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CENTRE MANIFOLDS OF FORCED DYNAMICAL SYSTEMS

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Abstract

Centre manifolds arise in a rational approach to the problem of forming low-dimensional models of dynamical systems with many degrees of freedom. When a dynamical system with a centre manifold is subject to a small forcing, $\mathbf{F}$, there are two effects: to displace the centre manifold; and to alter the evolution thereon. We propose a formal scheme for calculating the centre manifold of such a forced dynamical system. Our formalism permits the calculation of these effects, with errors of order $|\mathbf{F}|^2$. We find that the displacement of the manifold allows a re-parameterisation of its description, and we describe two "natural" ways in which this can be carried out. We give three examples: an introductory example; a five-mode model of the atmosphere to display the quasi-geostrophic approximation; and the forced Kuramoto-Sivashinsky equation.

1. Introduction

In many physical systems with a large number of degrees of freedom, the long-time behaviour, after the exponential decay of transients, is dominated by the relatively slow evolution of a small number of "modes". These modes may be, for example, the components of a Fourier decomposition of the system. The class of motions described by these modes forms some low-dimensional invariant manifold of the system. It is useful then to reduce the original system to a model system with only a few degrees of freedom, but with the same long-time behaviour as the original, larger system. One example of a rigorous theory which allows such a reduction in the number of degrees of freedom is the theory of centre manifolds.

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Perhaps the simplest approximation for a dissipative nonlinear dynamical system is to ignore, that is set to zero, all the linearly damped modes, and to consider just the *marginal* modes—namely, the modes for which a linear stability analysis predicts a zero temporal growth-rate. This results in the approximation of the full system by its *centre eigenspace* which is the (linear) vector space spanned by the marginal modes. The centre eigenspace is not in general invariant under the evolution of the full nonlinear system and so cannot contain exact solutions of the system. However, its natural modification, called the *centre manifold*, is invariant under the evolution of the full system and becomes tangent to the centre eigenspace as the amplitudes of the marginal modes tend to zero. (If there are any linearly unstable modes in the dynamics then these should also be included to form the *centre-unstable eigenspace* which is tangent to the evolution-invariant *centre-unstable manifold*.) The simple approximation of neglecting the linearly damped modes then may be viewed as the centre manifold (or the *centre-unstable manifold*) being approximated by its tangent space at the origin. Once it is appreciated that these manifolds lie at the heart of many accepted numerical and analytical approximations then it is clear that to approximate them better is a worthwhile aim. The purpose of a model system which is derived through the theory of centre manifolds is to describe accurately the long-time behaviour of the original dynamical system, and in practice even the crudest improvement to the tangent-space approximation for the centre manifold, M, typically improves this description significantly. The value of centre manifold theory is well illustrated by its ability to provide suitable initial conditions for the model from the initial conditions of the full system [13].

Centre-manifold theory is usually applied to autonomous dynamical systems in the so-called "standard form", although Coullet & Spiegel [3] and Roberts [12] have described how to avoid the tiresome changes of variables which are necessary in general to put a system into standard form, and instead perform algebraic manipulations directly on the system *as given*, to find the centre manifold, M_0. If we choose physically-meaningful parameters to describe M_0, then the model system resulting from the reduction to the centre manifold captures all the interesting dynamics of the full physical system. Many traditional approximations of physics and engineering involve the construction of such a "coarse" approximation from a "fine" exact theory [10]. Such approximations may be derived from a centre-manifold analysis: examples are shear dispersion [8] and beam theory [9, 15]. In each case, the traditional approximation is found as a crude low-order description of the centre manifold, and the approximation is improved by corrections which take into account more details of the shape of the centre manifold. Whereas the traditional approximation can be, and often is, derived from physical

considerations (heuristics), the (small) correction terms which are necessary for a fuller description of the centre manifold are much harder to derive heuristically.

In this paper we describe a procedure for the systematic calculation of centre manifolds for forced dynamical systems. There are two effects of forcing: firstly, to shift the centre manifold $\mathcal{M}$ to which trajectories are attracted as $t \to \infty$; and secondly, to alter the slow evolution on $\mathcal{M}$. The second effect has been quantified by Roberts [13, 15], who has shown how to derive the appropriate forcing for the marginal modes on the centre manifold from the full forcing of the original system. In this paper we show how his methods may be extended to quantify the first effect (giving the position of the forced centre manifold), and to better understand the second effect.

