J. Inst. Math. Jussieu (2024), **23**(1), 347–377

doi:10.1017/S1474748022000391

VARIANTS OF A MULTIPLIER THEOREM OF KISLYAKOV

ANDREAS DEFANT[1], MIECZYSŁAW MASTYŁO[2] AND ANTONIO PÉREZ-HERNÁNDEZ[3]

[1] *Institut für Mathematik, Carl von Ossietzky Universität, Postfach 2503 D-26111 Oldenburg, Germany*
(andreas.defant@uni-oldenburg.de)

[2] *Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Poznań, Uniwersytetu Poznańskiego 4, 61-614 Poznań, Poland*
(mieczyslaw.mastylo@amu.edu.pl)

[3] *Departamento de Matemática Aplicada I, Escuela Técnica Superior de Ingenieros Industriales, Universidad Nacional de Educación a Distancia (UNED), 28040 Madrid, Spain*
(antperez@ind.uned.es)

(Received 15 July 2021; revised 24 June 2022; accepted 28 June 2022; first published online 1 September 2022)

Abstract We prove stronger variants of a multiplier theorem of Kislyakov. The key ingredients are based on ideas of Kislyakov and the Kahane–Salem–Zygmund inequality. As a by-product, we show various multiplier theorems for spaces of trigonometric polynomials on the n-dimensional torus $\mathbb{T}^n$ or Boolean cubes $\{-1,1\}^N$. Our more abstract approach based on local Banach space theory has the advantage that it allows to consider more general compact abelian groups instead of only the multidimensional torus. As an application, we show that various recent ℓ_1-multiplier theorems for trigonometric polynomials in several variables or ordinary Dirichlet series may be proved without the Kahane–Salem–Zygmund inequality.

Key words and phrases: Fourier analysis on groups; multipliers; Sidon sets; Boolean functions; multivariable and Dirichlet polynomials

2020 Mathematics subject classification: Primary 43A46; 43A75; 06E30; 42A16

1. Introduction

Let $\mathbb{T}$ be the torus in the complex plane, that is, the compact abelian group of all $z \in \mathbb{C}$ with $|z| = 1$, which carries the normalised Lebesgue measure ν on $\mathbb{T}$ as its Haar measure. By $\mathbb{T}^\infty$, we denote the countable product of $\mathbb{T}$, which, again, forms a compact abelian group (the Haar measure is the countable product of ν), and identify its dual group with $\mathbb{Z}^{(\mathbb{N})}$, all finite multi indices $\alpha \in \mathbb{Z}^n, n \in \mathbb{N}$. We write $\mathbb{N}_0^{(\mathbb{N})}$ for all $\alpha \in \mathbb{Z}^{(\mathbb{N})}$ with entries in $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

As usual, $H_\infty(\mathbb{T}^\infty)$ stands for the Banach space of all functions $f \in L_\infty(\mathbb{T}^\infty)$, such that $\widehat{f}(\alpha) = 0$ for all $\alpha = (\alpha_i) \in \mathbb{Z}^{(\mathbb{N})}$ with $\alpha_j < 0$ for some j. We call $H_\infty(\mathbb{T}^\infty)$ Hardy space on the infinite dimensional torus and denote its closed subspace of all continuous functions by $C^A(\mathbb{T}^\infty)$.

A sequence $\xi = (\xi_\alpha)_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}}$ of scalars is a bounded $\ell_1(\mathbb{N}_0^{(\mathbb{N})})$-multiplier of $C^A(\mathbb{T}^\infty)$, whenever the mapping $M_\xi \colon C^A(\mathbb{T}^\infty) \to \ell_1(\mathbb{N}_0^{(\mathbb{N})})$ given by:

$$M_\xi(f) := \big(\widehat{f}(\alpha)\xi_\alpha\big)_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}}, \quad f \in C^A(\mathbb{T}^\infty)$$

is bounded. The following necessary condition for such multipliers is due to Kislyakov [15, Theorem 6], and it, in fact, is the main motivation of this paper.

Theorem 1.1. *Let $\xi = (\xi_\alpha)_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}}$ be a bounded $\ell_1(\mathbb{N}_0^{(\mathbb{N})})$-multiplier of $C^A(\mathbb{T}^\infty)$. Then:*

$$\sup_{n,d \in \mathbb{N}} \frac{1}{\sqrt{n \log(1+dn)}} \Big(\sum_{\alpha \in \mathbb{N}_0^n \colon \max\{\alpha_1,\dots,\alpha_n\} \le d} |\xi_\alpha|^2 \Big)^{1/2} < \infty. \tag{1.1}$$

Inspired by this result, in particular, the techniques from local Banach space theory which Kislyakov uses to prove it, we study the following more general (but also more vague) question:

Let G be a compact abelian group with Haar measure ν and Γ a subset of characters in the dual group $\widehat{G}$. Moreover, let $X(\Gamma)$ be a Banach sequence space over the set Γ. A (real or complex) sequence $\xi = (\xi_\gamma)_{\gamma \in \Gamma}$ is an $X(\Gamma)$-multiplier of a closed subspace $V \subset L_p(G)$ whenever:

$$(\widehat{f}(\gamma)\xi_\gamma)_{\gamma \in \Gamma} \in X(\Gamma), \quad f \in V.$$

The problem then is to find necessary and sufficient conditions for such $X(\Gamma)$-multipliers of V. Our applications mainly focus on multiplier theorems for the n-dimensional torus $\mathbb{T}^n$, its Boolean counterpart, the n-dimensional Boolean cube $\{-1,1\}^n$, as well as their countable counterparts $\mathbb{T}^\infty := \mathbb{T}^{\mathbb{N}}$ and $\{-1,1\}^\infty := \{-1,1\}^{\mathbb{N}}$.

Observe that by a simple closed graph argument, the study of $X(\Gamma)$-multipliers for $V \subset L_\infty(G)$ means to study concrete inequalities: ξ is an $X(\Gamma)$-multiplier for $V \subset L_\infty(G)$ if and only if there is a constant $C = C(\xi) > 0$, such that:

$$\sum_{\gamma \in \Gamma} |\widehat{f}(\gamma)\xi_\gamma| \le C\|f\|_\infty, \quad f \in V.$$

We note that, only in very few cases, a full description of the set of all X-multipliers ξ of $V \subset L_\infty(G)$ is possible. In most of our applications, we are able to give necessary and/or sufficient conditions in terms of the asymptotic decay of ξ.

In the first section, we show that Theorem 1.1 is in fact a consequence of the Kahane–Salem–Zygmund inequality, and in Theorem 3.2 and Corollary 3.3, we extend Kislyakov's multiplier theorem to certain analytic subspaces of $L_\infty(\mathbb{T}^\infty)$ instead of $C^A(\mathbb{T}^\infty)$. It should be mentioned here that Kislyakov's approach to Theorem 1.1 is different, and in the second and third sections, we analyse his cycle of ideas from local Banach space theory — the main advantage is that they apply to more general compact abelian groups than only multidimensional tori.

https://doi.org/10.1017/S1474748022000391 Published online by Cambridge University Press

The crucial link, which makes this possible, comes from Lemma 4.1 showing that, given a compact abelian group G, a finite subset Γ in $\widehat{G}$ and a Banach space $F := (\mathbb{C}^\Gamma, \|\cdot\|)$, for every finite sequence $\xi = (\xi_\gamma)_{\gamma\in\Gamma}$, one has:

$$\pi_2(M_\xi : \mathcal{P}_\Gamma \to F) = \sup_{\|\mu\|_{\ell_2(\Gamma)}\le 1} \|(\mu_\gamma\xi_\gamma)_{\gamma\in\Gamma}\|_F,$$

where $\mathcal{P}_\Gamma$ stands for the Banach space of all finite polynomials $\sum_{\gamma\in\Gamma}\xi_\gamma\gamma$ endowed with the sup norm, M_ξ for the multiplier, which assigns to every finite polynomial $\sum_{\gamma\in\Gamma}\mu_\gamma\gamma$, the finite sequence $(\mu_\gamma\xi_\gamma)_{\gamma\in\Gamma} \in F$ and $\pi_2(M_\xi)$ for the 2-summing norm of this operator.

In the Theorems 4.4, 5.2 and 5.5, fundamental knowledge on 2-summing operators leads to improvements of Theorem 1.1.

In the last section, we apply our results to study multiplier theorems for spaces of functions on multidimensional tori and Boolean cubes. We focus on topics like Sidon constants, Bohr radii, monomial convergence, as well as Dirichlet series.

Using Kislyakov's ideas, we prove new results, but we also reprove recent known results, which were originally proved through the use of the Kahane–Salem–Zygmund inequality.

At first glance, this might look surprising, but on the other hand, we already remarked that our starting point, Theorem 1.1, is a consequence of the Kahane–Salem–Zygmund inequality, and, conversely, we show in the recent paper [10] that Kislyakov's ideas are of great relevance within a further study of the Kahane–Salem–Zygmund inequality.

2. Preliminaries

Banach spaces. Let X, Y be Banach spaces. We denote by B_X the closed unit ball of X and by X^* its dual Banach space. If we write $X \hookrightarrow Y$, then we assume that $X \subset Y$ and the inclusion map $\mathrm{id}\colon X \to Y$ is bounded. If $X = Y$ with equality of norms, then we write $X \cong Y$. As usual, $C(K)$ denotes the Banach space of all continuous functions on a compact Hausdorff space K, with the sup norm $\|\cdot\|_\infty$.

We denote by $L(X,Y)$ the space of all bounded linear operators $T\colon X \to Y$ with the usual operator norm. An operator $T \in L(X,Y)$ is said to be an isomorphic embedding of X into Y whenever there exists $C > 0$, such that $\|Tx\|_Y \ge C\|x\|_X$ for every $x \in X$. Thus, T^{-1} is an isomorphism from $(TX, \|\cdot\|_Y)$ onto X. Given a real number $1 \le \lambda < \infty$, we say that X λ-embeds into Y whenever there exists an isomorphic embedding T of X into Y, such that $\|T\|\,\|T^{-1}\| \le \lambda$. In this case, we call T a λ-embedding of X into Y.

Let Γ be a nonempty set. We denote by $\ell_\infty(\Gamma)$ the space of all bounded functions on Γ, endowed with the sup norm. For any $\xi,\eta \in \ell_\infty(\Gamma)$, we denote by $\xi\cdot\eta$ their pointwise product. Let E and F be linear subspaces of $\mathbb{K}^\Gamma$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Any $\xi \in \mathbb{K}^\Gamma$, such that $\xi\eta := \xi\cdot\eta \in F$ for all $\eta \in E$, defines a *diagonal operator* $D_\xi\colon E \to F$ given by $D_\xi(\eta) := \xi\eta$ for all $\eta \in E$. We write $\mathcal{D}(E,F)$ for the vector space of all such maps. If E and F are Banach spaces, such that the inclusion maps from E and F into $\mathbb{K}^\Gamma$ are continuous, then $\mathcal{D}(E,F)$ equipped with the norm:

$$\|D_\xi\|_{\mathcal{D}(E,F)} := \sup_{\|(\eta_\gamma)\|_E \le 1} \|(\xi_\gamma\eta_\gamma)\|_F$$

is a Banach space.

https://doi.org/10.1017/S1474748022000391 Published online by Cambridge University Press

We apply standard methods from local Banach space theory. Recall that a Banach space X has cotype 2 whenever there is a constant C, such that for each choice of finitely many $x_1,\dots,x_n \in X$:

$$\Big(\sum_{k=1}^{n} \|x_k\|^2\Big)^{\frac{1}{2}} \le C\Big(\int_0^1 \Big\|\sum_{k=1}^{n} r_k(t)x_k\Big\|^2 dt\Big)^{\frac{1}{2}},$$

where r_k denotes the kth Rademacher function. The least possible value of this constant is denoted $C_2(X)$.

An operator $T\colon X \to Y$ between Banach spaces is said to be absolutely p-summing (p-summing for short) with $1 \le p < \infty$ if there is a constant $C > 0$, such that, for each $n \in \mathbb{N}$ and for all sequences $(x_k)_{k=1}^n$ in X, we have:

$$\Big(\sum_{k=1}^{n} \|Tx_k\|_Y^p\Big)^{1/p} \le C \sup_{x^*\in B_{X^*}} \Big(\sum_{k=1}^{n} |x^*(x_k)|^p\Big)^{\frac{1}{p}}.$$

The least such constant C is denoted by $\pi_p(T\colon X \to Y)$ ($\pi_p(T)$ for short) and is called the absolutely p-summing norm of T. We refer to the theory of p-summing operators to [13] and [19].

We recall that if K is a compact Hausdorff space, X a closed subspace of $C(K)$ and Y a Banach space, then the so-called Pietsch domination theorem states that $T\colon X \to Y$ is p-summing if and only if there is a constant C and a probability Borel measure μ on K, such that:

$$\|Tf\|_Y \le C\Big(\int_K |f|^p\, d\mu\Big)^{\frac{1}{p}}, \quad f \in X.$$

In that case, $\pi_p(T)$ coincides with the smallest constant C satisfying the previous inequality.

Compact abelian groups. In the following, we fix some compact abelian group $G := (G,\cdot)$. A linear subspace X of $\mathbb{K}^G$ is said to be translation invariant whenever for every $f \in X$ and every $h \in G$, the translation $f_h \in X$, where $f_h(g) := f(g\cdot h)$ for every $g \in G$.

As usual, we write $\widehat{G}$ for the dual group of G (i.e. the set of all continuous characters on G), and we denote by ν the (normalised) Haar measure on G, a unique translation invariant regular Borel probability measure. Recall that the translation invariance of ν is equivalent to the formula:

$$\int_G f(g)\, d\nu(g) = \int_G f_h(g)\, d\nu(g), \quad f \in L_1(G,\nu),\ h \in G.$$

The following well-known result from [18] is central for our purposes; for the sake of completeness, we include a simple proof.

