Now consider

$$S_{-m} = \{1 + 1/2^m + 1/3^m + \dots + 1/(n-1)^m\}(\overline{n-1}!)^m.$$

We have

$$\begin{aligned} 2\,S_{-m} &= (\overline{n-1}!)^m \Sigma\{1/r^m + 1/(n-r)^m\} \\ &= (\overline{n-1}!)^m \Sigma \frac{(n-r)^m + r^m}{r^m(n-r)^m} \\ &= (\overline{n-1}!)^m \Sigma \frac{n^m - {}_mC_1 n^{m-1} r \dots + {}_mC_1 n r^{m-1}}{r^m(n-r)^m} \\ &= (\overline{n-1}!)^m \Sigma \frac{Pn^2 + {}_mC_1 n}{r(n-r)^m}. \end{aligned}$$

We have thus to show that ${}_mC_1(\overline{n-1}!)^m\Sigma\{1/r(n-r)^m\}$ is divisible by n. We shall suppose, for the sake of clearness, that m is less than n; but the following method will be applicable, even if m be greater than n.

Assume $\quad (\overline{n-1}!)^m/\{(n-r)r^m\} \equiv a_r \pmod{n}$

$\therefore \quad (\overline{n-1}!)^m \equiv a_r(n-r)r^m$

$\therefore \quad a_r(n-r)r^m \equiv -1 \quad$ (by Wilson's theorem).

Now since $\quad (n-r)^{n-1} \equiv r^{n-1} \equiv 1$, we get

$$a_r \equiv -r^{n-m-1}(n-r)^{n-2}$$

$\therefore \quad a_r \equiv r^{n-m-1} r^{n-2} \pmod{n}$

$$\equiv r^{n-m-2}$$

Hence $\quad \Sigma a_r \equiv \Sigma r^{n-m-2}$

$$\equiv 0, \text{ if } n-m-2 \neq 0.$$

It follows that S_{-m} is divisible by n_2, m being subject to the restriction $n-m-2$ be not zero. If we remove the condition that m is to be less than n, we shall easily find that the general restriction as to the value of m, is that $m+1$ must not be a multiple of $n-1$.

In the paper referred to before in a footnote, Mr Leudesdorf considers the case where n is not prime and S_m denotes the sum of the m^{th} powers of the numbers less than n and prime to it. His method however cannot be considered rigorous, as it involves the use of divergent series.

Note on normals to conics.

By R. H. Pinkerton, M.A.

1. The following condition may be new; it does not appear in any of the books:—

The condition that the straight line

$$lx+my+n=0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

should be a *normal* to the general conic,

$$S \equiv ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \ldots \quad \ldots \quad (2),$$

referred to rectangular axes, is

$$\Sigma(al^2+2hlm+bm^2)=\Delta(l^2+m^2)^2 \quad \ldots \quad \ldots \quad (\alpha),$$

where Σ is written for

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm,$$

and Δ, A, B, C, F, G, H have their usual meanings.

We know* that the equation to the tangents to S at the points where (1) cuts S, is

$$S\Sigma=\Delta(lx+my+n)^2,$$

or
$$x^2(a\Sigma-\Delta l^2)+2xy(h\Sigma-\Delta lm)+y^2(b\Sigma-\Delta m^2)$$
$$+\text{terms of lower degree}=0 \quad \ldots \quad \ldots \quad \ldots \quad (3).$$

Now the straight line (1) will be normal to S if it is perpendicular to one of the straight lines (3). The condition for this is

$$l^2(a\Sigma-\Delta l^2)+2lm(h\Sigma-\Delta lm)+m^2(b\Sigma-\Delta m^2)=0,$$

or
$$\Sigma(al^2+2hlm+bm^2)=\Delta(l^4+2l^2m^2+m^4), \text{ which is } (\alpha).$$

2. If the line (1) be given as passing through a given point (α, β) its equation will be

$$lx+my-(l\alpha+m\beta)=0.$$

If in the condition (α) we substitute $-l\alpha-m\beta$ for n, we shall get the following biquadratic in l/m—

$$(al^2+2hlm+bm^2)\times$$
$$[Al^2+Bm^2+C(l\alpha+m\beta)^2-2Fm(l\alpha+m\beta)-2Gl(l\alpha+m\beta)+2Hlm]$$
$$=\Delta(l^2+m^2)^2,$$

or
$$l^4[a(A+C\alpha^2-2G\alpha)-\Delta]$$
$$+2l^3m[a(C\alpha\beta-F\alpha-G\beta+H)+h(A+C\alpha^2-2G\alpha)]$$
$$+l^2m^2[a(B+C\beta^2-2F\beta)+b(A+C\alpha^2-2G\alpha)$$
$$+4h(C\alpha\beta-F\alpha-G\beta+H)-2\Delta]$$
$$+2lm^3[b(C\alpha\beta-F\alpha-G\beta+H)+h(B+C\beta^2-2F\beta)]$$
$$+m^4[b(B+C\beta^2-2F\beta)-\Delta]=0 \quad \ldots \quad \ldots \quad \ldots \quad (\beta).$$

This biquadratic gives the directions of the four normals which can be drawn from the point (α, β) to the conic S.

3. If (x, y) be any point on any one of the four normals from (α, β) to the conic S, then

$$l(x-\alpha)+m(y-\beta)=0$$

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* *Salmon's Conics*—Fifth Edition—Art. 374.

if l/m be one of the roots of (β).

Hence $$l/(y-\beta) = -m/(x-a).$$

If then for l and m we substitute in (β) $y-\beta$ and $-(x-a)$ respectively, we shall obtain the equation to the four normals which can be drawn from (a,β) to the conic S.

4. It may be worth considering what the condition (a) becomes for the circle—

$$x^2+y^2+2gx+2fy+c=0.$$

Here
$$a=b=1\ ;\ h=0$$
$$\Delta = c-f^2-g^2,$$
$$\mathrm{A}=c-f^2, \mathrm{B}=c-g^2, \mathrm{C}=1,$$
$$\mathrm{F}=-f,\quad \mathrm{G}=-g,\ \mathrm{H}=fg.$$

Hence in this case (a) becomes

$$(l^2+m^2)[l^2(c-f^2)+m^2(c-g^2)+n^2-2fmn-2gnl+2fglm]$$
$$=(l^2+m^2)^2(c-f^2-g^2).$$

Dividing by l^2+m^2, we obtain

$$c(l^2+m^2)-f^2l^2-g^2m^2+n^2-2fmn-2gnl+2fglm$$
$$=c(l^2+m^2)-f^2l^2-f^2m^2-g^2l^2-g^2m^2,$$

or
$$g^2l^2+f^2m^2+2fglm-2n(gl+fm)+n^2=0,$$

or
$$(gl+fm-n)^2=0.$$

Hence the condition that $lx+my+n=0$ should be a normal to the circle is $$gl+fm-n=0.$$

But this is the tangential equation to the point $(-g,-f)$, the centre of the circle.

Third Meeting, 10th January 1890.

George A. Gibson, Esq., M.A., Ex-President, in the Chair.

Note on a curious operational Theorem.

By Professor Tait.

The idea in the following note is evidently capable of very wide development, but it can be made clear by a very simple example.

Whatever be the vectors a, β, γ, δ, we have always

$$\mathrm{V.V}a\beta\mathrm{V}\gamma\delta = a\mathrm{S}.\beta\gamma\delta - \beta\mathrm{S}.a\gamma\delta.$$