We begin in Section 2 by examining the effects of forcing on a set of two ordinary differential equations. With this simple example we introduce the concepts which are used in later sections to characterise the centre manifolds of more general dynamical systems. The example is somewhat artificial: the unforced system can be solved analytically [12, 13], and the reduction in dimension which is achieved by the centre-manifold analysis is rather insignificant. However, many of the ideas which we use in later sections were developed by working with this simple system. Section 3 generalises the results of Section 2 to any dynamical system in standard form. As an example of the analysis of Section 3, we consider in Section 4 a modified version of Lorenz & Krishnamurthy's [7] five-mode model of the general atmospheric circulation and find the first few terms in the power-series expansion of the centre manifold, $\mathcal{M}$. This particular centre manifold corresponds to the quasi-geostrophic, or slow manifold, and meteorologists need to "balance" their raw data by projecting onto it—but they can do this only if they know where the manifold is located.

The calculation of the forced centre manifold for dynamical systems which are not in standard form is described in Section 5. The structure of the equation for $\mathcal{M}$ is rather more complicated than when the original dynamical system is in standard form, but $\mathcal{M}$ can still be found by power-series expansion or by an iteration scheme. We illustrate this in Section 6 by the calculation of $\mathcal{M}$ for the forced Kuramoto-Sivashinsky equation (a model used in, for example, flames and viscous fluid flow), and describe how a pitchfork bifurcation is modified by the forcing.

Once a problem has been suitably posed for centre-manifold analysis, the calculation of $\mathcal{M}$ is entirely systematic, and is usually achieved through power series or by iteration. In practice, it is convenient to relegate the details of solving the equations for the centre manifold to some computer algebra package, as we have done with REDUCE. This is a major benefit, which in some

cases allows sufficiently many terms in a power series to be calculated to determine the radius of convergence, and thereby obtain definite estimates of the usefulness of the approximation [8, 15]. Although only the first correction in the magnitude of the forcing is discussed here, it would be possible to extend the analysis to arbitrary order in the forcing. However, the complications in detail upon doing so are considerable.

A result of the standard-form analysis is that both the unforced and forced centre manifolds, $\mathcal{M}_0$ and $\mathcal{M}$ respectively, are necessarily parameterised by the amplitudes of the marginal modes. For systems in a more general form, as examined in Section 5, the choice of parameters for $\mathcal{M}_0$ and $\mathcal{M}$ is at our disposal. In certain circumstances it is sensible to use the same parameters to describe both $\mathcal{M}_0$ and $\mathcal{M}$: this is particularly advantageous if the parameters have some physical meaning. However, if the forcing is time-dependent then the evolution equation has a memory in that the evolution on $\mathcal{M}$ depends not only on the instantaneous value of the forcing, but also upon previous values. In other cases a rather complicated, and less meaningful, choice of parameters may, in fact, be more appropriate as the awkward "memory" effect disappears from the evolution equation.

2. A system with an exact centre manifold

Before embarking on the general formalism which we propose for forced systems, we consider a simple example [12, 13] where the unforced system can be solved exactly, in particular the centre manifold is known analytically. This example illustrates the principal ideas with little algebraic detail to cloud the picture.

When there is no forcing, that is when $\mathbf{F} = (F_x, F_y)^{\mathsf{T}} = \mathbf{0}$, the system

$$\dot{x} = -xy + F_x, \quad \dot{y} = -y + x^2 - 2y^2 + F_y \tag{1}$$

has a centre manifold, $\mathcal{M}_0$, given by

$$y = x^2. \tag{2}$$

All nearby trajectories approach $\mathcal{M}_0$ exponentially quickly, and once on $\mathcal{M}_0$ their long-term evolution is much slower, given by

$$\dot{x} = -x^3. \tag{3}$$

The centre manifold is unique up to exponentially small terms of order $e^{-1/2x^2}$ [12].

2.1 Constant forcing

When $\mathbf{F}$ is steady and small, but non-zero, we still expect a centre manifold to which neighbouring trajectories are attracted, but two aspects of the asymptotic behaviour of (1) change: the position of this centre manifold

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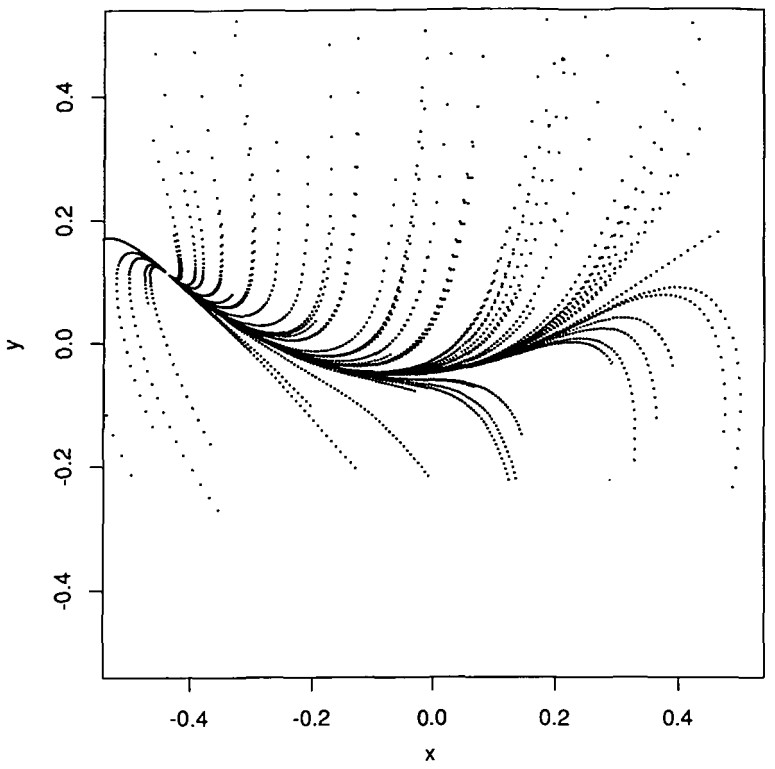