Lemma 2.1. *Let G be a compact abelian group with normalised Haar measure ν, let X be a closed translation invariant subspace of $C(G)$ and let Y be an arbitrary Banach space. Suppose that $T\colon X \to Y$ is a p-summing operator which satisfies that $\|Tf_h\|_Y = \|Tf\|_Y$*

https://doi.org/10.1017/S1474748022000391 Published online by Cambridge University Press

for all $f \in X$, $h \in G$. *Then*:

$$\|Tf\|_Y \le \pi_p(T)\Big(\int_G |f(g)|^p\, d\nu(g)\Big)^{1/p}, \quad f \in X.$$

Proof. Pietsch's domination theorem combined with Fubini's theorem show that there is a probability Borel measure μ on G, such that for all $f \in X$:

$$\begin{aligned}\|Tf\|_Y^p = \int_G \|Tf_h\|_Y^p\, d\nu(h) &\le \pi_p(T)^p \int_G \Big(\int_G |f_h(g)|^p\, d\mu(g)\Big)\, d\nu(h)\\ &= \pi_p(T)^p \int_G \Big(\int_G |f_h(g)|^p\, d\nu(h)\Big)\, d\mu(g).\end{aligned}$$

Since the Haar measure ν is translation invariant, it follows that:

$$\begin{aligned}\|Tf\|_Y^p &= \pi_p(T)^p \int_G \Big(\int_G |f_g(h)|^p\, d\nu(h)\Big)\, d\mu(g)\\ &= \pi_p(T)^p \int_G \Big(\int_G |f(h)|^p\, d\nu(h)\Big)\, d\mu(g) = \pi_p(T)^p \int_G |f(h)|^p\, d\nu(h).\end{aligned}$$

□

As usual, the Fourier transform of $f \in L_1(G,\nu)$ is defined by:

$$\widehat{f}(\gamma) := \int_G f(g)\overline{\gamma(g)}\, d\nu, \quad \gamma \in \widehat{G}.$$

Let X be any subspace of $L_1(G,\nu)$ and $\Gamma \subset \widehat{G}$ a nonempty subset. Then:

$$X_\Gamma := \{f \in X\colon\ \widehat{f}(\gamma) = 0 \quad \text{for all } \gamma \notin \Gamma\}.$$

Clearly, X_Γ is a translation invariant subspace of X. Note that for $X = C(G)$ or $X = L_p(G,\nu)$ with $1 \le p < \infty$, every translation invariant subspace of X has the form X_Γ for some $\Gamma \subset \widehat{G}$.

In what follows, for simplicity of notation, we write C_Γ instead of $C(G)_\Gamma$. By $\widehat{C_\Gamma} \subset \mathbb{K}^\Gamma$, we denote the linear space of all $(\widehat{f}(\gamma))_{\gamma\in\Gamma}$, $f \in C_\Gamma$, which equipped with the norm:

$$\|(\widehat{f}(\gamma))_{\gamma\in\Gamma}\|_{\widehat{C_\Gamma}} := \|f\|_{C(G)}, \quad f \in C_\Gamma$$

forms a Banach space. Throughout the paper, if $\xi \in \mathbb{K}^\Gamma$ and $F = (\mathbb{K}^\Gamma, \|\cdot\|)$ are Banach spaces, then the mapping $M_\xi\colon C_\Gamma \to F$ (which we call multiplier) is given by:

$$M_\xi f := (\xi_\gamma \widehat{f}(\gamma))_{\gamma\in\Gamma}, \quad f \in C_\Gamma.$$

The space of all such multipliers is denoted by $\mathcal{M}(C_\Gamma, F)$, and it obviously identifies with the space $\mathcal{D}(\widehat{C_\Gamma}, F)$ of all diagonal operators from $\widehat{C_\Gamma}$ into F.

A subset Γ of $\widehat{G}$ is called a p-Sidon set ($1 \le p < \infty$) if there is a constant C, such that:

$$\Big(\sum_{\gamma\in\Gamma} |\widehat{f}(\gamma)|^p\Big)^{\frac{1}{p}} \le C\|f\|_\infty, \quad f \in C_\Gamma.$$

The least possible value of this constant is denoted $S_p(\Gamma)$ and called the p-Sidon constant of G.

https://doi.org/10.1017/S1474748022000391 Published online by Cambridge University Press

Let $\{-1,1\}$ be the compact discrete group with the Haar measure $\sigma_1(\{-1\}) = \sigma_1(\{1\}) = 1/2$, and let $\mathbb{T}$ be a unit circle equipped with normalised Lebesgue measure. We will primarily be interested in the case when $G = \mathbb{T}^n$, $G = \{-1,1\}^N$, $G = \{-1,1\}^\infty$ or $G = \mathbb{T}^\infty$.

The countable product $\{-1,1\}^\infty$ is a compact abelian group with the pointwise product and the product topology. The Haar measure on $\{-1,1\}^\infty$ is then just the countable product of the measure σ_1. Clearly, the dual group of $\{-1,1\}^{\mathbb{N}}$ is the set $\{\chi_S \colon S \in \mathcal{P}_{fin}(\mathbb{N})\}$, where $S \in \mathcal{P}_{fin}(\mathbb{N})$ means that $|S| := \mathrm{card}(S) < \infty$ and $\chi_S \colon \{-1,1\}^\infty \to \{-1,1\}$ is defined by $\chi_S(x) = \prod_{n\in S} x_n$ for all $x = (x_n)_{n\in\mathbb{N}}$ in $\{-1,1\}^\infty$.

The compact abelian group $\mathbb{T}^\infty$ carries the pointwise product, the product topology and the product of the normalised Lebesgue measure on $\mathbb{T}^\infty$ as its Haar measure. Denote by $\mathbb{Z}^{(\mathbb{N})}$ the set of all sequences $\alpha = (\alpha_1,\dots,\alpha_n,\dots)$ of integers which vanish for n large enough. Then we have $\widehat{\mathbb{T}^\infty} = \mathbb{Z}^{(\mathbb{N})}$, where each $\alpha \in \mathbb{Z}^{(\mathbb{N})}$ is identified with the character $\gamma(z) = z^\alpha := \prod_{j=1}^\infty z^{\alpha_j}$, $z \in \mathbb{T}^\infty$.

The subset of all sequences $\alpha \in \mathbb{Z}^{(\mathbb{N})}$, for which all entries are either 0 or natural, is denoted by $\mathbb{N}_0^{(\mathbb{N})}$. We write $\Lambda^{\leq}(m,n) \subset \mathbb{N}_0^{(\mathbb{N})}$ for the subset of all α's of length n and with order $|\alpha| = \sum_{j=1}^n \alpha_j \leq m$, whereas $\Lambda^{=}(m,n)$ consists of all α's of length n but with order $|\alpha| = m$.

Trigonometric polynomials. Given $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$, we denote $T(m,n)$ the set of all multi indices $\{\alpha \in \mathbb{Z}^n \colon |\alpha| \leq m\}$ and $\mathcal{T}_{\leq m}(\mathbb{T}^n)$ the space of all trigonometric polynomials:

$$P(z) = \sum_{\alpha\in T(m,n)} c_\alpha z^\alpha, \quad z \in \mathbb{T}^n$$

on the n-dimensional torus $\mathbb{T}^n$ which have degree $\deg(P) = \max\{|\alpha|;\, c_\alpha \neq 0\} \leq m$. Clearly, $\mathcal{T}_{\leq m}(\mathbb{T}^n)$ together with the sup norm $\|\cdot\|_{\mathbb{T}^n}$ (also denoted by $\|\cdot\|_\infty$) form a Banach space.

By $\mathcal{P}_{\leq m}(\mathbb{T}^n)$, we denote the closed subspace of $\mathcal{T}_{\leq m}(\mathbb{T}^n)$ of all trigonometric analytic polynomials $P(z) = \sum_{\alpha\in\Lambda^{\leq}(m,n)} c_\alpha z^\alpha$ for all $z \in \mathbb{T}^n$. The space $\mathcal{P}_{=m}(\mathbb{T}^n)$ is defined to be the closed subspace of all m-homogeneous polynomials P given by $P(z) = \sum_{\alpha\in\Lambda^{=}(m,n)} c_\alpha z^\alpha$.

Moreover, we are going to make use of the so-called 'hypercontractive' Bohnenblust–Hille inequality: There is some universal constant $C > 0$, such that, for each m,n and $P \in \mathcal{P}_{\leq m}(\mathbb{T}^n)$:

$$\Bigg(\sum_{\alpha\in\Lambda^{\leq}(m,n)} |\widehat{P}(\alpha)|^{\frac{2m}{m+1}} \Bigg)^{\frac{m+1}{2m}} \leq C^{\sqrt{m\log m}} \|P\|_\infty\,; \tag{2.1}$$

we refer to [5] (original form), [8] (hypercontractive form), [2] (subexponential form) and within the context of Dirichlet series and holomorphic functions in infinitely many variables to the monograph [9].

A well-known consequence of Bernstein's inequality (see, e.g. [20, Corollary 5.2.3]) is that, for all positive integers n,m, there is a subset $F \subset \mathbb{T}^n$ of cardinality card

$F \le (1+20m)^n$, such that, for every $P \in \mathcal{T}_{\le m}(\mathbb{T}^n)$, we have:

$$\sup_{z\in\mathbb{T}^n} |P(z)| \le 2 \sup_{z\in F} |P(z)|.$$

In other terms, for $N = (1+20m)^n$, the linear mapping:

$$I\colon \mathcal{T}_{\le m}(\mathbb{T}^n) \to \ell_\infty^N, \ \ I(P) := (P(z))_{z\in F} \tag{2.2}$$

is a 2-embedding of $\mathcal{T}_{\le m}(\mathbb{T}^n)$ into ℓ_∞^N.

3. A probabilistic proof

We use the famous Kahane–Salem–Zygmund inequality (the KSZ-inequality, see, e.g. [9, 10], [14] or [20]) to improve Theorem 1.1. Our argument is different from the original proof of Theorem 1.1, which we are going to analyse in the next section.

Theorem 3.1. *Let $(\Omega,\mathcal{A},\mathbb{P})$ be a probability measure space. Then there is a positive constant C, such that, for each $m,n \in \mathbb{N}$ and for every trigonometric random polynomial $P(\omega,z) = \sum_{\alpha\in T(m,n)} \varepsilon_\alpha(\omega) c_\alpha z^\alpha$, $(\omega,z) \in \Omega\times\mathbb{T}^n$, one has:*

$$\int_\Omega \|P(\omega,\cdot)\|_{\mathcal{T}_{\le m}(\mathbb{T}^n)}\, d\mathbb{P}(\omega) \le C\sqrt{n\log(1+m)}\, \|(c_\alpha)\|_{\ell_2(T(m,n))},$$

where $(\varepsilon_\alpha)_{\alpha\in T(m,n)}$ is a sequence of Bernoulli variables on $(\Omega,\mathcal{A},\mathbb{P})$.

As we will see in Theorem 3.3, the following consequence of the KSZ-inequality gives Kislyakov's multiplier Theorem 1.1 as a particular case.

Theorem 3.2. *There is a positive constant C, such that for all $n,m \in \mathbb{N}$, all Banach spaces $F = (\mathbb{C}^{T(m,n)}, \|\cdot\|)$ and for all sequences $\xi = (\xi_\alpha)_{\alpha\in T(m,n)}$:*

$$\sup_{\|\mu\|_{\ell_2(T(m,n))}\le 1} \|(\mu_\alpha \xi_\alpha)_{\alpha\in T(m,n)}\|_F \le C\sqrt{n\log(1+m)}\, \big\|M_\xi\colon \mathcal{T}_{\le m}(\mathbb{T}^n) \to F\big\|.$$

Proof. Let $(\varepsilon_\alpha)_{\alpha\in T(m,n)}$ be a sequence of Bernoulli variables on some probability space $(\Omega,\mathcal{A},\mathbb{P})$, and consider the following three operators:

$$R_\xi\colon \ell_2(T(m,n)) \to F,\ (c_\alpha) \mapsto (c_\alpha \xi_\alpha),$$

$$\phi_{KSZ}\colon \ell_2(T(m,n)) \to L_1(\mathbb{P}, \mathcal{T}_{\le m}(\mathbb{T}^n)),\ (c_\alpha) \mapsto \sum_{\alpha\in T(m,n)} \varepsilon_\alpha(\cdot) c_\alpha z^\alpha,$$

$$L_\xi\colon L_1(\mathbb{P}, \mathcal{T}_{\le m}(\mathbb{T}^n)) \to F,\ \ \sum_{\alpha\in T(m,n)} \varepsilon_\alpha(\cdot) c_\alpha z^\alpha \mapsto (c_\alpha \xi_\alpha).$$

Clearly, $R_\xi = L_\xi \circ \phi_{KSZ}$ and:

$$\|R_\xi\| = \sup_{\|\mu\|_{\ell_2(T(m,n))}\le 1} \|(\mu_\alpha \xi_\alpha)_{\alpha\in T(m,n)}\|_F.$$

Moreover, by the KSZ-inequality from Theorem 3.1, we get:

$$\|\phi_{KSZ}\| \le C\sqrt{n\log(1+m)}.$$

We claim that:

$$\|L_\xi\| \le \big\|M_\xi\colon \mathcal{T}_{\le m}(\mathbb{T}^n) \to F\big\|.$$

Indeed, given a random polynomial $P \in L_1(\mathbb{P}, \mathcal{T}_{\le m}(\mathbb{T}^n))$ given by:

$$P(\omega,z) = \sum_{\alpha \in T(m,n)} \varepsilon_\alpha(\omega) c_\alpha z^\alpha, \quad (\omega,z) \in \Omega \times \mathbb{T}^n.$$

Then, for every $\omega \in \Omega$, we have:

$$\|(c_\alpha \xi_\alpha)\|_F \le \big\|D_\xi\colon \mathcal{T}_{\le m}(\mathbb{T}^n) \to F\big\| \big\|P(\omega,\cdot)\big\|_{\mathcal{T}_{\le m}(\mathbb{T}^n)}.$$

Now integrating proves the claim. All together yields:

$$\begin{aligned}\sup_{\|\mu\|_{\ell_2(T(m,n))}\le 1} &\|(\mu_\alpha \xi_\alpha)_{\alpha \in T(m,n)}\|_F \\ &\le \|L_\xi\|\,\|\phi_{KSZ}\| \le C\sqrt{n\log(1+m)}\,\big\|M_\xi\colon \mathcal{T}_{\le m}(\mathbb{T}^n) \to F\big\|,\end{aligned}$$

and so this completes the proof. □

Now by Theorem 3.2, we deduce that Kislyakov's original Theorem 1.1 is a special case of the following more general result.

