For, at any such point P_r, we must have $Q(P_r) = 0 = \kappa_r$ (otherwise μ_+ would have a pole of order 2 at P_r). Thus (20) and (21) hold for all pole divisors. Note that this implies that, if $P_r = P_r(t)$ is a ramification point, then $(\mu_+ + \mu_-)(t, P_r)$ is finite.

Now plug (14) and (15) into (13), then use (16) and (17). One obtains $2e = i(\mu_1 - \tilde{\mu}_1)$, $2a = -(\mu_1 + \tilde{\mu}_1) = -2q_g$ for all $t \in B$. Hence

$$e(t) = \tfrac{1}{2} \sum_{j=1}^{2g+2} \lambda_j - \sum_{r=1}^{g+1} P_r(t), \tag{22}$$

$$a(t) = -\sum_{j=1}^{g+1} \frac{\kappa(P_j(t))}{\prod_{r \ne j} P_j(t) - P_r(t)}. \tag{23}$$

We see that a and e are continuous in t, and, more importantly, are symmetric rational functions of the pole divisor. This completes step 1.

We proceed to step 2. Introduce the $g+1$ Abel–Jacobi coordinates

$$\gamma_r(t)=\sum_{j=1}^{g+1}\int_{P_j(0)}^{P_j(t)}\frac{\lambda^{r-1}d\lambda}{\kappa(p)},\qquad \lambda=\pi(p),1\le r\le g+1. \tag{24}$$

Note that $(\lambda^g d\lambda)/\kappa$ is not a holomorphic differential.

Now, each pole $t\to P_j(t)$ is absolutely continuous and in fact C^1 in t; we omit the proof, which uses continuity of $a(t)$ and $e(t)$, the simplicity of the poles, and the inverse function theorem. Hence we have

$$\gamma_r'=\sum_{j=1}^{g+1}\frac{P_j^{r-1}P_j'}{\kappa_j},\qquad \kappa_j=\kappa(P_j). \tag{25}$$

Here and below, the prime $'$ always means d/dt.

Note in addition that $Q(t,\lambda)$ and $H(t,\lambda)$ are C^1 in t, so we have for all $p\in C$ with $\lambda=\pi(p)\neq\infty$:

$$\mu_+'=\left(\frac{\kappa+Q}{H}\right)'=\frac{Q'H-(\kappa+Q)H'}{H^2}=(-\lambda+e)-2a\left(\frac{\kappa+Q}{H}\right)-(\lambda+e)\left(\frac{\kappa+Q}{H}\right)^2.$$

We multiply by H^2 in the last equality, then substitute $P_r\equiv\pi(P_r(t))$: recalling that $H(P_r)=0$, $Q(P_r)=\kappa_r$, $e(t)=\frac{1}{2}\sum_{j=1}^{2g+2}\lambda_j-\sum_{s=1}^{g+1}P_s(t)$, we get

$$\frac{H'(P_r)}{\kappa_r}=\sum_{j=1}^{2g+2}\lambda_j-2\sum_{s\neq r}P_s. \tag{26}$$

Now $H'(P_r)=\prod_{j=1}^{g+1}(\lambda-P_j)'_{\lambda=P_r}=-P_r'\prod_{s\neq r}(P_r-P_s)$. From (26) we get

$$\frac{P_r'\prod_{s\neq r}(P_r-P_s)}{\kappa_r}=c_1-c_2\sum_{s\neq r}P_s, \tag{27}$$

where $c_1=-\sum_{j=1}^{2g+2}\lambda_j$, $c_2=-2$.

Next observe that $\prod_{s\neq r}(P_r-P_s)=\sum_{j=0}^{g}(-1)^{g-j}\sigma_{g-j}^{(r)}P_r^j$, where $\sigma_{g-j}^{(r)}$ is the $(g-j)$th symmetric function in $(P_1,\dots,\hat{P}_r,\dots,P_{g+1})$ (i.e. omit P_r). Therefore, letting

$$A=\begin{pmatrix}1 & -\sigma_1^{(1)} & \cdots & (-1)^g\sigma_g^{(1)}\\ \vdots & & & \\ 1 & -\sigma_1^{(g+1)} & \cdots & (-1)^g\sigma_g^{(g+1)}\end{pmatrix},$$

we can rewrite (27) as follows:

$$A\cdot\begin{pmatrix}P_1^gP_1'/\kappa_1 & \cdots & P_{g+1}^gP_{g+1}'/\kappa_{g+1}\\ \vdots & & \vdots\\ P_1'/\kappa_1 & & P_{g+1}'/\kappa_{g+1}\end{pmatrix}=\operatorname{diag}\left(c_1-c_2\sum_{s\neq r}P_s\right). \tag{28}$$