FIGURE 1(a). Evolution of a set of points from random initial conditions under (1) when the forcing is $\mathbf{F} = (-0.05, -0.05)^{\mathsf{T}}$. The positions of the points are plotted every 0.1 time-units, and they clearly converge to the centre manifold, $\mathcal{M}$.

(which for the forced system we denote by $\mathcal{M}$, rather than $\mathcal{M}_0$); and the evolution on the centre manifold $\mathcal{M}$.

Figure 1(a) shows a definite example: trajectories evolving under (1) from a series of random initial conditions are plotted at discrete times for the case $F_x = F_y = -0.05$. It is clear that there is some central curve to which trajectories are quickly attracted. Figure 1(b) on p. 406 shows the positions of the points after a time integration of six time units (marked as crosses). Solutions no longer approach the curve $y = x^2$ (the dotted line in Figure 1(b)) as $t \to \infty$: instead, as shown in the figure, the long-time behaviour of (1) is confined to a curve $\mathcal{M}$ of the form

$$y = x^2 + \eta_x(x)F_x + \eta_y(x)F_y + O(|\mathbf{F}|^2), \tag{4}$$

with the evolution on $\mathcal{M}$ given by the x-evolution equation of (1), with y as in (4). We may now derive two expressions for $\dot{y}$ on $\mathcal{M}$: the first follows from an application of the chain rule ($\dot{y} = \dot{x}\partial y/\partial x$) to (4), and the

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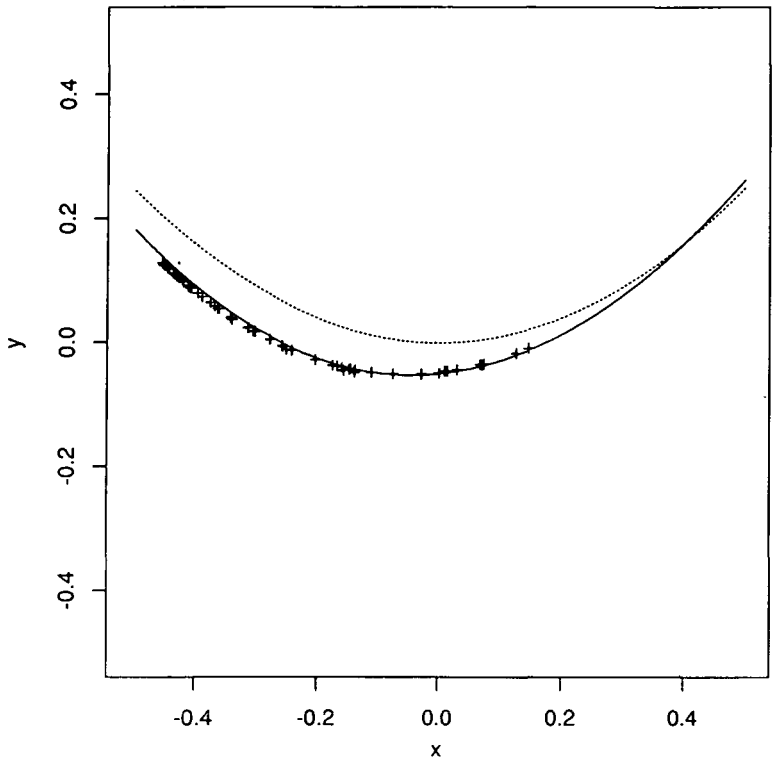

FIGURE 1(b). Crosses mark the position after six time-units of the points whose trajectories are shown in (a). The solid line is the centre manifold, calculated to $O(|\mathbf{F}|)$, from (5). The dotted line is the position of the unforced centre manifold M_0.

x-evolution equation of (1); the second follows from direct substitution of (4) into the y-evolution equation of (1). By equating these two expressions, we find that the centre manifold for (1) is

$$y = x^2 - 2xc(x)F_x + \left(1 - 2x^2\right) F_y + O(|\mathbf{F}|^2), \tag{5}$$

where $c(x) = \int_0^{1/2x^2} \sqrt{1 - 2x^2\tau}\ e^{-\tau} d\tau$. The centre manifold M is parameterised by x, whose evolution is, from (1),

$$\dot{x}(t) = -x^3 + \left(1 + 2x^2 c(x)\right) F_x - x(1 - 2x^2)F_y + O(|\mathbf{F}|^2). \tag{6}$$

The forced centre manifold M is shown by the solid line in Figure 1(b). There is good agreement between our analysis and the numerical results for the full system (1).