Corollary 3.3. *Let V be a closed subspace of $L_\infty(\mathbb{T}^\infty)$. Then for every $\ell_1(\mathbb{Z}^{(\mathbb{N})})$-multiplier $\xi = (\xi_\alpha)_{\alpha \in \mathbb{Z}^{(\mathbb{N})}}$ of V, one has:*

$$\sup_{n,m\in\mathbb{N}} \frac{1}{\sqrt{n\log(1+m)}} \Bigg(\sum_{\alpha \in \Gamma \cap T(m,n)} |\xi_\alpha|^2\Bigg)^{1/2} < \infty,$$

where:

$$\Gamma = \bigcup_{f \in V} \operatorname{supp} \widehat{f} \subset \mathbb{Z}^{(\mathbb{N})}.$$

In particular, every $\ell_1(\mathbb{N}_0^{(\mathbb{N})})$-multiplier of $H^\infty(\mathbb{T}^\infty)$ satisfies the preceding estimate whenever we replace $\Gamma \cap T(m,n)$ by $\Lambda^{\le}(m,n)$ and every $\ell_1(\mathbb{N}_0^{(\mathbb{N})})$-multiplier of $C^A(\mathbb{T}^\infty)$ satisfies the estimate from (1.1).

Proof. Let us prove the first statement. For a fixed $\xi = (\xi_\alpha)_{\alpha \in \mathbb{Z}^{(\mathbb{N})}}$, we can define a new sequence $\xi' = (\xi'_\alpha)_{\alpha \in \mathbb{Z}^{(\mathbb{N})}}$ with $\xi'_\alpha = \xi_\alpha$ if $\alpha \in \Gamma$ and $\xi'_\alpha = 0$ otherwise. It is clear that ξ is a $\ell_1(\mathbb{Z}^{(\mathbb{N})})$-multiplier if and only if so does ξ'. Thus, applying Theorem 3.2 to ξ', we immediately conclude the result. For the second statement, note that:

$$\begin{aligned}\Bigg(\sum_{\alpha \in \mathbb{N}_0^n\colon \max\{\alpha_1,\dots,\alpha_n\} \le m} |\xi_\alpha|^2\Bigg)^{\frac{1}{2}} &\le \Bigg(\sum_{|\alpha| \le mn} |\xi_\alpha|^2\Bigg)^{\frac{1}{2}} \\ &\le C\sqrt{n\log(1+mn)}\,\big\|M_\xi\colon \mathcal{P}_{\le mn}(\mathbb{T}^n) \to \ell_1(\Lambda^{\le}(mn,n))\big\| \\ &\le C\sqrt{n\log(1+mn)}\,\big\|M_\xi\colon C^A(\mathbb{T}^n) \to \ell_1(\mathbb{N}_0^n)\big\|,\end{aligned}$$

which completes the argument. □

Note that Corollary 4.5 below recovers the preceding result — using a different technique of proof with the advantage of slightly more explicit constants.

4. Variants for compact abelian groups

Based on local Banach space theory and inspired by ideas from [15], we improve Theorem 3.2 and its Corollary 3.3. Our more abstract approach has the advantage that it allows to consider more general compact abelian groups instead of only the multidimensional torus. The main result here is Theorem 4.4 below.

We start with the following basic lemma which is a simple consequence of Lemma 2.1 and crucial for our purpose.

Lemma 4.1. *Let G be a compact abelian group, Γ a finite subset in $\widehat{G}$ and $F := (\mathbb{C}^\Gamma, \|\cdot\|)$ a Banach space. Then for every $\xi = (\xi_\gamma)_{\gamma\in\Gamma}$:*

$$\pi_2(M_\xi \colon C_\Gamma \to F) = \|D_\xi \colon \ell_2(\Gamma) \to F\| = \sup_{\|\mu\|_{\ell_2(\Gamma)}\leq 1} \|(\mu_\gamma \xi_\gamma)_{\gamma\in\Gamma}\|_F.$$

Proof. The second equality is obvious, and from Lemma 2.1 and the orthogonality of the characters in $L_2(\nu)$ (where ν denotes the normalised Haar measure on G), we easily deduce that:

$$\sup_{\|\mu\|_{\ell_2(\Gamma)}\leq 1} \|(\mu_\gamma \xi_\gamma)_{\gamma\in\Gamma}\|_F \leq \pi_2(M_\xi \colon C_\Gamma \to F).$$

To see the reverse estimate, note that for every $f \in C_\Gamma$, one has:

$$\begin{aligned}\|M_\xi f\|_F = \|(\xi_\gamma \widehat{f}(\gamma))\|_F &\leq \|D_\xi \colon \ell_2(\Gamma) \to F\| \, \|(\widehat{f}(\gamma))\|_{\ell_2(\Gamma)} \\ &= \|D_\xi \colon \ell_2(\Gamma) \to F\| \Big(\int_G |f(x)|^2 d\nu(x) \Big)^{\frac{1}{2}},\end{aligned}$$

where we have used the Plancherel theorem for groups. Then, Pietsch domination theorem yields that:

$$\pi_2(M_\xi \colon C_\Gamma \to F) \leq \|D_\xi : \ell_2(\Gamma) \to F)\|. \qquad \square$$

As a very first application of the preceding lemma, we obtain an interesting reformulation of Theorem 3.2.

Corollary 4.2. *There is $C > 0$, such that for all $n, m \in \mathbb{N}$, all Banach spaces $F = (\mathbb{C}^{T(m,n)}, \|\cdot\|)$ and all $\xi = (\xi_\alpha)_{\alpha\in T(m,n)}$:*

$$\pi_2\big(M_\xi \colon \mathcal{T}_{\leq m}(\mathbb{T}^n) \to F\big) \leq C \sqrt{n \log(1+m)} \, \big\|M_\xi \colon \mathcal{T}_{\leq m}(\mathbb{T}^n) \to F\big\|.$$

In view of Bernstein's theorem from (2.2), the next result is a strong extension despite the cotype assumption. It will allow us to get the Kislyakov type multiplier theorems (in particular, Theorem 4.4) for compact abelian groups different from the multidimensional torus.

https://doi.org/10.1017/S1474748022000391 Published online by Cambridge University Press

Proposition 4.3. *Let $I\colon X \hookrightarrow \ell_\infty^N$ be a λ-embedding and F a Banach space. Then for every operator $T\colon X \to F$:*

$$\pi_2(T) \le e^2 \lambda C_2(F) \sqrt{1+\ln N}\,\|T\|.$$

Proof. Define for each $N \in \mathbb{N}$ the N-th harmonic number $h_N := \sum_{j=1}^N \frac{1}{j}$ and the discrete measure μ_N on the power set of $\{1,\dots,N\}$ given by $\mu_N(\{j\}) := \frac{1}{j}$. for each $j \in \{1,\dots,N\}$. In what follows, we need the elementary observations that $\log N < h_N \le 1+\log N$ and for every $\xi = (\xi_i)_{i=1}^N \in \mathbb{C}^N$:

$$\|\xi\|_{\ell_\infty^N} \le e\,\|\xi\|_{L_{h_N}(\mu_N)}.$$

Indeed, if $\|\xi\|_{\ell_\infty^N} = |\xi_k|$ for some $k \in \{1,\dots,N\}$, then:

$$\Big(\sum_{j=1}^N \frac{1}{j}|\xi_j|^{h_N}\Big)^{\frac{1}{h_N}} \ge \frac{1}{k^{1/h_N}}\,|\xi_k| = e^{-\frac{\log k}{h_N}}\,|\xi_k| \ge e^{-\frac{\log k}{\log N}}\,|\xi_k| \ge e^{-1}\,|\xi_k|.$$

It is well known that, for every operator $S\colon E \to F$ between Banach spaces and $2 \le p < \infty$, one has:

$$\pi_2(S) \le K_p C_2(F)\,\pi_p(S),$$

where $K_p \le \sqrt{p}$ is the best constant from the right-hand side of Khinchine's inequality for Rademacher p-averages (see [19, Theorem 5.15]).

Consider the following obvious factorisation of an operator $T\colon X \to F$:

$$T\colon X \xrightarrow{I} I(X) \xrightarrow{I^{-1}} X \xrightarrow{T} F.$$

Thus, the above facts combined with the ideal properties of p-summing operators yield:

$$\begin{aligned}
\pi_2(T) &\le \|I\|\,\pi_2(T\circ I^{-1}) \le \|I\|\,\sqrt{h_N}\,C_2(F)\,\pi_{h_N}(T\circ I^{-1})\\
&\le \|I\|\,\|I^{-1}\|\,\|T\|\,\sqrt{h_N}\,C_2(F)\,\pi_{h_N}(\mathrm{id}_{I(X)})\\
&\le \|T\|\,C_2(F)\,\sqrt{h_N}\,\lambda\,\pi_{h_N}(\mathrm{id}_{\ell_\infty^N}).
\end{aligned}$$

Since $\|\mathrm{id}\colon L_{h_N}(\mu_N) \to \ell_\infty^N\| \le e$, it follows that:

$$\pi_{h_N}(\mathrm{id}_{\ell_\infty^N}) \le e\pi_{h_N}(\mathrm{id}\colon \ell_\infty^N \to L_{h_N}(\mu_N)) = e h_N^{\frac{1}{h_N}} \le e^2. \qquad \square$$

The following theorem is the main result of this section, and we want to mention once again that its proof is very much inspired by [15, Theorem 6].

Theorem 4.4. *Let G be a compact abelian group and Γ a finite subset in $\widehat{G}$. Assume that $I\colon C_\Gamma \to \ell_\infty^N$ is a λ-embedding and $F := (\mathbb{C}^\Gamma, \|\cdot\|)$ a Banach space. Then, for every $\xi = (\xi_\gamma)_{\gamma\in\Gamma} \in \mathbb{C}^\Gamma$, one has:*

$$\sup_{\|\mu\|_{\ell_2(\Gamma)}\le 1} \|(\mu_\gamma\xi_\gamma)\|_F \le e^2\,\lambda\,C_2(F)\,\sqrt{1+\log N}\,\big\|M_\xi\colon C_\Gamma \to F\big\|.$$

https://doi.org/10.1017/S1474748022000391 Published online by Cambridge University Press

Comparing with Theorem 3.2, we see that the price we pay for the fact that the theorem applies to more general groups than the multidimensional tori is that the estimate involves the cotype constant of F.

Proof. By Lemma 4.3 and Bernstein's embedding from (2.2), we get:

$$\pi_2(M_\xi\colon C_\Gamma \to F) \le e^2\,\lambda\, C_2(F)\,\sqrt{1+\log N}\,\|M_\xi\colon C_\Gamma \to F\|.$$

Thus, it follows from Lemma 2.1 that for all $f \in C_\Gamma$:

$$\begin{aligned}\|(\widehat{f}(\gamma)\xi_\gamma)\|_F &\le e^2\,\lambda\, C_2(F)\,\sqrt{1+\log N}\,\|M_\xi\colon C_\Gamma \to F\|\Big(\int_G |f(g)|^2\,d\nu\Big)^{\frac12}\\ &= e^2\,\lambda\, C_2(F)\,\sqrt{1+\log N}\,\|M_\xi\colon C_\Gamma \to F\|\Big(\sum_{\gamma\in\Gamma} |\widehat{f}(\gamma)|^2\Big)^{\frac12}.\end{aligned}$$

Then the conclusion follows by duality. □

We finish with the following improvement of Corollary 3.3; its proof is an immediate consequence of Theorem 4.4, Bernstein's embedding from (2.2), the fact that $C_2(\ell_p(\mathbb{Z}^n)) \le \sqrt{2}$ for $1 \le p \le 2$ and, that by Hölder's inequality, we have:

$$\Big(\sum_{\gamma\in\Gamma} |\xi_\gamma|^r\Big)^{1/r} = \sup_{\|\mu\|_{\ell_2(\Gamma)}\le 1} \|(\mu_\gamma\xi_\gamma)\|_{\ell_p(\Gamma)}, \quad (\xi_\gamma) \in \ell_r(\Gamma)$$

for all $1 \le p \le 2$ with $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$.

Corollary 4.5. *Let $1 \le p \le 2$ and $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$. Then, for each $n, m \in \mathbb{N}$, and for every $\xi = (\xi_\alpha)_{\alpha\in\mathbb{Z}^n, |\alpha|\le m}$, the following estimate holds*:

$$\begin{aligned}&\Big(\sum_{\alpha\in\mathbb{Z}^n, |\alpha|\le m} |\xi_\alpha|^r\Big)^{1/r}\\ &\qquad\le 2\sqrt{2}e^2\,\sqrt{n\log(1+20m)}\,\big\|M_\xi\colon \mathcal{T}_{\le m}(\mathbb{T}^n) \to \ell_p\big(\{\alpha\in\mathbb{Z}^n\colon |\alpha|\le m\}\big)\big\|.\end{aligned}$$

The following remarks suggest that this corollary leaves some space for improvements. Given $1 \le p < \infty$, consider again the multiplication operator:

$$M_\xi\colon \mathcal{P}_{\le m}(\mathbb{T}^n) \to \ell_p(\Lambda^{\le}(m,n)), \quad f \mapsto \big(\widehat{f}(\alpha)\xi_\alpha\big)_{\alpha\in\Lambda^{\le}(m,n)}.$$

If $2 \le p < \infty$, then:

$$\sup_{\alpha\in\Lambda^{\le}(m,n)} |\xi_\alpha| = \|M_\xi\|,$$

whereas for $\frac{2m}{m+1} \le p \le 2$ by the hypercontractive BH-inequality (2.1):

$$\sup_{\alpha\in\Lambda^{\le}(m,n)} |\xi_\alpha| \le \|M_\xi\| \le C^m \sup_{\alpha\in\Lambda^{\le}(m,n)} |\xi_\alpha|.$$

Hence, in view of Corollary 4.5, the complex interpolation between the extreme cases $\ell_1(\Gamma)$ and $\ell_{\frac{2m}{m+1}}(\Gamma)$ suggests the following conjecture.

https://doi.org/10.1017/S1474748022000391 Published online by Cambridge University Press

Conjecture 4.6. *Let* $1 \leq p \leq \frac{2m}{m+1}$ *and* $\frac{1}{s} = \frac{m}{m-1}\big(\frac{1}{p} - \frac{m+1}{2m}\big)$. *Then, there exists a universal constant* $C > 0$, *such that for each* $n,m \in \mathbb{N}$, *and for every* $\xi = (\xi_\alpha)_{\alpha \in \Lambda^{\leq}(m,n)}$,

$$\Big(\sum_{\alpha \in \Lambda^{\leq}(m,n)} |\xi_\alpha|^s\Big)^{1/s} \leq C^{\frac{m}{s}} (n \log(1+20m))^{\frac{1}{s}} \, \|M_\xi : \mathcal{P}_{\leq m}(\mathbb{T}^n) \to \ell_p(\Lambda^{\leq}(m,n))\|.$$

5. Variants by interpolation

The purpose of this section is to prove two variants of Kislyakov's Theorem 3.3, both based on Theorem 4.4 and interpolation methods. We at first recall some notation from interpolation theory (see, e.g. [3]). The pair $\vec{X} = (X_0, X_1)$ of Banach spaces is called a Banach couple if there exists a Hausdorff topological vector space $\mathcal{X}$, such that $X_j \hookrightarrow \mathcal{X}$, $j = 0,1$. A mapping $\mathcal{F}$, acting on the class of all Banach couples, is called an interpolation functor if for every couple $\vec{X} = (X_0, X_1)$, $\mathcal{F}(\vec{X})$ is a Banach space which is intermediate with respect to $\vec{X}$ (i.e. $X_0 \cap X_1 \subset \mathcal{F}(\vec{X}) \subset X_0 + X_1$) and $T\colon \mathcal{F}(\vec{X}) \to \mathcal{F}(\vec{Y})$ is bounded for every operator $T\colon \vec{X} \to \vec{Y}$ (meaning $T\colon X_0 + X_1 \to Y_0 + Y_1$ is linear and its restrictions $T\colon X_j \to Y_j$, $j = 0,1$ are defined and bounded). If, additionally, there is a constant $C > 0$, such that for each $T\colon \vec{X} \to \vec{Y}$:

$$\|T\colon \mathcal{F}(\vec{X}) \to \mathcal{F}(\vec{Y})\| \leq C \, \|T\colon \vec{X} \to \vec{Y}\|,$$

where $\|T\colon \vec{X} \to \vec{Y}\| := \max\{\|T\colon X_0 \to Y_0\|, \|T\colon X_1 \to Y_1\|\}$, then $\mathcal{F}$ is called bounded. Clearly, $C \geq 1$, and if $C = 1$, then $\mathcal{F}$ is called exact.