We used the fact that $\sum_{j=0}^{g}P_r^j\sigma_{g-j}^{(s)}(-1)^{g-j}=0$ if $s\neq r$.

We multiply (28) on the right by $\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$, obtaining

$$A\begin{pmatrix}\gamma_{g+1}'\\ \vdots\\ \gamma_1'\end{pmatrix}=\begin{pmatrix}c_1-c_2\sigma_1^{(1)}\\ \vdots\\ c_1-c_2\sigma_1^{(g+1)}\end{pmatrix},\qquad \sigma_1^{(r)}=\sum_{s\neq r}P_s. \tag{29}$$

Since $\det A = \pm\prod_{i<j}(P_i - P_j) \neq 0$, we can apply A^{-1} in (29); noting that

$$A^{-1}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} = \begin{pmatrix}1\\ 0\\ \vdots\\ 0\end{pmatrix}, \qquad A^{-1}\begin{pmatrix}-\sigma_1^{(1)}\\ \vdots\\ -\sigma_1^{(g+1)}\end{pmatrix} = \begin{pmatrix}0\\ 1\\ \vdots\\ 0\end{pmatrix},$$

we obtain finally

$$\vec{\gamma}' = \begin{pmatrix}\gamma_1'\\ \vdots\\ \gamma_g'\\ \gamma_{g+1}'\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ c_2\\ c_1\end{pmatrix}. \tag{30}$$

That is, the motion of the poles in the γ-coordinates is linear. This completes step 2.

The third step is to construct a real torus T of dimension $g+1$ on which $\gamma_1, \ldots, \gamma_{g+1}$ are angular variables. To do so, we introduce the generalized Jacobian $J(C_0)$ corresponding to a singularization C_0 of C; then $J(C_0)$ is a $g+1$-dimensional complex analytic variety, and T will be a subset of $J(C_0)$. We only sketch the construction of $J(C_0)$; for a detailed discussion see [**12**] and [**11**, p. 124]; see also [**30**, defn. 1.10].

Define C_0 to be C with ∞_+ and ∞_- identified to a point. This introduces an extra hole in C, so intuitively the genus of C_0 is $g+1$. Let $a_1, \ldots, a_g, b_1, \ldots, b_g$ be a normalized basis of $H_1(C, \mathbb{Z})$; we suppose that none of these curves contain $\infty_\pm$. Let $\omega_1, \ldots, \omega_g$ be the holomorphic differentials $d\lambda/\kappa(p), \ldots, \lambda^{g-1}\,d\lambda/\kappa(p)$ respectively, and let $\omega_{g+1} = (\lambda^g\,d\lambda)/\kappa(p)$. Let a_{g+1} be a small circle centred at ∞_+. Let Λ be the lattice in $\mathbb{C}^{g+1}$ generated by the $g+1$ vectors $(\int_{a_i}\omega_1, \ldots, \int_{a_i}\omega_{g+1})$ $(1 \le i \le g+1)$ and the g vectors $(\int_{b_i}\omega_1, \ldots, \int_{b_i}\omega_{g+1})$ $(1 \le i \le g)$. Then it can be shown that Λ has rank $2g+1$. The generalized Jacobian is defined to be $\mathbb{C}^{g+1}/\Lambda$.

There is an equivalent algebraic definition: $J(C_0)$ is the set of divisors d in $C\backslash\{\infty_\pm\} = \check{C}$ under the equivalence relation $d_1 \sim d_2 \Leftrightarrow d_1 - d_2$ is the divisor of a meromorphic function f on C such that $f(\infty_+) = f(\infty_-) \neq \infty$.