The centre manifold M_0 of the unforced system may be described parametrically as

$$x = s, \quad y = s^2. \tag{7}$$

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Roberts [13] gave a simple argument which showed, by considering the projection of the forcing $\mathbf{F}$ on $\mathcal{M}_0$, that for (1) (for small $\mathbf{F}$) the evolution of s should be given by

$$\dot{s} = -s^3 + \left(1 + 2s^2\right) F_x - sF_y + O(|\mathbf{F}|^4). \tag{8}$$

This modified evolution equation gives information abut the dynamics of the forced system, its bifurcations and the long-time behaviour, but does not indicate the position of $\mathcal{M}$. Clearly, knowledge of this position is necessary for quantitative prediction using the model (that is, using the centre manifold $\mathcal{M}$ as an approximation to the full system (1)). Furthermore, the evolution equations (6) and (8) appear different. However, there is no inconsistency since the two equations are equivalent under a near-identity transformation $x = x(s)$, which corresponds to a re-parameterisation of $\mathcal{M}$. Clearly, the choice of parameter s used to describe $\mathcal{M}$ determines the form of the evolution equation for s.

In order to calculate the forced centre manifold, $\mathcal{M}$, of (1), we pose that $\mathcal{M}$ is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s \\ s^2 \end{bmatrix} + \begin{bmatrix} \xi_x(s) & \xi_y(s) \\ \eta_x(s) & \eta_y(s) \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix} + O(|\mathbf{F}|^2), \tag{9}$$

where s evolves according to the equation

$$\dot{s} = -s^3 + [\,\sigma_x(s) \quad \sigma_y(s)\,] \begin{bmatrix} F_x \\ F_y \end{bmatrix} + O(|\mathbf{F}|^2). \tag{10}$$

Then substitution into (1) and consideration of the terms of order $|\mathbf{F}|$ gives the four equations

$$-s^3 d\xi_x/ds = -s^2\xi_x - s\eta_x - \sigma_x + 1, \tag{11}$$

$$-s^3 d\xi_y/ds = -s^2\xi_y - s\eta_y - \sigma_y, \tag{12}$$

$$-s^3 d\eta_x/ds = -\eta_x + 2s\xi_x - 4s^2\eta_x - 2s\sigma_x, \tag{13}$$

$$-s^3 d\eta_y/ds = -\eta_y + 2s\xi_y - 4s^2\eta_y - 2s\sigma_y + 1. \tag{14}$$

These are four equations in six unknown functions, and the remaining two degrees of freedom left to us correspond to the freedom to choose the new parameterisation of $\mathcal{M}$ as we wish. If we choose to identify s to be x, as in the unforced problem, then ξ_x and ξ_y are each identically equal to zero—these would be our two remaining equations. Then by solving (13) and (14) we arrive at (5) and (6).

Before we describe the parameterisation by s of $\mathcal{M}$ for which (8), rather than (6), is the appropriate evolution equation, let us follow the argument of Roberts [13] which gives rise to (8).

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In general, a centre manifold is approached exponentially quickly by neighboring trajectories, and to each solution $(x(t), y(t))$ of the full system (in this case (1)) there corresponds a solution $s(t)$ of the evolution equation on the centre manifold M (defined parametrically by $(x, y) = (X(s), Y(s))$) for which $(X(s(t)), Y(s(t)))$ has the same behaviour as $(x(t), y(t))$ to an exponentially decaying error as $t \to \infty$. Similarly, to each point s_0 on M there corresponds a one-dimensional *isochronic* manifold which consists of those initial points (x_0, y_0) whose subsequent evolutions have the same long-term behaviour under the full system as does the evolution starting from s_0 on M. For a general system, if M has dimension m then the isochronic manifold corresponding to each s_0 has codimension m. Now imagine applying an impulsive forcing so that the solution $s(t)$ is "kicked" a small distance off the centre manifold. If the direction of the impulse is such that the point remains on the same isochronic manifold then, by definition, its long-term behaviour is unaffected by the impulse. It is only the component of any impulse orthogonal to the isochronic manifold which significantly affects the long-term behaviour. (Note that this is not the same thing as ignoring the component of the impulse which is normal to M, since the isochronic manifolds are not in general orthogonal to M.) Any continuous forcing can be viewed as the limit of a sequence of such impulses, and therefore the correct evolution equation for s on M for a forced system is obtained by considering only the components of the forcing normal to the isochronic manifolds. From these considerations, (8) can be derived.