For an exact interpolation functor $\mathcal{F}$, we define the fundamental function $\psi_{\mathcal{F}}$ of $\mathcal{F}$ by:

$$\psi_{\mathcal{F}}(s,t) = \sup \|T\colon \mathcal{F}(\vec{X}) \to \mathcal{F}(\vec{Y})\|, \quad s,t > 0,$$

where the supremum is taken over all Banach couples $\vec{X}$, $\vec{Y}$ and all operators $T\colon \vec{X} \to \vec{Y}$, such that $\|T\colon X_0 \to Y_0\| \leq s$ and $\|T\colon X_1 \to Y_1\| \leq t$.

Theorem 5.1. *Let* G *be a compact abelian group and* Γ *a finite subset in* $\widehat{G}$. *Suppose that* $\mathcal{F}$ *is an exact interpolation functor with the fundamental function* $\psi_{\mathcal{F}}$. *Given two norms* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *on* $\mathbb{C}^\Gamma$ *define:*

$$F := \mathcal{F}\big((\mathbb{C}^\Gamma, \|\cdot\|_1), (\mathbb{C}^\Gamma, \|\cdot\|_2)\big).$$

Then, for every $(\xi_\gamma)_{\gamma \in \Gamma} \in \mathbb{C}^\Gamma$, *one has:*

$$\sup_{\|\mu\|_{\ell_2(\Gamma)} \leq 1} \|(\mu_\gamma \xi_\gamma)_{\gamma \in \Gamma}\|_F \leq \psi_{\mathcal{F}}(1,1)\, \psi_{\mathcal{F}}\big(\pi_2(M_\xi\colon C_\Gamma \to (\mathbb{C}^\Gamma, \|\cdot\|_1)), \pi_2(M_\xi\colon C_\Gamma \to (\mathbb{C}^\Gamma, \|\cdot\|_2))\big).$$

Note that Lemma 4.1 shows:

$$\pi_2(M_\xi\colon C_\Gamma \to (\mathbb{C}^\Gamma, \|\cdot\|_i)) = \|D_\xi\colon \ell_2(\Gamma) \to (\mathbb{C}^\Gamma, \|\cdot\|_i)\|, \quad i = 1,2.$$

Hence, the proof of Theorem 5.1 is straightforward: It follows from the definition of the function $\psi_{\mathcal{F}}$ that, for any operator $T\colon \vec{X} \to \vec{Y}$ between the Banach couples $\vec{X} = (X_0, X_1)$

https://doi.org/10.1017/S1474748022000391 Published online by Cambridge University Press

and $\vec{Y} = (Y_0, Y_1)$, we have:

$$\big\|T\colon \mathcal{F}(\vec{X}) \to \mathcal{F}(\vec{Y})\big\| \le \psi_{\mathcal{F}}\big(\|T\colon X_0 \to Y_0\|, \|T\colon X_1 \to Y_1\|\big).$$

In particular, this implies that, for any Banach space Y,

$$\|\mathrm{id}\colon Y \to \mathcal{F}(Y,Y)\| \le \psi_{\mathcal{F}}(1,1).$$

5.1. Variant I

Theorem 5.2. *Let G be a compact abelian group, Γ a finite subset in $\widehat{G}$ and $I\colon C_\Gamma \to \ell_\infty^N$ a λ-embedding. Suppose that $\mathcal{F}$ is an exact interpolation functor with the fundamental function $\psi_{\mathcal{F}}$, and let:*

$$X = \mathcal{F}(\ell_1(\Gamma), \ell_2(\Gamma)).$$

Then, for every $\xi = (\xi_\gamma)_{\gamma\in\Gamma}$, one has:

$$\sup_{\|\mu\|_{\ell_2(\Gamma)}\le 1} \|(\xi_\gamma \mu_\gamma)_{\gamma\in\Gamma}\|_X \le K\, \psi_{\mathcal{F}}\big(\|M_\xi\colon C_\Gamma \to \ell_1(\Gamma)\|, \|\xi\|_{\ell_\infty(\Gamma)}\big)\, \psi_{\mathcal{F}}\big(\sqrt{1+\log N}, 1\big),$$

where $K = e^2\lambda\sqrt{2}\,\psi_{\mathcal{F}}(1,1)$.

Proof. Fix $\xi = (\xi_\gamma) \in \mathbb{C}^\Gamma$. Since $\ell_1(\Gamma)$ has cotype 2 with $C_2(\ell_1(\Gamma) \le \sqrt{2}$, it follows from Proposition 4.3 that:

$$\pi_2(M_\xi\colon C_\Gamma \to \ell_1(\Gamma)) \le e^2\sqrt{2}\lambda\sqrt{1+\ln N}\, \|M_\xi\colon C_\Gamma \to \ell_1(\Gamma)\|.$$

Now observe that for any finite sequence $(f_i)_{i=1}^N$ in C_Γ, we have (where for a given $g\in G$, the Dirac functional $\delta_g \in B_{C(G)^*}$ is given by $\delta_g(f) := f(g)$ for all $f \in C(G)$):

$$\begin{aligned}\sum_{i=1}^N \|M_\xi f_i\|_{\ell_2(\Gamma)}^2 &\le \|\xi\|_\infty \int_G \sum_{i=1}^N |f_i(g)|^2\, d\nu\\ &\le \|\xi\|_\infty \sup_{g\in G} \sum_{i=1}^N |\delta_g(f_i)|^2 \le \|\xi\|_\infty \sup_{\|x^*\|_{C(G)^*}\le 1} \sum_{i=1}^N |x^*(f_i)|^2.\end{aligned}$$

This shows that $\pi_2(M_\xi\colon C_\Gamma \to \ell_2(\Gamma)) \le \|\xi\|_\infty$. Since $\|\xi\|_\infty = \|M_\xi\colon C_\Gamma \to \ell_2(\Gamma)\| \le \pi_2(M_\xi\colon C_\Gamma \to \ell_2(\Gamma))$, we get:

$$\pi_2(M_\xi\colon C_\Gamma \to \ell_2(\Gamma)) = \|\xi\|_\infty. \tag{5.1}$$

Applying Proposition 5.1, we conclude (by submultiplicativity of $\psi_{\mathcal{F}}$) that:

$$\begin{aligned}\pi_2(M_\xi\colon C_\Gamma \to X) &\le \psi_{\mathcal{F}}(1,1)\, \psi_{\mathcal{F}}\big(\pi_2(M_\xi\colon C_\Gamma \to \ell_1(\Gamma)), \pi_2(M_\xi\colon C_\Gamma \to \ell_2(\Gamma)\big)\\ &\le K\, \psi_{\mathcal{F}}\big(\sqrt{1+\log N}, 1\big)\, \psi_{\mathcal{F}}\big(\|M_\xi\colon C_\Gamma \to \ell_1(\Gamma)\|, \|\xi\|_\infty\big).\end{aligned}$$

This estimate combined with Lemma 2.1 yield the required statement. □

In order to see a first consequence, we apply the preceding theorem to $G = \mathbb{T}^n$ and $C_\Gamma = \mathcal{P}_{\le m}(\mathbb{T}^n)$ with $\Gamma := \Lambda^{\le}(m,n)$. For simplicity of notation, we for $1 \le p \le \infty$ write below ℓ_p instead of $\ell_p(\Lambda^{\le}(m,n))$.

https://doi.org/10.1017/S1474748022000391 Published online by Cambridge University Press

Corollary 5.3. *Let* $\frac{1}{p_\theta} = (1-\theta) + \frac{\theta}{2}$ *for* $\theta \in (0,1)$. *Then for every* $\xi = (\xi_\gamma) \in \mathbb{C}^{\Lambda^{\leq}(m,n)}$, *one has*:

$$\sup_{\|(\mu_\gamma)\|_{\ell_2}\leq 1} \big\|(\mu_\gamma \xi_\gamma)\big\|_{\ell_{p_\theta}} \leq 2e^2\sqrt{2}\left(\sqrt{n\log(1+20m)}\,\|M_\xi\colon \mathcal{P}_{\leq m}(\mathbb{T}^n) \to \ell_1\|\right)^{1-\theta}\|\xi\|_\infty^\theta.$$

In particular, one has:

$$\sup_{\|(\mu_\gamma)\|_{\ell_2}\leq 1} \big\|(\mu_\gamma \xi_\gamma)\big\|_{\ell_{\frac{2m}{m+1}}}$$
$$\leq 2e^2\sqrt{2}\left(\sqrt{n(\log(1+20m)}\,\|M_\xi\colon \mathcal{P}_{\leq m}(\mathbb{T}^n) \to \ell_1\|\right)^{\frac{1}{m}}(\|\xi\|_\infty)^{1-\frac{1}{m}}.$$

Proof. It is well known that the complex method $\mathcal{F}_\theta := [\,\cdot\,]_\theta$ of interpolation has the fundamental function $\psi_F(s,t) = s^{1-\theta}t^\theta$ for all $s,t > 0$. Applying the well-known interpolation formula for couples of ℓ_p-spaces, we get:

$$\left[\ell_1,\ell_2\right]_\theta \cong \ell_{p_\theta}.$$

Since by Bernstein's result from (2.2) there exists a 2-embedding of $\mathcal{P}_{\leq m}(\mathbb{T}^n)$ into ℓ_∞^N with $N = (1+20m)^n$, Theorem 5.2 gives the first conclusion. To get the second assertion, we take $\theta = 1-\frac{1}{m}$. □

For another seemingly interesting consequence, we recall the definition of the abstract Lorentz space $\Lambda_\varphi(\vec{X})$.

For a given function $\varphi \in \mathcal{Q}$ and Banach couple $\vec{X} = (X_0,X_1)$, the abstract Lorentz space $\Lambda_\varphi(\vec{X})$ is defined to be the space of all $x \in X_0 + X_1$, such that:

$$x = \sum_{n\in\mathbb{Z}} x_n, \quad (\text{convergence in } X_0 + X_1),$$

where $x_n \in X_0 \cap X_1$ and $\sum_{n\in\mathbb{Z}} \varphi(\|x_n\|_{X_0},\|x_n\|_{X_1}) < \infty$. The norm on $\Lambda_\varphi(\vec{X})$ is defined by:

$$\|x\|_{\Lambda_\varphi(\vec{X})} = \inf \sum_{n\in\mathbb{Z}} \varphi(\|x_n\|_{X_0},\|x_n\|_{X_1}),$$

where the infimum is taken over all series described above. It is easily verified that Λ_φ defines an exact interpolation functor.

Corollary 5.4. *Under the notation and assumption of Theorem 5.2, we get for* $\varphi := \psi_{\mathcal{F}}$:

$$\big\|\mathrm{id}\colon \Lambda_\varphi(\mathcal{M}(\mathrm{C}_\Gamma,\ell_1(\Gamma)),\ell_\infty(\Gamma)) \to \mathcal{D}(\ell_2(\Gamma),\mathcal{F}(\ell_1(\Gamma),\ell_2(\Gamma))\big\| \leq K\varphi\big(\sqrt{1+\log N},1\big).$$

Proof. From Theorem 5.2, for any $\xi = (\xi_\gamma) \in \mathbb{C}^\Gamma$, we get that:

$$\|D_\xi\colon \ell_2(\Gamma) \to \mathcal{F}(\ell_1(\Gamma),\ell_2(\Gamma))\|$$
$$\leq K\,\varphi\big(\sqrt{1+\log N},1\big)\,\varphi(\|\xi\|_{\mathcal{M}(C_\Gamma,\ell_1(\Gamma))},\|\xi\|_{\ell_\infty(\Gamma)}).$$

By the construction of the abstract Lorentz space Λ_φ, this completes the proof. □

https://doi.org/10.1017/S1474748022000391 Published online by Cambridge University Press

5.2. Variant II

Theorem 5.5. *Let $\mathcal{F}$ be an exact interpolation functor with a fundamental function $\psi_{\mathcal{F}}$ and for $m,n \in \mathbb{N}$:*

$$X = \mathcal{F}\big(\ell_1(\Lambda^{\leq}(m,n)), \ell_{\frac{2m}{m+1}}(\Lambda^{\leq}(m,n))\big).$$

Then for every $\xi = (\xi_\alpha)_{\alpha \in \Lambda(m,n)}$, one has:

$$\sup_{\|\mu\|_{\ell_2(\Lambda^{\leq}(m,n))} \leq 1} \|(\xi_\alpha \mu_\alpha)_{\alpha \in \Lambda^{\leq}(m,n)}\|_X \leq C(m)\sqrt{n}\,\|M_\xi \colon \mathcal{P}_{\leq m}(\mathbb{T}^n) \to \ell_1(\Lambda^{\leq}(m,n))\|,$$

where $C(m) = e^2 2\sqrt{2}\,\psi_{\mathcal{F}}(1,1)\,\psi_{\mathcal{F}}\big(\sqrt{\log(1+20m)},1\big)$.