We define $\check{C}^{(g+1)}$ to be the set of unordered $(g+1)$-tuples of points in C. We define an Abel map $I : \check{C}^{(g+1)} \to J(C_0)$ adapted to present purposes: if $(p_1, \ldots, p_{g+1}) \in \check{C}^{(g+1)}$, then

$$I(p_1, \ldots, p_{g+1}) = \sum_{j=1}^{g+1}\left(\int_{p^*}^{p_j}\omega_1, \ldots, \int_{p^*}^{p_j}\omega_{g+1}\right),$$

where $p^* \neq \infty_\pm$ is an element of C. It can be shown that I is a holomorphic diffeomorphism from $\{(p_1, \ldots, p_{g+1}) \mid p_i \neq \sigma(p_j) \text{ if } i \neq j\} \subset \check{C}^{(g+1)}$ onto an open dense subset of $J(C_0)$. In particular, let $T_C = \{(p_1, \ldots, p_{g+1}) \in \check{C}^{(g+1)} \mid p_i \in \pi^{-1}[\lambda_{2i-1}, \lambda_{2i}],\ 1 \le i \le g+1\}$. Our pole divisor $(P_1(t), \ldots, P_{g+1}(t))$ is in T_C. Recalling (24) and (30), we see that

$$I(P_1(t), \ldots, P_{g+1}(t)) = \vec{\gamma}(0) + (0, \ldots, 0, c_2, c_1), \tag{31}$$

where $\vec{\gamma}(0) = I(P_1(0), \ldots, P_{g+1}(0))$. Thus I straightens out the pole motion, which becomes linear in $I(T_C) \subset J(C_0)$.

Define algebraic functions A, E on an open dense subset of $J(C_0)$ by $A \circ I(p_1, \ldots, p_{g+1}) = a(P_1, \ldots, P_{g+1})$, $E \circ I(p_1, \ldots, p_{g+1}) = e(p_1, \ldots, p_{g+1})$ (see (22) and (23)). From (31),

$$a(t) = A(\bar{\gamma}(0) + (0, \ldots, 0, c_2 t, c_1 t)),$$
$$e(t) = E(\bar{\gamma}(0) + (0, \ldots, 0, c_2 t, c_1 t)). \tag{32}$$

Now assume that our fixed point $y \in Y$ has a dense orbit (3.1). Note that $\mathrm{cls}\{\bar{\gamma}(t) \mid -\infty < t < \infty\}$ is a sub-torus T of $I(T_C)$. Also $y(t)$ is obtained by evaluating A and E along the dense orbit $\bar{\gamma}$ in T. It is now easy to check that the closure Y in $\mathscr{L}$ of the orbit $\{\tau_t(y) \mid t \in \mathbb{R}\}$ is homeomorphic to T under the map $\phi : y \to I(P_1(y), \ldots, P_{g+1}(y))$. Moreover ϕ sends the flow τ on Y to the rectilinear winding on T determined by Λ, c_1, and c_2. Since normalized Haar measure h on T is the only measure invariant under the rectilinear winding, we also have $\phi(\nu) = h$. We have thus completely described (Y, τ, ν) if $b \equiv 0$.

We proceed to step 4: if a and e are defined as above, then $\Sigma = (-\infty, \lambda_1] \cup \cdots \cup [\lambda_{2g+2}, \infty)$ and $\beta(\lambda) = 0$ for a.a. $\lambda \in \Sigma$.

Precisely, let κ, C, c_1, c_2, Λ, $J(C_0)$, I, A, E be as above. Given a divisor

$$d = (P_1, \ldots, P_{g+1}) \in T_C \overset{\text{def}}{=} \{(p_1, \ldots, p_{g+1}) \in \check{C}^{(g+1)} \mid p_i \in \pi^{-1}[\lambda_{2i-1}, \lambda_{2i}] (1 \le i \le g+1)\},$$

define $H(\lambda) = \prod_{i=1}^{g+1} (\lambda - P_i)$, and let $Q(\lambda)$ be that polynomial of degree g such that $Q(\pi(P_i)) = \kappa(P_i)$ $(1 \le i \le g+1)$.