However, this method of projecting the forcing has drawbacks: firstly, the isochronic manifolds are typically as difficult to calculate exactly as the original full problem is to solve; and secondly, the position of the new, displaced centre manifold for the forced system is not found. The first of these can be overcome in a neighbourhood of M since it turns out to be straightforward to calculate the tangent-space approximations to the isochronic manifolds [13]. A knowledge of these tangent spaces is sufficient for the projection of small forces, where we ignore terms that are nonlinear in the forcing (that is, a linear projection is sufficient). The second problem is addressed below where we extend Roberts' argument to calculate M.

It turns out that (8) is the appropriate evolution equation if we choose the parameterisation of M in such a way that points on M_0 and M which lie on the same isochronic manifold have the same value of the parameter s. Since points which both start on the one isochronic manifold remain on the same isochronic manifold (albeit a different isochronic manifold at different times), then any component of the forcing which is parallel to the isochronic manifolds must make zero contribution to (10), so that solutions on M_0 and M remain "synchronised". Now a solution $\mathbf{u}_F = (x_F(s), y_F(s))$ of the forced

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system on M, and a solution $\mathbf{u}_0 = (x_0(s), y_0(s))$ of the unforced system on M_0 differ by an amount given by the product on the right-hand side of (9). This difference, by the definition of our parameterisation of M, is tangent to the isochronic manifold, and so the condition which we have described above, that the component of $\mathbf{F}$ parallel to the isochronic manifolds should vanish, is

$$[\sigma_x \quad \sigma_y] \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix} = 0. \tag{15}$$

Since this must hold for all F_x and F_y we conclude that the remaining two equations for our six unknown functions are

$$\sigma_x \xi_x + \sigma_y \eta_x = 0, \tag{16}$$

$$\sigma_x \xi_y + \sigma_y \eta_y = 0. \tag{17}$$

The following (or rather, just *one* solution, correct to terms of order $e^{-1/2s^2}$) is

$$\xi_x = -2s^2 c(s), \tag{18}$$

$$\xi_y = s(1 - 2s^2), \tag{19}$$

$$\eta_x = -2s(1 + 2s^2)c(s), \tag{20}$$

$$\eta_y = (1 - 4s^4), \tag{21}$$

$$\sigma_x = 1 + 2s^2, \tag{22}$$

$$\sigma_y = -s, \tag{23}$$

where, as before, $c(s) = \int_0^{1/2s^2} \sqrt{1 - 2s^2\tau}\ e^{-\tau} d\tau$. This gives the centre manifold M as

$$\left.\begin{aligned} x &= s - 2s^2 c(s) F_x + s(1 - 2s^2) F_y \\ y &= s^2 - 2s(1 + 2s^2)c(s) F_x + (1 - 4s^4) F_y \end{aligned}\right\}, \tag{24}$$

with the evolution on M given by (8). We see that this particular parameterisation of the forced centre manifold gives rise to an evolution equation which is identical to that derived by Roberts' projection arguments. (These results are consistent with (5).)

A more straightforward approach than the above to obtain the centre manifold for (1) might be to change the variables so that a fixed point of (1) for $\mathbf{F} \neq \mathbf{0}$ lies at the new origin, and then to apply standard centre manifold analysis. However, such changes of basis are tedious to carry out in practice for systems of any complexity. We wish instead to describe a method which avoids such changes—algebraic manipulations are performed on the full system, *as given*, to find the displaced centre manifold and evolution thereon. In particular our approach allows us to treat unsteady forcing easily.

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2.2 Time-dependent forcing

When $\mathbf{F}$ is small but unsteady, we would expect that there is (in some sense) a *time-dependent* centre manifold to which nearby trajectories are exponentially attracted, and on which there is a slower evolution. (Note that currently the theory of centre manifolds does not strictly admit time-dependent centre manifolds.) It is relatively straightforward to show that this is indeed the case, and that in this example trajectories are attracted to the curve

$$y = x^2 - 2xC_x(x, t) \star F_x(t) + C_y(x, t) \star F_y(t) + O(|\mathbf{F}|^2), \qquad (25)$$

where $C_y(x, t) = (1 - 2x^2 t)e^{-t}$ and $C_x(x, t) = \sqrt{1 - 2x^2 t}\, e^{-t}$, and where the convolutions indicated by "$\star$" are defined by

$$C(x, t) \star F(t) = \int_0^{1/(2x^2)} C(x, \tau) F(t - \tau) d\tau .$$

If we choose to parameterise the centre manifold $\mathcal{M}$ by x, the evolution is, from (1),

$$\dot{x}(t) = -x^3 + 2x^2 C_x(x, t) \star F_x(t) - xC_y(x, t) \star F_y(t) + F_x(t) + O(|\mathbf{F}|^2). \quad (26)$$