In order to prove this result (see the end of this subsection), we apply Theorem 5.1 in combination with Proposition 4.3 and the following lemma.

Lemma 5.6. *For each $m,n \in \mathbb{N}$ and for every $\xi \in \mathbb{C}^{\Lambda^{\leq}(m,n)}$,*

$$\pi_2\big(M_\xi \colon \mathcal{P}_{\leq m}(\mathbb{T}^n) \to \ell_{\frac{2m}{m+1}}(\Lambda^{\leq}(m,n))\big) \leq |\Lambda^{\leq}(m,n)|^{\frac{1}{2m}}\,\|\xi\|_\infty.$$

In particular, if $\xi = \mathbf{1}$,

$$\sqrt{1+\frac{n-1}{m}} \leq \pi_2\big(M_\xi \colon \mathcal{P}_{\leq m}(\mathbb{T}^n) \to \ell_{\frac{2m}{m+1}}(\Lambda^{\leq}(m,n))\big)$$
$$= |\Lambda^{\leq}(m,n)|^{\frac{1}{2m}} \leq 2\sqrt{2e}\sqrt{1+\frac{n-1}{m}}.$$

Proof. We apply Lemma 4.1 in the case $G = \mathbb{T}^n$ with $\Gamma = \Lambda^{\leq}(m,n) \subset \widehat{G} = \mathbb{Z}^n$ and $F = \ell_{\frac{2m}{m+1}}(\Lambda^{\leq}(m,n))$. Clearly, for each $m,n \in \mathbb{N}$, we have:

$$\big\|\mathrm{id} \colon \ell_2(\Lambda^{\leq}(m,n)) \to \ell_{\frac{2m}{m+1}}(\Lambda^{\leq}(m,n))\big\|$$
$$= |\Lambda^{\leq}(m,n)|^{\frac{1}{2m}} = \left(\sum_{k=0}^{m}\binom{k+n-1}{k}\right)^{\frac{1}{2m}} \leq (m+1)^{\frac{1}{2m}}\binom{m+n-1}{m}^{\frac{1}{2m}}.$$

Applying the well-known quantitative version of Stirling's formula yields:

$$\binom{m+n-1}{m} \leq 2e^m\Big(1+\frac{n-1}{m}\Big)^m.$$

Now observe that, for any $x \geq y > z > 0$, $\frac{x-z}{y-z} \geq \frac{x}{y}$ and whence:

$$\binom{N}{k} = \frac{N}{k}\,\frac{N-1}{k-1}\cdots\frac{N-k+1}{1} \geq \Big(\frac{N}{k}\Big)^k, \quad 1 \leq k \leq N.$$

In consequence, we deduce that:

$$\sqrt{1+\frac{n-1}{m}} \leq |\Lambda^{\leq}(m,n)|^{\frac{1}{2m}} \leq 2\sqrt{2e}\sqrt{1+\frac{n-1}{m}},$$

and this gives the required estimates. □

Finally, we are prepared to give the proof of Theorem 5.5.

https://doi.org/10.1017/S1474748022000391 Published online by Cambridge University Press

Proof. (of Theorem 5.5). Note that:

$$\|\xi\|_\infty \le \beta := \|M_\xi\colon \mathcal{P}_{\le m}(\mathbb{T}^n \to \ell_1(\Lambda^{\le}(m,n))\|.$$

Since $\ell_1(\Lambda^{\le}(m,n)$ has cotype 2 with $C_2(\ell_1(\Lambda^{\le}(m,n)) \le \sqrt{2}$, it follows from Proposition 4.3 and the embedding from (2.2) that:

$$\pi_2(M_\xi\colon \mathcal{P}_{\le m}(\mathbb{T}^n) \to \ell_1(\Lambda^{\le}(m,n))) \le e^2 2\sqrt{2}\beta\sqrt{n\log(1+20m)}.$$

From Lemma 5.6 we know that:

$$\pi_2(M_\xi\colon \mathcal{P}_{\le m}(\mathbb{T}^n) \to \ell_{\frac{2m}{m+1}}(\Lambda^{\le}(m,n)) \le 2\sqrt{2e}\sqrt{n}\beta.$$

Applying Theorem 5.1, we conclude (by submulitiplicativity of $\psi_{\mathcal{F}}$) that with $C = \psi_{\mathcal{F}}(1,1)$, it holds that:

$$\begin{aligned}
&\sup_{\|\mu\|_{\ell_2(\Lambda^{\le}(m,n))}\le 1} \|(\mu_\gamma\xi_\gamma)_{\gamma\in\Gamma}\|_F\\
&\le C\psi_{\mathcal{F}}\big(\pi_2(M_\xi\colon \mathcal{P}_{\le m}(\mathbb{T}^n) \to \ell_1(\Lambda^{\le}(m,n))), \pi_2(M_\xi\colon \mathcal{P}_{\le m}(\mathbb{T}^n) \to \ell_2(\Lambda^{\le}(m,n))\big)\\
&\le C\psi_{\mathcal{F}}\big(e^2 2\sqrt{2}\sqrt{n\log(1+20m)}\beta, 2\sqrt{2e}\sqrt{n}\beta\big)\\
&\le Ce^2 2\sqrt{2}\beta\sqrt{n}\psi_{\mathcal{F}}\big(\sqrt{\log(1+20m)},1\big).
\end{aligned}$$

□

The following corollary 'interpolates' the estimates from Corollary 3.2 ($p=1$) and Lemma 5.6 ($p = 2m/(m+1)$).

Corollary 5.7. *For $m\in\mathbb{N}$, let $1\le p\le \frac{2m}{m+1}$. Then, for each $n\in\mathbb{N}$ and every $\xi = (\xi_\alpha)_{\alpha\in\Lambda^{\le}(m,n)}$,*

$$\Big(\sum_{\alpha\in\Lambda^{\le}(m,n)} |\xi_\alpha|^r\Big)^{1/r} \le 2\sqrt{2}e^2\sqrt{n}\sqrt{\log(1+20m)}^{\beta_m}\,\|M_\xi\colon \mathcal{P}_{\le m}(\mathbb{T}^n) \to \ell_p(\Lambda^{\le}(m,n))\|,$$

where $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$ and $\beta_m = 1 - \frac{1-1/p}{1-\frac{m+1}{2m}}$.

Proof. Define $\theta_m\in(0,1)$ by $\frac{1}{p} = \frac{1-\theta_m}{1} + \frac{\theta_m}{\frac{2m}{m+1}}$. Then $1-\theta_m = \beta_m$. Hence, by Theorem 5.5 and as in the proof of Corollary 5.3, it follows that for every $\xi = (\xi_\alpha)_{\alpha\in\Lambda^{\le}(m,n)}$, we get:

$$\Big(\sum_{\alpha\in\Lambda^{\le}(m,n)} |\xi_\alpha|^r\Big)^{1/r} \le C(m)\sqrt{n}\|M_\xi\colon \mathcal{P}_{\le m}(\mathbb{T}^n) \to \ell_1(\Lambda^{\le}(m,n))\|,$$

where $C(m) = 2\sqrt{2}e^2(\log(1+20m))^{\beta_m}$. □

6. Applications

6.1. Multipliers of analytic trigonometric polynomials

Recall from (2) the definition of $\mathcal{P}_{\le m}(\mathbb{T}^n)$, all trigonometric analytic polynomials of degree $\le m$ on the n-dimensional torus $\mathbb{T}^n$ and its subspace $\mathcal{P}_{=m}(\mathbb{T}^n)$ of all m-homogeneous polynomials.

We also recall a definition, which we already touched on in the introduction, the Hardy space:

$$H_\infty(\mathbb{T}^\infty) := \Big\{ f \in L_\infty(\mathbb{T}^\infty) \colon \ \widehat{f}(\alpha) = 0, \ \forall \alpha \in \mathbb{Z}^{(\mathbb{N})} \setminus \mathbb{N}_0^{(\mathbb{N})} \Big\}$$

and its m-homogeneous part:

$$H_\infty^m(\mathbb{T}^\infty) := \Big\{ f \in H_\infty(\mathbb{T}^\infty) \colon \ \widehat{f}(\alpha) \neq 0 \Rightarrow |\alpha| = m \Big\}.$$

6.1.1. Sidon constants I. Let $1 \leq p < \infty$. By $\chi_p\big(\mathcal{P}_{\leq m}(\mathbb{T}^n)\big)$, we denote the p-Sidon constant of $\mathcal{P}_{\leq m}(\mathbb{T}^n)$, that is, the best constant $c > 0$, such that, for all polynomials $f(z) = \sum_{\alpha \in \Lambda^{\leq}(m,n)} \widehat{f}(\alpha) z^\alpha$, we have:

$$\Big(\sum_{\alpha \in \Lambda^{\leq}(m,n)} |\widehat{f}(\alpha)|^p \Big)^{\frac{1}{p}} \leq c \|f\|_\infty .$$

Similarly, we define $\chi_p\big(\mathcal{P}_{= m}(\mathbb{T}^n)\big)$, but in this case, we only consider m-homogeneous trigonometric polynomials instead of all trigonometric polynomials of degree less than or equal to m. Since for every $f \in \mathcal{P}_{\leq m}(\mathbb{T}^n)$:

$$\Big(\sum_{|\alpha| \leq m} |\widehat{f}(\alpha)|^2 \Big)^{\frac{1}{2}} = \Big(\int_{\mathbb{T}^n} |f(z)|^2 \, dz \Big)^{\frac{1}{2}} \leq \|f\|_\infty,$$

it follows that, for each $m,n \in \mathbb{N}$,

$$\chi_p\big(\mathcal{P}_{=m}(\mathbb{T}^n)\big) = \chi_p\big(\mathcal{P}_{\leq m}(\mathbb{T}^n)\big) = 1, \quad 2 \leq p \leq \infty.$$

Observe also that the Bohnenblust–Hille inequality (2.1) combined with Hölder's inequality imply that there exists a constant $C > 0$, such that, for each $m,n \in \mathbb{N}$, we have:

$$1 \leq \chi_p\big(\mathcal{P}_{=m}(\mathbb{T}^n)\big) \leq \chi_p\big(\mathcal{P}_{\leq m}(\mathbb{T}^n)\big) \leq C^m, \quad \frac{2m}{m+1} \leq p < 2. \tag{6.1}$$

Theorem 6.1. *Let $1 \leq p \leq \frac{2m}{m+1}$ and $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$. Then there is a universal positive constant γ, such that, for each $m,n \in \mathbb{N}$:*

$$\frac{1}{\gamma^m} \Big(\frac{n}{m} \Big)^{\frac{m}{r} - \frac{1}{2}} \leq \chi_p\big(\mathcal{P}_{=m}(\mathbb{T}^n)\big) \leq \chi_p\big(\mathcal{P}_{\leq m}(\mathbb{T}^n)\big) \leq \gamma^m \Big(\frac{n}{m} \Big)^{\frac{m}{r} - \frac{1}{2}}.$$

Note that $\frac{m}{r} - \frac{1}{2} \geq 0$ whenever $1 \leq p \leq \frac{2m}{m+1}$ and $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$. Moreover, as it should be, we have that $\frac{m}{r} - \frac{1}{2} = \frac{m-1}{2}$ for $p = 1$ and $\frac{m}{r} - \frac{1}{2} = 0$ for $p = \frac{2m}{m+1}$.

For the homogeneous case and $p = 1$, this result is proved in [8] (see also [9, Theorem 9.10]) — the upper estimate is based on the hypercontractivity of the Bohnenblust–Hille inequality and the lower estimate on the Kahane–Salem–Zygmund inequality. Here, we deduce the upper inequality from the case $p = 1$ by applying the complex interpolation method and the lower estimate by the following independently interesting lemma based on Corollary 4.5.

https://doi.org/10.1017/S1474748022000391 Published online by Cambridge University Press

Lemma 6.2. *Let $1 \le p \le \frac{2m}{m+1}$ and $z \in \mathbb{C}^n$, and denote by z^* the decreasing rearrangement of z. Then, for each $m \in \mathbb{N}$, one has:*

$$(z_n^*)^m \le 2\sqrt{2}e^2 \sqrt[r]{m!}\sqrt{\log(1+20m)}\,\frac{1}{n^{\frac{m}{r}-\frac{1}{2}}}\,\big\|M_z : \mathcal{P}_{=m}(\mathbb{T}^n) \to \ell_p(\Lambda^{=}(m,n))\big\|,$$

where $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$ and:

$$M_z : \mathcal{P}_{=m}(\mathbb{T}^n) \to \ell_p(\Lambda^{=}(m,n)),\ f \to \big(\widehat{f}(\alpha)(z^*)^\alpha\big).$$

Proof. From Corollary 4.5, we deduce that:

$$\begin{aligned}(z_n^*)^{rm}\sum_{|\alpha|=m} 1 &= \sum_{|\alpha|=m}\big((z_n^*)^{\alpha_1}\cdots(z_n^*))^{\alpha_n}\big)^r \\ &\le \sum_{|\alpha|=m}(z^*)^{r\alpha} \le (2\sqrt{2}e^2)^r\,(n\log(1+20m))^{\frac{r}{2}}\,\|M_z\|^r.\end{aligned}$$

Since $\dim \mathcal{P}_m(\mathbb{T}^n) = \binom{n+m-1}{m}$, we get:

$$(z_n^*)^{rm}\frac{n^m}{m!} \le (z_n^*)^{rm}\binom{n+m-1}{m} \le (2\sqrt{2}e^2)^r\,(n\log(1+20m))^{\frac{r}{2}}\,\|M_z\|^r,$$

and the desired result follows by taking roots. $\square$

We are ready to give the proof of the theorem.