Define $M(d, p) = (\kappa(p) + Q(\lambda))/H(\lambda)$ $(\lambda = \pi(p))$, so that $M(d, \sigma(p)) = (Q(\lambda) - \kappa(p))/H(\lambda)$. Let $\Gamma \subset I(T_C)$ be the closure of some orbit $\bar{\gamma}_1$: $\bar{\gamma}_1(t) = \bar{\gamma}_1(0) + (0, \ldots, 0, c_2 t, c_1 t)$ $(t \in \mathbb{R})$. Corresponding to each $\bar{\gamma} \in \Gamma$, there is a quasi-periodic function

$$y(t) = y_\gamma(t) = \begin{pmatrix} a(t) & e(t) \\ e(t) & -a(t) \end{pmatrix},$$

where $a(t) = A(\bar{\gamma}(t))$, $e(t) = E(\bar{\gamma}(t))$ and $\bar{\gamma}(t) = \bar{\gamma} + (0, \ldots, 0, c_2 t, c_1 t)$. For $\bar{\gamma} \in \Gamma$, let $d(t) = I^{-1}(\bar{\gamma}(t))$, and let $\mu_+(t, p) = M(d(t), p)$, $\mu_-(t, p) = M(d(t), \sigma(p))$.

We will eventually show that, if $\bar{\gamma} \in \Gamma$ and if $\lambda = \pi(p)$ has a non-zero imaginary part, then $\mu_\pm$ satisfy the Riccati equation

$$\frac{d\mu_\pm}{dt} = (-\lambda + e(t)) - 2a(t)\mu_\pm - (\lambda + e(t))\mu_\pm^2. \tag{33}$$

We first assume this has been done, and verify (H1) and (H2) for the stationary ergodic process (Y, τ, ν) given as follows: $Y = \{y_\gamma \mid \bar{\gamma} \in \Gamma\}$, τ is translation, and ν is the image of normalized Haar measure h on Γ under the map $\psi : \Gamma \to Y : \bar{\gamma} \to y_\gamma$. (Note that ψ commutes with the flows on Γ and Y.)

For $\lambda \in \mathbb{C}$, define a flow $\hat{\tau} = \hat{\tau}_\lambda$ on $\Gamma \times \mathbb{C}^2$ by solving equations $(2)_{y_\gamma, \lambda}$, just as in § 2. Let $\hat{\tau}$ be the induced flow on $\Gamma \times \mathbb{P}^1(\mathbb{C})$. Given $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda \neq 0$, choose $p \in \pi^{-1}(\lambda)$ so that $\operatorname{Im} \lambda \cdot \operatorname{Im} \kappa(p) > 0$, and define $m_+(\bar{\gamma}, \lambda) = M(I^{-1}(\bar{\gamma}), p)$, $m_-(\bar{\gamma}, \lambda) = M(I^{-1}(\bar{\gamma}), \sigma(p))$. By (33), the sections $S^\pm(\lambda) = \{(\lambda, m_\pm(\bar{\gamma}, \lambda)) \mid \bar{\gamma} \in \Gamma\} \subset \Gamma \times \mathbb{P}^1(\mathbb{C})$ are invariant under $\tilde{\tau}$ if $\operatorname{Im} \lambda \neq 0$. By a continuity argument, $S^\pm(\lambda)$ can be extended to define invariant sections also for $\lambda \in \mathbb{R}$.

Now let $\lambda \in (-\infty, \lambda_1) \cup \cdots \cup (\lambda_{2g+2}, \infty)$. Then $\operatorname{Im} m_+ > 0 > \operatorname{Im} m_-$, so $S^{\pm}(\lambda)$ are *distinct.* Also $\Gamma \times \mathbb{P}^1(\mathbb{R})$ is invariant under $\tilde{\tau}$. Arguing as in the proof of 3.6, we see that all solutions of $(2)_{y_\gamma, \lambda}$ are bounded, for all $\vec{\gamma} \in \Gamma$. It follows that $\beta(\lambda) = 0$, and that λ is in the spectrum of K_{y_γ} $(\gamma \in \Gamma)$.