Note that the position of $\mathcal{M}$ and the evolution of x depend not only on the present value of the forcing, $\mathbf{F}(t)$, but also upon previous values, $\mathbf{F}(t - \tau)$. A similar "memory" has been noted in asymptotic descriptions of shear dispersion. There, the aim is to calculate the effective diffusion coefficient for a shear flow in a channel which may have varying breadth and where the fluid may have varying longitudinal dispersion. The local diffusion coefficient has a memory not only of the upstream conditions [16], but also of the diffusivity of the fluid at previous times [8].

Figure 2(a) shows the evolution of a random set of initial points under (1) when $F_x = 0$ and $F_y = -0.5 \sin 10t$. The points represent a stroboscopic view of the evolving system, where we plot the position of each point at every period of the flow, that is, at times $0, 2\pi/10, 4\pi/10, \ldots$. Clearly in this stroboscopic view the points are attracted to a curve, which is the instantaneous position of $\mathcal{M}$ at the strobe times. (Whenever the forcing is periodic then the position of $\mathcal{M}$ is a period function of time.) The crosses in Figure 2(b) on p. 412 indicate the positions of the points after eleven periods of the forcing, and the solid line is the curve (25), with the appropriate phase. There is clearly good agreement between our analysis and the numerical results for the full system. Note that although the forcing has zero mean and at the origin is perpendicular to the centre manifold it results in the de-stabilisation of the origin, and the birth of two finite-amplitude fixed points on $\mathcal{M}$. However, the fixed point nearest the origin is nonhyperbolic, just as the origin is when $\mathbf{F} = \mathbf{0}$, and so in order to calculate the two new fixed points, which are

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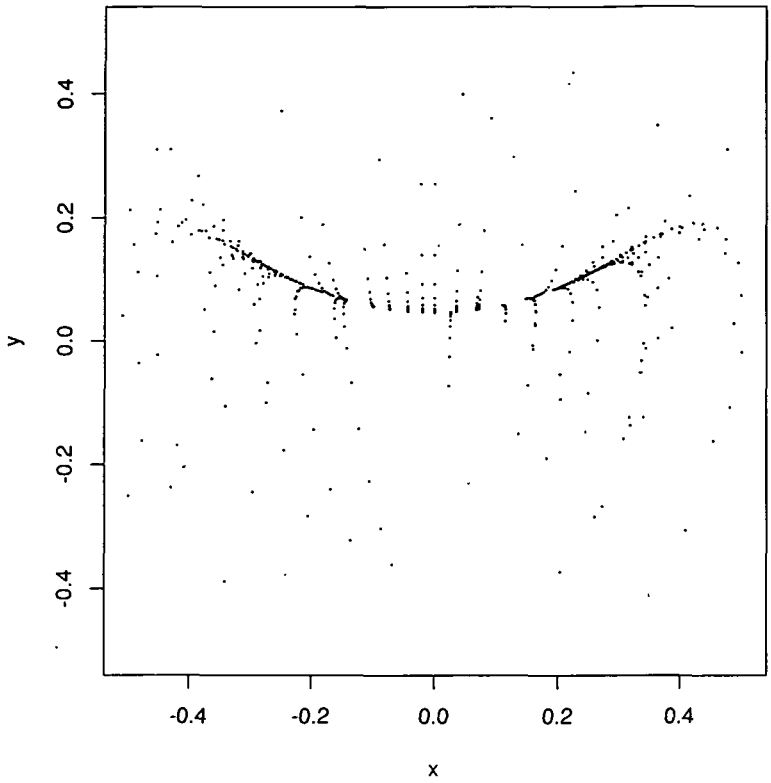

FIGURE 2(a). Evolution of a set of points from random initial conditions under (1) when the forcing is time-dependent. In this example, $\mathbf{F}^{\mathsf{T}} = (0, -0.5\sin 10t)$. Points are plotted at every period of the forcing, that is at intervals of $2\pi/10 \approx 0.63$ time-units. In this stroboscopic picture the points are clearly converging to a curve, which is the position of the centre manifold $M(t)$ at the strobe times.

$(x, y) = O(|F_y|) = \left(\pm a/\sqrt{1+\omega^2}, -a\omega/(1+\omega^2)\right) + O(a^2)$ when the forcing is $\mathbf{F} = (0, a\sin\omega t)$, we need to consider the quantities of order $|\mathbf{F}|^2$ in (25) and (26). These two comments hold whenever the forcing is solely in the y-direction, and has zero mean. If we choose instead to force the system again only in the y-direction, but so that F_y has non-zero mean, $\overline{F}_y \neq 0$, then the fixed point on the y-axis is stable or unstable as $\overline{F}_y$ is positive or negative, respectively, and the new fixed points which exist when $\overline{F}_y < 0$ lie a distance of order $\sqrt{-\overline{F}_y}$ from the origin. This extends the example of *steady* y-forcing, which has been remarked upon by Roberts [13]. As there, the surprising thing is that a small force perpendicular to the centre manifold, whose averaged component along M is towards the fixed point, may actually de-stabilise the fixed point.