Proof. (of Theorem 6.1). Lower bound: If $\mathbf{1} = (1,\ldots,1) \in \mathbb{C}^{\Lambda^{=}(m,n)}$, then by the definition we obtain:

$$\|M_{\mathbf{1}} : \mathcal{P}_{=m}(\mathbb{T}^n) \to \ell_p(\Lambda^{=}(m,n))\| = \chi_p\big(\mathcal{P}_{=m}(\mathbb{T}^n)\big).$$

Then, by Lemma 6.2:

$$1 \le 2\sqrt{2}e^2\sqrt[r]{m!}\sqrt{\log(1+20m)}\,\chi_p\big(\mathcal{P}_{=m}(\mathbb{T}^n)\big)\frac{1}{n^{\frac{m}{r}-\frac{1}{2}}},$$

and so:

$$\begin{aligned}\frac{1}{\gamma^m}\left(\frac{n}{m}\right)^{\frac{m}{r}-\frac{1}{2}} &\le \frac{1}{2\sqrt{2}e^2 m^{\frac{1}{2}}(1+\log m)^{\frac{1}{2}}}\left(\frac{n}{m}\right)^{\frac{m}{r}-\frac{1}{2}} \\ &\le \frac{m^{\frac{m}{r}-\frac{1}{2}}}{2\sqrt{2}e^2(m!)^{\frac{1}{r}}(1+\log m)^{\frac{1}{2}}}\left(\frac{n}{m}\right)^{\frac{m}{r}-\frac{1}{2}} \le \chi_p\big(\mathcal{P}_{=m}(\mathbb{T}^n)\big).\end{aligned}$$

Upper bound: Define $0 \le \theta \le 1$ by:

$$\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{\frac{2m}{m+1}},$$

then:

$$\frac{m-1}{2}\theta = \frac{m}{r} - \frac{1}{2}.$$

https://doi.org/10.1017/S1474748022000391 Published online by Cambridge University Press

We know that:

$$\|M_{\mathbf{1}}\colon \mathcal{P}_{\leq m}(\mathbb{T}^n) \to \ell_1(\Lambda^{\leq}(m,n))\| \leq C_1^m \left(\frac{n}{m}\right)^{\frac{m-1}{2}},$$

and by (6.1):

$$\|M_{\mathbf{1}}\colon \mathcal{P}_{\leq m}(\mathbb{T}^n) \to \ell_{\frac{2m}{m+1}}(\Lambda^{\leq}(m,n))\| \leq C_2^m.$$

Applying the complex interpolation method, we get:

$$\|M_{\mathbf{1}}\colon \mathcal{P}_{\leq m}(\mathbb{T}^n) \to \ell_p(\Lambda^{\leq}(m,n))\| \leq \left(C_1^m \left(\frac{n}{m}\right)^{\frac{m-1}{2}}\right)^{\theta} \left(C_2^m\right)^{1-\theta} \leq \gamma^m \left(\frac{n}{m}\right)^{\frac{m}{r}-\frac{1}{2}},$$

and so the conclusion follows. □

Why would a positive answer to Conjecture 4.6 not lead to a better lower bound? In this case, we, for $\frac{1}{s} = \frac{m}{m-1}\left(\frac{1}{p} - \frac{m+1}{2m}\right)$, would get that:

$$\frac{1}{\gamma^m}\left(\frac{n}{m}\right)^{\frac{m}{s}-\frac{1}{s}} \leq \chi_p\big(\mathcal{P}_{=m}(\mathbb{T}^n)\big).$$

But $\frac{m}{s} - \frac{1}{s} = \frac{m}{r} - \frac{1}{2}$, where, again, $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$, and so we would not arrive at a contradiction.

6.1.2. Bohr radii. Denote by K_n the nth Bohr radius, that is the best $0 < r \leq 1$, such that, for every $f \in H_\infty(\mathbb{T}^\infty)$, we have:

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |\widehat{f}(\alpha)|\, r^{|\alpha|} \leq \|f\|_\infty.$$

It is known that:

$$\lim_{n\to\infty} \frac{K_n}{\sqrt{\frac{\log n}{n}}} = 1; \tag{6.2}$$

this was established in [2], extending an earlier result of [8], which basically proved that the limit is between 1 and $\sqrt{2}$. For a detailed account on all this, see the monograph [9].

The original proof of the lower estimate in (6.2) is based on the Kahane–Salem–Zygmund inequality (see, e.g. [9, Theorem 7.1]). Let us indicate an alternative argument based on Theorem 4.4. We have that for each $m \in \mathbb{N}$:

$$K_n \leq K_n^m,$$

where K_n^m is defined like K_n, only taking into account functions from $H_m(\mathbb{T}^\infty)$ instead of all functions from $H_m(\mathbb{T}^\infty)$. Then a simple reformulation shows that:

$$K_n \leq K_n^m = \frac{1}{\sqrt[m]{\chi(m,n)}}$$

(see [9, (9.15)]). Using the lower estimate from Theorem 6.1 (which was proved with Kislyakov's ideas), and following the proof given in [9, Theorem 8.22], we obtain an

https://doi.org/10.1017/S1474748022000391 Published online by Cambridge University Press

alternative approach to the upper bound in (6.2) without using the Kahane–Salem–Zygmund inequality.

6.1.3. Monomial convergence I. Let V be a subset of $H_\infty(\mathbb{T}^\infty)$, then:

$$\operatorname{mon} V := \left\{ z \in \mathbb{C}^{\mathbb{N}} \colon \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |\widehat{f}(\alpha) z^\alpha| < \infty \text{ for all } f \in V \right\}$$

is called the set of monomial convergence of V. If V is a closed subspace of $H_\infty(\mathbb{T}^n)$, then a simple closed graph argument shows that $z \in \operatorname{mon} V$ if and only if there is $C = C(z) > 0$, such that, for every $f \in V$,

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |\widehat{f}(\alpha) z^\alpha| \leq C \|f\|_\infty .$$

A sequence $\xi = (\xi_\alpha)_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}}$ is said to be multiplicative whenever for all $\alpha, \beta \in \mathbb{N}_0^{(\mathbb{N})}$, we have $\xi_{\alpha+\beta} = \xi_\alpha \xi_\beta$. Equivalently, ξ is multiplicative if and only if there is $z \in \mathbb{C}^{\mathbb{N}}$, such that, for all $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$, we have $\xi_\alpha = z^\alpha$, for all $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ (if ξ is multiplicative, then define $z_k = \xi_{e_k}$ for each $k \in \mathbb{N}$).

Remark 6.3. Let $V \subset H_\infty(\mathbb{T}^\infty)$ be a closed subspace. Then $\operatorname{mon} V$ equals the set of all multiplicative $\ell_1(\Gamma)$-multipliers $\xi = (\xi_\alpha)_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}}$ for V, where:

$$\Gamma := \bigcup_{f \in V} \operatorname{supp} \widehat{f} \subset \mathbb{Z}^{(\mathbb{N})}.$$

We refer to the monograph [9] for a detailed exposition on the sets of monomial convergence of $H_\infty(\mathbb{T}^\infty)$ and its closed subspace $H_\infty^m(\mathbb{T}^\infty)$ (together with all its consequences for spaces of holomorphic functions in infinitely many variables and ordinary Dirichlet series). In particular, the following results from [1] (see also [9, Theorems 10.1 and -10.15]) give two (almost) complete description of both sets.

Theorem 6.4.

(i) $\operatorname{mon} H_\infty^m(\mathbb{T}^\infty) = \ell_{\frac{2m}{m-1},\infty}$ *for each* $m \in \mathbb{N}$;

(ii) *For every* $z \in \mathbb{C}^{\mathbb{N}}$, *the following two statements hold*:

(a) *If* $\limsup_{n\to\infty} \dfrac{1}{\log n} \sum_{j=1}^n z_j^{*2} < 1$, *then* $z \in \operatorname{mon} H_\infty(\mathbb{T}^\infty)$.

(b) *If* $z \in \operatorname{mon} H_\infty(\mathbb{T}^\infty)$, *then* $\limsup_{n\to\infty} \dfrac{1}{\log n} \sum_{j=1}^n z_j^{*2} \leq 1$. *Moreover, here, the converse implication is false.*

In fact, the 'lower inclusions' for $\operatorname{mon} H_\infty(\mathbb{T}^\infty)$ and $\operatorname{mon} H_\infty^m(\mathbb{T}^\infty)$ follow from the Kahane–Salem–Zygmund theorem. Alternatively, the following two proofs show that these lower inclusions also may be deduced from Theorem 4.4.

Proof. (alternative proof of $\ell_{\frac{2m}{m-1}} \subset \operatorname{mon} H_\infty^m(\mathbb{T}^\infty)$ in Theorem 6.4). Fix $z \in \operatorname{mon} H_\infty(\mathbb{T}^\infty)$, and recall that then the decreasing rearrangement $z^* \in \operatorname{mon} H_\infty(\mathbb{T}^\infty)$ (see [9, Remark 10.4]). Then:

$$M_{z^*} : H_\infty(\mathbb{T}^\infty) \to \ell_1(\mathbb{N}_0^{(\mathbb{N})}),\ f \to \big(\widehat{f}(\alpha)(z^*)^\alpha\big)$$

is bounded. Consequently, by Lemma 6.2, there is some universal constant $\gamma > 0$, such that, for each $m \in \mathbb{N}$,

$$(z_n^*)^m \leq \gamma\sqrt{m!}\log m \,\frac{1}{n^{\frac{m-1}{2}}}\,\|M_{z^*}\|.$$

Taking the mth-rot, we conclude that:

$$z_n^* \leq \gamma^{\frac{1}{m}}\sqrt{m!}^{\frac{1}{m}}\,(\log m)^{\frac{1}{m}}\,\|M_{z^*}\|^{\frac{1}{m}}\,\frac{1}{n^{\frac{m-1}{2m}}} \ll_m \frac{1}{n^{\frac{m-1}{2m}}},$$

and this completes the proof. □

Proof. (alternative proof of (2b) **in Theorem 6.4).** With the closed graph argument from the preceding proof and Theorem 3.2, we see that for some constant $\gamma > 0$ and each $m,n \in \mathbb{N}$, we have:

$$\Bigg(\sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| = m}} |(z^*)^\alpha|^2\Bigg)^{\frac{1}{2}} \leq \gamma\sqrt{n\log m}.$$

Then the proof finishes exactly as in [9, Section 10.5.1]. □

6.1.4. Bohr–Bohnenblust–Hille theorem. Referring to a couple of results, which were worked out in the recent monograph [9], we intend to show that our alternative approach to Theorem 6.4.(ii.b) leads to an alternative solution of Bohr's famous absolute convergence problem on ordinary Dirichlet series.

This problem asked for the largest possible width S of the strip in the complex plane on which an ordinary Dirichlet series $D = \sum a_n n^{-s}$ converges uniformly but not absolutely (see [9, Section 1]).

Bohr, himself, established the upper estimate $S \leq\ 1/2$ (see [9, Proposition 1.10]), but he was not able to decide whether this upper bound is optimal. A nontrivial reformulation (still due to Bohr, see [9, Proposition 1.24]) shows that:

$$S = \sup_{D \in \mathcal{H}_\infty} \sigma_a(D), \tag{6.3}$$

where $\mathcal{H}_\infty$ denotes the Banach space of all Dirichlet series D, which on the positive half plane converge pointwise to a bounded (and then necessarily holomorphic) function and $\sigma_a(D) \in \mathbb{R}$ defines the abscissa of absolute convergence of D.

On the other hand,

$$\mathcal{H}_\infty = H_\infty(\mathbb{T}^\infty),$$

https://doi.org/10.1017/S1474748022000391 Published online by Cambridge University Press

that is, there is an isometric linear bijection between both Banach spaces preserving Dirichlet and Fourier coefficients. More precisely, if this bijection identifies $f \in H_\infty(\mathbb{T}^\infty)$ and $D = \sum a_n n^{-s} \in \mathcal{H}_\infty$, then $a_n = \widehat{f}(\alpha)$, whenever $n = \mathfrak{p}^\alpha$ (here, $\mathfrak{p} = (p_n)$ stands for the sequence of primes (see [9, Corollary 5.3]). Then this fact combined with (6.3) show that:

$$S = \inf\left\{\sigma > 0\colon\ \frac{1}{\mathfrak{p}^\sigma} \in \operatorname{mon} H_\infty(\mathbb{T}^\infty)\right\}$$

(see [9, (10.5)]). As a consequence, we see that Theorem 6.4.(iib), for which we gave an alternative proof using Kislyakov's ideas condensed in Theorem 4.4, immediately proves that $S \geq 1/2$.

So, all in all, we arrive at a new proof of the so-called Bohr–Bohnenblust–Hille theorem: $S = \frac{1}{2}$, which, in fact, is in the very centre of the discussion in the monograph [9] (for the highly nontrivial original proof due to Bohnenblust and Hille from 1931, see [9, Section 2]). We finally remark that this monograph also contains a couple of other proofs — none of them being trivial.

6.2. Multipliers of functions on Boolean cubes

For $N \in \mathbb{N}$, let $\mathcal{B}_N$ be the set of all functions $f : \{-1,1\}^N \to \mathbb{R}$. Recall for $f \in \mathcal{B}_N$, the expectation is given by:

$$\mathbb{E}[f] := \frac{1}{2^N} \sum_{x \in \{-1,1\}^N} f(x).$$

The dual group of $\{-1,1\}^N$ actually consists of the set of all Walsh functions χ_S for $S \subset [N]$, which allows to associate to each such $f \in \mathcal{B}_N$ its Fourier-Walsh expansion:

$$f(x) = \sum_{S \subset [N]} \widehat{f}(S)\, x^S, \quad x \in \{-1,1\}^N, \tag{6.4}$$

where $x^S := \chi_S(x) := \prod_{n \in S} x_n$ are the Walsh functions and the coefficients are given by $\widehat{f}(S) = \mathbb{E}[f\chi_S]$. Thereby, a nonzero function f of degree d satisfies that $\widehat{f}(S) = 0$ provided $|S| > d$. We say that f is m-homogeneous whenever $\widehat{f}(S) = 0$ provided $|S| \neq m$.