On the other hand, it follows from invariance of $S^{\pm}(\lambda)$ that $m_{\pm}(\vec{\gamma}, \lambda)$ depend only on $y = \psi(\vec{\gamma})$. We write $m_{\pm}(y, \lambda)$. Using the uniqueness of the invariant bundles in 2.1, we see that $\lambda \to m_{\pm}(y, \lambda)$ are the Weyl functions for K_y $(\operatorname{Im} \lambda \neq 0, y \in Y)$. Now $m_{\pm}(y, \lambda)$ extend meromorphically through each resolvent interval $(\lambda_{2i-1}, \lambda_{2i})$ $(1 \le i \le g+1)$. By [**15**], the essential resolvent of the half-line operator K_y^+ contains $\bigcup_{i=1}^{g+1} (\lambda_{2i-1}, \lambda_{2i})$ $(y \in Y)$. This means (2.5) that $\Sigma = (-\infty, \lambda_1] \cup \cdots \cup [\lambda_{2g+2}, \infty)$ contains the spectrum of each K_y.

We conclude that Σ equals the spectrum of K_y for all $y \in Y$, and that $\beta(\lambda) = 0$ for a.a. $\lambda \in \Sigma$. This shows that (H1) and (H2) follow from (33).

Let us verify (33). We consider only μ_+. For $\gamma \in \Gamma$, let $(P_1(t), \ldots, P_{g+1}(t)) = I^{-1}(\vec{\gamma}(t))$. Explicitly, we want to show that

$$\frac{Q'H - (\kappa + Q)H'}{H^2} = (-\lambda + e) - 2a\frac{\kappa + Q}{H} - (\lambda + e)\frac{(\kappa + Q)^2}{H^2} \tag{34}$$

if $\operatorname{Im} \lambda \neq 0$. Multiplying by H^2 and reorganizing, we transform (34) into

$$Q'H - QH' - (e - \lambda)H^2 + 2aQH + (e + \lambda)(\kappa^2 + Q^2) = \kappa A, \tag{35}$$

where $A = A(\lambda, t) \stackrel{\text{def}}{=} H' - 2aH - 2(\lambda + e)Q$. We will show that (35) holds for *all* $\lambda \in \mathbb{C}$.

We claim that $A(\lambda, t)$ is identically zero. To see this, fix t, write $P_i = P_i(t)$ $(1 \le i \le g+1)$, and note that $A(\pi(P_i), t) = H'(\pi(P_i)) - 2(\pi(P_i) + e)\kappa(P_i) = 0$ (see (26) and the calculation (27)-(30) following it). Hence A is a polynomial with at least $g+1$ distinct roots. On the other hand, A has degree at most $g+1$, and furthermore the coefficient of λ^{g+1} is $-2a - 2q_g = 0$ (see (23); here q_g is the coefficient of λ^g in $Q(\lambda)$). Thus $\deg A \le g$, hence A is identically zero.

Next we rewrite (35): note that

$$-Q(H' - 2aH - (e + \lambda)Q) = -Q(A + (e + \lambda)Q) = -Q^2(e + \lambda);$$

thus (35) is equivalent to

$$S(\lambda, t) \stackrel{\text{def}}{=} Q'H - (e - \lambda)H^2 + (e + \lambda)(\kappa^2 - Q^2) = 0. \tag{36}$$

To prove (36), note first that $\deg S \le 2g+3$. In fact $\deg S \le 2g+1$, because the coefficient of λ^{2g+3} is $1 - 1 = 0$, and the coefficient of λ^{2g+2} is $-e - 2\sum_{r=1}^{g+1} \pi(P_r) - e + \sum_{i=1}^{2g+2} \lambda_i = 0$ (see (22)). Moreover, it is easy to see that $\pi(P_1), \ldots, \pi(P_{g+1})$ are zeros of S.

The idea now is to show that each zero of S is *double*; this clearly implies (36). First of all, it is clear that P_r is a double zero of $(e - \lambda)H^2$ $(1 \le r \le g+1$; we agree to confuse P_r with $\pi(P_r)$). So we prove that

$$G(\lambda, t) = Q'H + (e + \lambda)(\kappa^2 - Q^2) = H \cdot F(\lambda, t)$$

has double zeros (here F is defined by the above factorization).

Consider $G' = Q''H + Q'H' + e'(\kappa^2 - Q^2) - 2(e+\lambda)QQ'$. We see that

(37) $$G'(P_r) = Q'(P_r)[H'(P_r) - 2(e+P_r)\kappa(P_r)] = 0 \qquad (1 \le r \le g+1);$$

here we have used (26). Therefore H divides G'. But also $G' = F'H + FH'$; hence H divides FH'.