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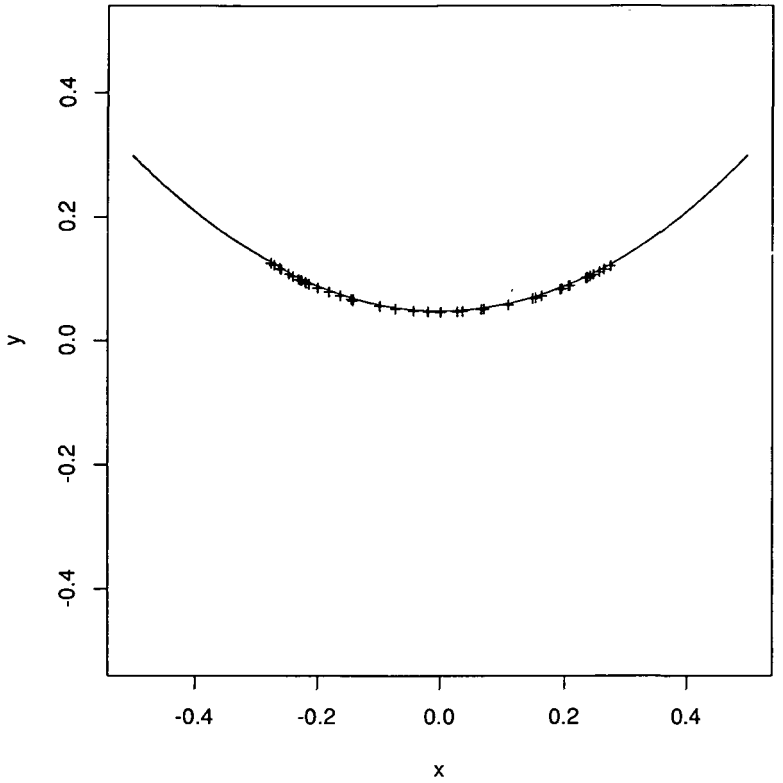

FIGURE 2(b). Crosses mark the position after eleven periods of the forcing of the points in (a). The solid line is the instantaneous position of the centre manifold, which is calculated to $O(|\mathbf{F}|)$, from (25).

Roberts' projection arguments imply that with an appropriate parameterisation the evolution on $\mathcal{M}$ under time-dependent forcing can be written as

$$\dot{s}(t) = -s^3 + \left(1 + 2s^2\right) F_x(t) - sF_y(t) + O(|\mathbf{F}|^2). \tag{27}$$

Here, in contrast to (26), there is no memory in the evolution equation on $\mathcal{M}$: the evolution depends only on the instantaneous value of the forcing, $\mathbf{F}(t)$. This apparent contradiction is easily resolved. The memory of previous forcing can be removed from (26) by a slight re-parameterisation of $\mathcal{M}$: if we write

$$s = x + 2x^2 \left\{\sqrt{1 - 2x^2 t}\, e^{-t}\right\} \star F_x(t) - x \left\{(1 - 2x^2 t)e^{-t}\right\} \star F_y(t) + O(|\mathbf{F}|^2), \tag{28}$$

then (26) and (27) are equivalent.

When there is time-dependent forcing, just as when $\mathbf{F}$ is constant, we can choose to project along the isochronic manifolds to define a parameterisation

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for M. In this case, we find that the evolution of s is given by (27). The displacement of M, however, "remembers" previous values of $\mathbf{F}$—in this case the (ξ, η)-matrix of (9) becomes a more general convolution operator on $\mathbf{F}(t)$, so that M is given to order $|\mathbf{F}|$ by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s \\ s^2 \end{bmatrix} + \begin{bmatrix} s \\ 1+2s^2 \end{bmatrix} (-2s \left\{ \sqrt{1-2s^2 t}\, e^{-t} \right\} \star F_x(t) + \left\{ (1-2s^2 t) e^{-t} \right\} \star F_y(t) K). \quad (29)$$

This description of M reduces to (8) and (24) when $\mathbf{F}$ is independent of time. This is one justification for what seems a complicated definition of parameters for the forced centre manifold: that by projecting along the isochronic manifolds we arrive at an evolution equation which depends only on the present value of the forcing and does not depend on previous values. If, rather than project along the isochronic manifolds, we choose any other parameterisation for M then we will in general derive an evolution equation with memory.