Given $m,d \in \mathbb{N}$ with $m,d \leq N$, we write $\mathcal{B}_N^{=m}$ for all m-homogeneous functions in $\mathcal{B}_N$ and $\mathcal{B}_N^{\leq d}$ for all functions of degree $\leq d$. Moreover, we define:

$$\mathcal{B} := \bigcup_N \mathcal{B}_N,\ \ \mathcal{B}^{=m} := \bigcup_N \mathcal{B}_N^{=m} \text{ and } \mathcal{B}^{\leq d} := \bigcup_N \mathcal{B}_N^{\leq d}.$$

Similar to (2.1), we have the following 'hypercontractive' Bohnenblust–Hille inequality for functions on the Boolean cube from [12]: There is a universal constant $C > 1$, such that, for each $d,N \in \mathbb{N}$ and for every $f \in \mathcal{B}_N^{\leq d}$, we have:

$$\Big(\sum_{S \subset [N]\colon |S| \leq d} |\widehat{f}(S)|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \leq C^{\sqrt{m \log m}} \|f\|_\infty. \tag{6.5}$$

https://doi.org/10.1017/S1474748022000391 Published online by Cambridge University Press

6.2.1. Multipliers. We start with variants of Theorem 4.4 and Corollary 3.2 for functions on the Boolean cubes.

Theorem 6.5. *Assume that $1 \le p \le 2$ and $d, N \in \mathbb{N}$ with $d \le N$. Define $1 \le r < \infty$ by $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$. Then,*

(i) *for every $\xi = (\xi_S)_{S \subset [N]}$,*

$$\frac{1}{\sqrt{1 + N\log 2}} \Big(\sum_{S \subset [N]} |\xi_S|^r \Big)^{\frac{1}{r}} \le 2\sqrt{2}e^2 \big\| M_\xi \colon \mathcal{B}_N \to \ell_p(\{S \colon S \subset [N]\}) \big\|.$$

(ii) *for every $\xi = (\xi_S)_{S \subset [N], |S| \le d}$,*

$$\frac{1}{\sqrt{1 + N\log(1+20d)}} \Big(\sum_{S \subset [N],\, |S| \le d} |\xi_S|^r \Big)^{\frac{1}{r}} \le 2\sqrt{2}e^2(1+\sqrt{2})^d \big\| M_\xi \colon \mathcal{B}_N^{\le d} \to \ell_p(\{S \colon |S| \le d\}) \big\|.$$

Moreover, in the homogeneous case $\xi = (\xi_S)_{S \subset [N], |S| = d}$, we may replace the constant on the right side by $2\sqrt{2}e^2 2^{d-1}$.

Proof. For the proof of both statements, we use the fact that $C_2(\ell_p(\Gamma)) \le \sqrt{2}$ for $1 \le p \le 2$. Consider then for the proof of (i) the canonical isometric embedding:

$$\mathcal{B}_N \ni f \mapsto (f(x))_{x \in \{-1,1\}^N} \in \ell_\infty^{2^N}.$$

Then the conclusion is immediate from Theorem 4.4. The proof of (ii) is slightly more involved: Denote by $\mathcal{P}_{\le d}([-1,1]^N)$ the space of all real polynomials $f(x) = \sum_{\alpha \in \mathbb{N}_0^n : |\alpha| \le d} c_\alpha x^\alpha, x \in \mathbb{R}^N$, and endow it with the supremum norm on the N-dimensional cube $[0,1]^N$. Since every $f \in \mathcal{B}_N$ can be viewed as a tetrahedral polynomial in $\mathcal{P}_{\le d}([-1,1]^N)$ with equal norm, the canonical embedding:

$$\mathcal{B}^{\le d} \longrightarrow \mathcal{P}_{\le d}([-1,1]^N)$$

is isometric. Next we look at the canonical embedding:

$$\mathcal{P}_{\le d}([-1,1]^N) \longrightarrow \mathcal{P}_{\le d}(\mathbb{T}^N),$$

which by a result of Klimek [16] is an $(1+\sqrt{2})^d$-embedding, and, finally, we recall that by (2.2), there is a 2-embedding:

$$I \colon \mathcal{P}_{\le d}(\mathbb{T}^N) \to \ell_\infty^M,$$

where $M = (1+20m)^N$. Then the conclusion again follows from Theorem 4.4. In the d-homogeneous case, we replace Klimek's with a result of Visser [21], which states that the canonical map from $\mathcal{P}_{=d}([-1,1]^N)$ into $\mathcal{P}_{=d}(\mathbb{T}^N)$ is a 2^{d-1}- embedding. This completes the proof. □

6.2.2. Sidon constants II. The preceding theorem is used to obtain the following analog of Theorem 6.1.

https://doi.org/10.1017/S1474748022000391 Published online by Cambridge University Press

Theorem 6.6. *Let $1 \le p \le \frac{2m}{m+1}$ and $r > 0$ with $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$. Then, there is a universal constant $\gamma \ge 1$, such that, for each positive integer $m \le N$:*

$$\frac{1}{\gamma^m}\left(\frac{N}{m}\right)^{\frac{m}{r}-\frac{1}{2}} \le \chi_p\big(\mathcal{B}_N^{=m}\big) \le \chi_p\big(\mathcal{B}_N^{\le m}\big) \le \gamma^m\left(\frac{N}{m}\right)^{\frac{m}{r}-\frac{1}{2}}.$$

Note that in a similar fashion as in Section 6.1.1, for each $m \le N$, one has:

$$\chi_p\big(\mathcal{B}_N^{=m}\big) = \chi_p\big(\mathcal{B}_N^{\le m}\big) = 1, \quad p \in [2,\infty],$$

and, combining the Bohnenblust–Hille inequality for functions on the Boolen cube from (6.5) with Hölder's inequality, there exists a constant $\gamma \ge 1$, such that:

$$\chi_p\big(\mathcal{B}_N^{=m}\big) \le \chi_p\big(\mathcal{B}_N^{\le m}\big) \le \gamma^m, \quad m \le N,\ \frac{2m}{m+1} \le p < 2.$$

We prepare the proof of the preceding theorem with a lemma similar to Lemma 6.2.

Lemma 6.7. *Let $1 \le p \le \frac{2m}{m+1}$ and $z \in \mathbb{C}^N$, and denote by z^* the decreasing rearrangement of z. Then, for each $m \le N$, the following estimate holds:*

$$(z_N^*)^m \le 4\sqrt{2}e^2 2^{m-1}\sqrt[r]{m!}\sqrt{\log(1+20m)}\,\frac{1}{N^{\frac{m}{r}-\frac{1}{2}}}\,\|M_z\|,$$

where $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$ and $M_z\colon \mathcal{B}_N^{=m} \to \ell_p(\{S\colon |S| = m\})$ is given by:

$$M_z f = \big(\widehat{f}(S)(z^*)^S\big)_{|S|=m}, \quad f \in \mathcal{B}_N^{=m}.$$

Proof. From Theorem 6.5, we get:

$$\begin{aligned}(z_N^*)^{rm}\sum_{S\subset[N],|S|=m} 1 &= \sum_{S\subset[N],|S|=m}\big((z_N^*)\ldots(z_N^*)\big)^r\\ &\le \sum_{S\subset[N],|S|=m}\big((z^*)^S\big)^r \le (4e^2 2^{m-1})^r\big(1+N\log(1+20m)\big)^{\frac{r}{2}}\|M_z\|^r.\end{aligned}$$

This yields:

$$(z_N^*)^{rm}\frac{N^m}{m!} \le (z_N^*)^{rm}\binom{N}{m} \le (4\sqrt{2}e^2 2^{m-1})^r\big(N\log(1+20m)\big)^{\frac{r}{2}}\|M_z\|^r,$$

as required. □

Proof. (of Theorem 6.6). Lower bound: For $\mathbf{1} = (1,\ldots,1) \in \mathbb{C}^{|\{S\colon |S|=m\}|}$, we by definition have:

$$\|M_{\mathbf{1}}\colon \mathcal{B}_N^{=m} \to \ell_p(\{S;|S| = m\})\| = \chi_p\big(\mathcal{B}_N^{=m}\big).$$

Then the conclusion follows if we, exactly as in Lemma 6.2, apply Lemma 6.7 to $z = \mathbf{1}$. Upper bound: Take $f \in \mathcal{B}_N^{\le m}$, and interpret it as a polynomial $F \in \mathcal{P}_{\le m}(\mathbb{T}^N)$. Then we

https://doi.org/10.1017/S1474748022000391 Published online by Cambridge University Press

know from Theorem 6.1 that:

$$\sum_{|S|\leq m}|\widehat{f}(S)|\leq \gamma^m\left(\frac{N}{m}\right)^{\frac{m}{r}-\frac{1}{2}}\|F\|_{\mathcal{P}_{\leq m}(\mathbb{T}^N)}\leq \gamma^m(1+\sqrt{2})^m\left(\frac{N}{m}\right)^{\frac{m}{r}-\frac{1}{2}}\|f\|_{\mathcal{B}_N^{\leq m}},$$

where the very last estimate, again, follows from a result of Klimek in [16]. □

6.2.3. Monomial convergence II. Given $V\subset\mathcal{B}$, we define the set of monomial convergence of V by:

$$\operatorname{mon}(V):=\Big\{x\in\mathbb{R}^{\mathbb{N}}:\ \ \exists C>0\,\forall f\in V:\quad \sum_{S\subset[N]}|\widehat{f}(S)x^S|\leq C\|f\|_\infty\Big\},$$

where recall that $x^S:=\prod_{n\in S}x_n$ for each $S\subset[N]$ and for all $x\in\mathbb{R}^{\mathbb{N}}$. Let us denote by $S\subset_{fin}\mathbb{N}$ the fact that S is a finite subset of $\mathbb{N}$. We say that a real sequence $(\xi_S)_{S\subset_{fin}\mathbb{N}}$ is multiplicative if:

$$\xi_R\,\xi_S=\xi_{R\cup S}\ \text{for all pairwise disjoint subsets}\ R,S\subset_{fin}\mathbb{N}.$$

Note that $(\xi_S)_{S\subset_{fin}\mathbb{N}}$ is multiplicative if and only if there exists $x\in\mathbb{R}^{\mathbb{N}}$, such that for all $S\subset_{fin}\mathbb{N}$, we have $x^S=\xi_S$. Hence, $\operatorname{mon}(V)$ consists exactly of all multiplicative $\ell_1(\{S\colon S\subset_{fin}\mathbb{N}\})$-multipliers of V.

Here are some basic properties of $\operatorname{mon}(\mathcal{B})$.

Proposition 6.8. *Let $x\in\operatorname{mon}(\mathcal{B})$ and $y\in\mathbb{R}^{\mathbb{N}}$. Then, each of the following conditions yields that $y\in\operatorname{mon}(\mathcal{B})$:*

(i) *y differs from x in a finite number of entries*;

(ii) *y is a permutation of x*;

(iii) *$|y_n|\leq|x_n|$ for each $n\in\mathbb{N}$.*

Proof. To prove the sufficiency of (i), it is enough to assume that y differs from x in one entry, $x_n=y_n$ for each $n\neq n_0\in\mathbb{N}$. Using that $g(x)=x_{n_0}f(x)$ also belongs to $\mathcal{B}$ with $\widehat{g}(S\setminus\{n_0\})=\widehat{f}(S)$ if $n_0\in S\subset_{fin}\mathbb{N}$, we get:

$$\sum_S|\widehat{f}(S)y_S|=\sum_{n_0\notin S}|\widehat{f}(S)x^S|+|y_{n_0}|\sum_{n_0\in S}|\widehat{f}(S)x_{S\setminus\{n_0\}}|<\infty.$$

It is a simple observation that for every $f\in\mathcal{B}$ and for each permutation σ of the natural numbers, the function f_σ defined by:

$$f_\sigma((x_n)_{n\in\mathbb{N}})=f((x_{\sigma(n)})_{n\in\mathbb{N}})$$

also belongs to $\mathcal{B}$, which proves that condition (ii) is sufficient. Finally, if (iii) is satisfied, then:

$$\sum_S|\widehat{f}(S)y_S|\leq\sum_S|\widehat{f}(S)x^S|<\infty.$$

□

https://doi.org/10.1017/S1474748022000391 Published online by Cambridge University Press

Our aim is to find nice descriptions of $\mathrm{mon}(V)$ for $V = \mathcal{B}$, $V = \mathcal{B}^{=m}$ and $V = \mathcal{B}^{\leq d}$. It is clear that:

$$\mathrm{mon}(\mathcal{B}) \subset \mathrm{mon}(\mathcal{B}^{\leq d}) \subset \mathrm{mon}(\mathcal{B}^{=m}).$$

Using the Bohnenblust–Hille inequality from (6.5) (together with Hölder's inequality and the multimonomial theorem) give that, for each $m \in \mathbb{N}$, we have:

$$\ell_{\frac{2m}{m-1}} \subset \mathrm{mon}(\mathcal{B}^{\leq m}) \subset \mathrm{mon}(\mathcal{B}^{=m}).$$

Similar to Theorem 6.4.(i), we even have the following full description.

Theorem 6.9. *For each positive integer m, one has*:

$$\mathrm{mon}(\mathcal{B}^{=m}) = \mathrm{mon}(\mathcal{B}^{\leq m}) = \ell_{\frac{2m}{m-1},\infty}.$$

Proof. We first prove that $\mathrm{mon}(\mathcal{B}^{=m}) = \ell_{\frac{2m}{m-1},\infty}$. Given an m-homogeneous function $f\colon \{-1,1\}^N \to \mathbb{R}$, we can find $F \in H_\infty^m(\mathbb{T}^\infty)$ with $\widehat{F}(\alpha) = \widehat{f}(\alpha)$ for all $\alpha \in \{0,1\}^{(\mathbb{N})}$ and $\widehat{F}(\alpha) = 0$, otherwise, and, such that:

$$\|F\|_{H_\infty(\mathbb{T}^\infty)} \leq 2^{m-1}\,\|f\|_{\mathcal{B}^{=m}},$$

where for this estimate, we use a result from [21] (see also the end of the proof of Theorem 6.5). This implies that $\mathrm{mon}\, H_\infty^m(\mathbb{T}^\infty) \subset \mathrm{mon}(\mathcal{B}^{=m})$, which by Theorem 6.4.(i) gives the lower inclusion:

$$\ell_{\frac{2m}{m-1},\infty} \subset \mathrm{mon}(\mathcal{B}^{=m}).$$

Conversely, if $x \in \mathrm{mon}(\mathcal{B}^{=m})$, then by Proposition 6.8, also its decreasing rearrangement $r = (x_n^*) \in \mathrm{mon}(\mathcal{B}^{=m})$. Thus, there is a constant $C > 0$, such that, for each $f \in \mathcal{B}^{=m}$, we get:

$$\sum_{|S|=m} |\widehat{f}(S)|\,|r_S| \leq C\,\|f\|_\infty.$$

We give a probabilistic argument and a Kislyakov type argument. The probabilistic argument: Let $A = \{S \subset [N], |S| = m\}$. By the Kahane–Salem–Zygmund theorem (see (6.8) below), there is a choice of signs $(\xi_S)_{S\in A}$, such that:

$$\sum_{S\in A} |\xi_S| r^S \leq C \Big\| \sum_{S\in A} \xi_S x^S \Big\|_\infty \ll \sqrt{N}\sqrt{\binom{N}{m}}.$$