Now, $H' = -\sum_{j=1}^{g+1} P_j' \prod_{s \neq j} (\lambda - P_s) \Rightarrow H'(P_r) = -P_r' \prod_{s \neq r} (P_r - P_s)$. Thus $H'(P_r) = 0$ iff $P_r' = 0$. Clearly if $P_r' \neq 0$ for all r, then H divides F, and (36) is proved.

Suppose first that $\sigma(P_r) \neq P_r$ $(1 \le r \le g+1)$. Then it follows from the facts that I is a diffeomorphism on T_C and that π is invertible near each P_r that $\pi(P_r') \neq 0$. Hence (36) holds in this case. If $\sigma(P_r) = P_r$ for some r, then we approximate the divisor $d = (P_1, \ldots, P_{g+1})$ by divisors $d_n = (P_1^{(n)}, \ldots, P_{g+1}^{(n)}) \in T_C$ for which $\sigma(P_r^{(n)}) \neq P_r^{(n)}$ $(1 \le r \le g+1)$. By a limiting argument, (36) holds for d. This completes step 4.

We finish the discussion by removing the assumption that $b \equiv 0$. Let (Y, τ, ν) be a stationary ergodic process consisting of elements

$$y = \begin{pmatrix} a & b+e \\ -b+e & -a \end{pmatrix}$$

for which (H1) and (H2) hold. We show that there are functions $\tilde{a}(t)$, $\tilde{e}(t)$ as constructed above and $\theta_0 \in [0, 2\pi)$ such that, with $B(t) = \theta_0 + \int_0^t b(s)\, ds$,

(38) $$\begin{aligned} a(t) &= \tilde{a}(t) \cos 2B(t) - \tilde{e}(t) \sin 2B(t) \\ e(t) &= \tilde{a}(t) \sin 2B(t) + \tilde{e}(t) \cos 2B(t) \end{aligned}$$

Conversely, if there is a point $y \in Y$ with dense orbit and functions $\tilde{a}(t)$, $\tilde{e}(t)$ as constructed above such that relations (38) hold, then (Y, τ, ν) satisfies (H1) and (H2).

To prove these assertions, let $\mathbb{K} = \{\rho = e^{i\theta} \mid 0 \le \theta \le 2\pi\}$, and define a flow τ^* on $Y \times \mathbb{K}$ by

$$\tau_t^*(y, \rho_0) = (\tau_t(y), \rho(t)),$$

where

$$\rho(t) \equiv \rho(t; y, \rho_0) = \rho_0 \exp\left(i \int_0^t b(s)\, ds\right).$$

Let $\eta: Y \times \mathbb{K} \to Y: (y, \rho) \to y$ be the projection, and let ν^* be a measure on $Y \times \mathbb{K}$ which is ergodic with respect to τ^* such that $\eta(\nu^*) = \nu$. Such a measure exists. Let $Y^* \subset Y \times \mathbb{K}$ be the support of ν^*. For each $(y, \rho_0) \in Y^*$ make the change of variable

(39) $$\vec{u} = \rho(t)\vec{v}$$

in equation $(2)_{y,\lambda}$, where $\rho \in \mathbb{K}$ is identified with

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

We obtain for each equation $(2)_{y,\lambda}$ and each $(y, \rho_0) \in Y^*$ the equation

$(40)_{(y,\rho_0),\lambda}$ $$\vec{v}' = \begin{pmatrix} \tilde{a}(t) & \lambda + \tilde{e}(t) \\ -\lambda + \tilde{e}(t) & -\tilde{a}(t) \end{pmatrix} \vec{v} \qquad ((y, \rho_0) \in Y^*),$$

where $\tilde{a}$, $\tilde{e}$ are determined by (38). Note that (H1), (H2) hold for (Y^*, τ^*, ν^*) if and only if they hold for (Y, τ, ν).

It is now straightforward to show that $y \in Y$ is characterized by relations (38). We remark only that the following fact is used: the group action $\rho_1 \cdot (y, \rho) = (y, \rho_1 \rho)$ commutes with τ^*.

We conclude that the most general operator K_y satisfying (H1) and (H2) arises from an operator constructed in steps 1-4 via the gauge transformation (39).

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