Since $(r_n)_{n\in\mathbb{N}}$ is decreasing, we get:

$$\binom{N}{m} r_N^m \ll \sqrt{N}\sqrt{\binom{N}{m}},$$

and so for some constant K_m independent of N, we have:

$$r_N^m \leq C\sqrt{N}\binom{N}{m}^{-\frac{1}{2}} \ll_m \frac{\sqrt{N}}{N^{m/2}}.$$

The Kislyakov argument: Consider the canonical embeddings:

$$\mathcal{B}^{=m} \longrightarrow \mathcal{P}_m([-1,1])^N) \longrightarrow \mathcal{P}_m(\mathbb{T}^N) \longrightarrow \ell_\infty^{(1+20m)^N},$$

where the first two embeddings are the canonical ones and the last comes from (2.2). The first one is isometric, the second one 2^{m-1}-isomorphic and the third 2-isomorphic. Then, we deduce from Theorem 3.2 that:

$$r_N^{2m}\binom{N}{m} \le \sum_{S\in A} |r^S|^2 \ll_m N.$$

Since $\left(\frac{N}{m}\right)^m \le \binom{N}{m}$, the argument completes. Finally, we prove that for each $d\in\mathbb{N}$, we have $\mathrm{mon}(\mathcal{B}^{\le d}) = \ell_{\frac{2d}{d-1}}$. Using the fact that, for every $f\in\mathcal{B}_N^{\le d}$, we get by [16],

$$\|f\|_\infty \le (1+\sqrt{2})^m \|f_m\|_\infty,$$

where $f_m = \sum_{|S|=m}\widehat{f}(S)\chi_S$ denotes the m-homogeneous part of f, it easily follows that:

$$\ell_{\frac{2d}{d-1},\infty} \subset \bigcap_{m=1}^{d} \mathrm{mon}(\mathcal{B}^{=m}) \subset \mathrm{mon}(\mathcal{B}^{\le d}) \subset \mathrm{mon}(\mathcal{B}^{=d}) \subset \ell_{\frac{2d}{d-1},\infty}. \qquad \square$$

Let us turn to the description of $\mathrm{mon}(\mathcal{B})$. We easily obtain:

$$\ell_2 \subset \mathrm{mon}(\mathcal{B}), \tag{6.6}$$

using the Cauchy–Schwarz inequality:

$$\sum_S |\widehat{f}(S)x^S| \le \Big(\sum_S |\widehat{f}(S)|^2\Big)^{\frac{1}{2}} \Big(\sum_S |x^S|^2\Big)^{\frac{1}{2}} \le \|f\|_\infty \Big(\prod_{n=1}^{\infty}(1+|x_n|^2)\Big)^{\frac{1}{2}}.$$

Regarding the estimations from above, using the results for the m-homogeneous functions, we have:

$$\mathrm{mon}(\mathcal{B}) \subset \bigcap_{m\in\mathbb{N}} \ell_{\frac{2m}{m-1},\infty}. \tag{6.7}$$

This result can be improved.

Proposition 6.10. *For every $x\in\mathrm{mon}(\mathcal{B})$, one has*:

$$\sup_{N\in\mathbb{N}} \frac{1}{\sqrt{N}} \sum_{n=1}^{N} |x_n| < +\infty$$

and, in particular:

$$\mathrm{mon}(\mathcal{B}) \subset \ell_{2,\infty}.$$

Proof. Using the majority function $\mathrm{Maj}_N(x)$ (in fact, only its 1-homogeneous part), we know from [17] that for all S with $|S|=1$:

$$\widehat{\mathrm{Maj}}_N(S) = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{N}}.$$

https://doi.org/10.1017/S1474748022000391 Published online by Cambridge University Press

Hence, for all $x \in \mathrm{mon}(\mathcal{B})$ and every $N \in \mathbb{N}$, we have:

$$\sum_{n=1}^{N} \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{N}} |x_n| = \sum_{|S|=1} |\widehat{\mathrm{Maj}}_N(S)| \, |x^S| \le \sum_{S \subset [N]} |\widehat{\mathrm{Maj}}_N(S)| \, |x^S| \le C_x,$$

the first estimate. Since the decreasing rearrangement x^* by Proposition 6.8 also belongs to $\mathrm{mon}(\mathcal{B})$, the 'in particular' follows. □

On the other hand, the use of the Kahane–Salem–Zygmund inequality (for Boolean functions, see, e.g. [11, Lemma 3.1]) does not improve the condition from Proposition 6.10. Indeed, this inequality yields that for every $N \in \mathbb{N}$ and every family $(c_S)_{S \subset [N]}$ in $\mathbb{R}$, there is a choice of signs $(\xi_S)_{S \subset [N]}$, such that:

$$\sum_{S \subset [N]} |c_S| |x^S| \le C_x \Big\| \sum_{S \subset [N]} \xi_S c_S x^S \Big\|_\infty \le C_x \, 6 \sqrt{\log 2} \sqrt{N} \Big(\sum_{S \subset [N]} |c_S|^2 \Big)^{1/2}. \tag{6.8}$$

Taking the supremum over all $(c_S)_{S \subset [N]}$ with ℓ_2-norm equal to one, we deduce that:

$$\prod_{n=1}^{N} (1 + x_n^2) = \Big(\sum_{S \subset [N]} |x^S|^2 \Big)^{1/2} \le C_x \, 6 \sqrt{\log 2} \sqrt{N}. \tag{6.9}$$

Since $(x_n)_{n \in \mathbb{N}}$ converges to zero, we can find a positive constant $\alpha > 0$, such that $\exp(\alpha |x_n|^2) \le 1 + |x_n|^2$ for every $n \in \mathbb{N}$, so that:

$$\exp\Big(\alpha \sum_{n=1}^{N} |x_n|^2\Big) \le \prod_{n=1}^{N} (1 + x_n^2) \le C_x \, 6 \sqrt{\log 2} \sqrt{N}.$$

It follows that:

$$\sup_{N \in \mathbb{N}} \frac{1}{\log N} \sum_{n=1}^{N} |x_n|^2 < +\infty. \tag{6.10}$$

But this condition is weaker than what we got in Proposition 6.10, since:

$$\frac{1}{\log N} \sum_{n=1}^{N} |x_n|^2 \le \frac{1}{\log N} \sum_{n=1}^{N} |x_n^*|^2 \le \frac{\|x\|_{\ell_{2,\infty}}^2}{\log N} \sum_{n=1}^{N} \frac{1}{n} \le \|x\|_{\ell_{2,\infty}}^2.$$

Finally, we establish the following analog of statement (2) from Theorem 6.4.

Proposition 6.11. *For each $x \in \mathrm{mon}(\mathcal{B})$, one has:*

$$\limsup_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} (x_n^*)^2 \le 1.$$

Proof. We write $r = (r_n)_{n \in \mathbb{N}}$ for the decreasing rearrangement of $(|x_n|)_{n \in \mathbb{N}}$. Then:

$$\sup_N \big\| M_r \colon \mathcal{B}_N \to \ell_1(\{S \colon S \subset [N]\}) \big\| < \infty,$$

https://doi.org/10.1017/S1474748022000391 Published online by Cambridge University Press

and, hence, by Theorem 6.5.(i), we have that there is a constant $C = C(r)$, such that for all N:

$$\sum_{S \subset [N]} r_S^2 \le C\, N;$$

again, there is an alternative proof using the Kahane–Salem–Zygmund inequality (in the form of (6.8) and (6.9)). We claim that:

$$\frac{(r_m^2 + \ldots + r_N^2)^m}{m!} \le \sum_{|S|=m} r_S^2.$$

We postpone the proof of the claim to end. The claim yields by Stirling's inequality that there is a positive constant C' depending just on r, such that:

$$\frac{r_m^2 + \ldots + r_N^2}{m} \le (C')^{\frac{1}{m}} m^{\frac{1}{2m}} \frac{N^{\frac{1}{m}}}{e}.$$

Putting $m = \log N$, we arrive to the inequality:

$$\limsup_N \frac{r_{\log N}^2 + \ldots + r_N^2}{\log N} \le 1.$$

Since r_n^2 converges to zero, we immediately conclude that:

$$\limsup_N \frac{r_1^2 + \ldots + r_N^2}{\log N} \le 1.$$

To prove the claim, note that every $S \subset \mathbb{N}$ with $|S| = m$ is determined by the subset S_1 of elements n with $n < m$ and the subset S_2 of those with $n \ge m$. Then, we can find unique finite sequences $m \le i_1, i_2, \ldots, i_k \le N$ and $m_1, m_2, \ldots, m_k$ in $\mathbb{N}$ with $m_1 + \ldots + m_k = m$ and, such that:

$$\begin{aligned} S^c \cap \{n\colon\ n < m\} &= \{m_1, m_1 + m_2, \ldots, m_1 + \ldots + m_{k-1}\} \\ S \cap \{n\colon\ n \ge m\} &= \{i_1 < i_2 < \ldots < i_k\}. \end{aligned}$$

We can then rewrite r_S as:

$$r_S = r_1 \cdot \ldots \cdot r_{m_1 - 1} \cdot r_{i_1} \cdot r_{m_1+1} \cdot \ldots \cdot r_{m_1+m_2-1} \cdot r_{i_2} \cdot r_{m_1+m_2+1} \cdot \ldots,$$

and using that $(r_n)_{n \in \mathbb{N}}$ is nonincreasing, we deduce that:

$$r_S \ge r_{i_1}^{m_1} \cdot r_{i_2}^{m_2} \cdot \ldots \cdot r_{i_k}^{m_k}.$$

Therefore:

$$\begin{aligned} m! \sum_{|S|=m} r_S^2 &\ge \sum_{(\alpha_m, \ldots, \alpha_N) \in \mathbb{N}_0^{N-m+1} |\alpha| = m} \binom{m}{\alpha} r_m^{\alpha_m} \cdot r_{m+1}^{\alpha_{m+1}} \cdot \ldots \cdot r_N^{\alpha_N} \\ &= (r_m + \ldots + r_N)^m, \end{aligned}$$

and the proof completes. □

https://doi.org/10.1017/S1474748022000391 Published online by Cambridge University Press

Acknowledgments. The second named author was supported by the National Science Centre, Poland, Grant no. 2019/33/B/ST1/00165. The third named author acknowledges support from the grants 'Juan de la Cierva Formación' (FJC2018-036519-I) and 2021-MAT11 (ETSI Industriales, UNED).

Competing Interests. The author(s) declare none.

References

[1] F. Bayart, A. Defant, L. Frerick, M. Maestre and P. Sevilla, Multipliers of Dirichlet series and monomial series expansion of holomorphic functions in infinitely many variables, *Math. Ann.* **368**(1–2) (2017), 837–876.

[2] F. Bayart, D. Pellegrino and J. Seoane-Sepulveda, The Bohr radius of the n-dimensional polydisk is equivalent to $\sqrt{\log(n)/n}$, *Adv. Math.* **264** (2014), 726–746.

[3] J. Bergh and J. Löfström, *Interpolation spaces. An introduction.* Crundlehren Math. Wiss. 223 (Springer, Berlin, Heidelberg, 1976).

[4] H. Boas and D. Khavinson, Bohr's power series theorem in several variables, *Proc. Amer. Math. Soc.* **125** (1997), 2975–2979.

[5] H. F. Bohnenblust and E. Hille, On the absolute convergence of Dirichlet series, *Ann. of Math.* **32**(3) (1931), 600–622.

[6] H. Bohr, A theorem concerning power series, *Proc. London Math. Soc.* **13**(2) (1914), 1–5.

[7] J. Bourgain, Bounded orthogonal systems and the $\Lambda(p)$-set problem, *Acta Math.* **162(3–4)** (1989), 227–245.

[8] A. Defant, L. Frerick, J. Ortega-Cerdá, M. Ounäies and K. Seip, The Bohnenblust-Hille inequality for homogeneous polynomials is hypercontractive, *Ann. of Math. (2)* **174**(1) (2011), 485–497.

[9] A. Defant, D. García, M. Maestre and P. Sevilla Peris, *Dirichlet Series and Holomorphic Functions in High Dimensions*, New Mathematical Monographs Series. Vol. **37** (Cambridge University Press, Cambridge, 2019).

[10] A. Defant and M. Mastyło, Aspects of the Kahane–Salem–Zygmund inequalities in Banach spaces, Preprint, 2022.

[11] A. Defant, M. Mastyło and A. Pérez, Bohr's phenomenon for functions on the Boolean cube, *J. Funct. Anal.* **275**(11) (2011), 3115–3147.

[12] A. Defant, M. Mastyło and A. Pérez, On the Fourier spectrum of functions on Boolean cubes, *Math. Ann.* **374**(1–2) (2019), 653–680.

[13] J. Diestel, H. Jarchow and A. Tonge, Absolutely summing operators, in *Cambridge Studies in Advanced Mathematics.* Vol **43** (Cambridge University Press, Cambridge, 1995).

[14] J.-P. Kahane, *Some Random Series of Functions*, 2nd ed., Cambridge Studies in Advanced Mathematics. Vol. **5** (Cambridge University Press, Cambridge, New York, 1985).

[15] S. Kislyakov, Fourier coeffcients of boundary values of analytic functions in the disc and bidisc II, *Tr. Mat. Inst. Steklova* **155** (1981), 77–91 (in Russian).

[16] M. Klimek, Metrics associated with extremal plurisubharmonic functions, *Proc. Amer. Math. Soc.* **123**(9) (1995), 2763–2770.

[17] R. O'Donnell, *Analysis of Boolean Functions* (Cambridge University Press, New York, 2014).

[18] A. Pełczyński, p-integral operators commuting with group representation and examples of quasi-p-integral operators which are not p-integral, *Studia Math.* **33**(1) (1969), 63–70.

[19] G. PISIER, *Factorization of Linear Operators and Geometry of Banach Spaces*, **60** (Amer. Math. Soc., CBMS, Providence, R.I., 1985).

[20] H. QUEFFÉLEC AND M. QUEFFÉLEC, *Diophantine Approximation and Dirichlet Series*, HRI Lecture Notes Series. Vol. **2** (Hindustan Book Agency, New Delhi, 2013).

[21] C. VISSER, A generalization of Tchebychef's inequality to polynomials in more than one variable, *Indagationes Math.* **8** (1946), 310–311.