3. Forcing of differential equations in standard form

Here we describe a general formal method for calculating the effects of forcing on the centre manifold, and evolution thereon. Consider a dynamical system written in the so-called "standard form"

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{y}), \qquad \mathbf{x} \in \mathbb{R}^m, \\ \dot{\mathbf{y}} &= B\mathbf{y} + \mathbf{g}(\mathbf{x}, \mathbf{y}), \qquad \mathbf{y} \in \mathbb{R}^n, \end{aligned} \quad (30)$$

where A is an $m \times m$ matrix whose m eigenvalues have zero real part, B is an $n \times n$ matrix whose n eigenvalues have negative real parts, and $\mathbf{f}$ and $\mathbf{g}$ are at least C^2, and $O(|(\mathbf{x}, \mathbf{y})|^2)$ as $|(\mathbf{x}, \mathbf{y})| \to 0$. This system has a centre manifold, M_0, given by $\mathbf{y} = \mathbf{h}_0(\mathbf{x})$, which is invariant under (30), and which nearby trajectories approach exponentially quickly [2]. This centre manifold satisfies

$$B\mathbf{h}_0(\mathbf{x}) + \mathbf{g}(\mathbf{x}, \mathbf{h}_0(\mathbf{x})) = \partial \mathbf{h}_0 / \partial \mathbf{x} (A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{h}_0(\mathbf{x}))), \quad (31)$$

and has the m-dimensional dynamics of

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{h}_0(\mathbf{x})). \quad (32)$$

Often m is small, so that by considering only the evolution (32) of the system on M_0 we greatly simplify the original system.

When forced, (30) becomes

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{y}) + \mathbf{F}_x(t), \qquad \mathbf{x} \in \mathbb{R}^m, \\ \dot{\mathbf{y}} &= B\mathbf{y} + \mathbf{g}(\mathbf{x}, \mathbf{y}) + \mathbf{F}_y(t), \qquad \mathbf{y} \in \mathbb{R}^n, \end{aligned} \quad (33)$$

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where the forcing $\mathbf{F}(t) = (\mathbf{F}_x, \mathbf{F}_y)^\mathsf{T}$ is assumed to be "small", and we describe its effects on the centre manifold M_0 of (30), to $O(|\mathbf{F}|)$.

3.1 Existence of a centre manifold under constant forcing

When $\mathbf{F}$ is non-zero and steady, the system (33) has a centre manifold, M, given by $\mathbf{y} = \mathbf{h}_0(\mathbf{x}) + \mathbf{h}_1(\mathbf{x}, \mathbf{F})$, where $\mathbf{h}_1(\mathbf{x}, \mathbf{F}) = O(|\mathbf{F}|)$, at least for small $\mathbf{F}$. We may demonstrate this by re-writing the forcing of (20) as $\epsilon^2\overline{\mathbf{F}}$, where $\overline{\mathbf{F}}$ is now of order one, and ϵ is a small constant. Then by adjoining the trivial equation $\dot{\epsilon} = 0$ to the forced system we effectively consider the parameter ϵ as an extra component of $\mathbf{x}$, and treat the forcing $\epsilon^2\overline{\mathbf{F}}$ as a nonlinear term. This is a standard trick for bifurcation problems [2]. Thus (33) can be put in the (unforced) standard form (30) as

$$\begin{aligned} \dot{\mathbf{x}}_* &= A_*\mathbf{x}_* + \mathbf{f}_*(\mathbf{x}_*, \mathbf{y}), \quad \mathbf{x}_* \in \mathbb{R}^{m+1}, \\ \dot{\mathbf{y}} &= B\mathbf{y} + \mathbf{g}_*(\mathbf{x}_*, \mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^n, \end{aligned} \tag{34}$$

where $\mathbf{x}_* = (\epsilon, \mathbf{x})$,

$$A_* = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{bmatrix}, \quad \mathbf{f}_* = \begin{bmatrix} 0 \\ \mathbf{f}(\mathbf{x}, \mathbf{y}) + \epsilon^2\overline{\mathbf{F}}_x \end{bmatrix}, \quad \mathbf{g}_* = \mathbf{g}(\mathbf{x}, \mathbf{y}) + \epsilon^2\overline{\mathbf{F}}_y. \tag{35}$$

We shall assume in what follows that the forcing is sufficiently small to ensure the existence of a centre manifold.

Substituting the *ansatz* $\mathbf{y} = \mathbf{h}_0(\mathbf{x}) + \mathbf{h}_1(\mathbf{x}, \mathbf{F})$ into (33), and equating terms of order $|\mathbf{F}|$, we find that $\mathbf{h}_1$ satisfies

$$B\mathbf{h}_1 + \frac{\partial \mathbf{g}}{\partial \mathbf{y}}\mathbf{h}_1 + \mathbf{F}_y = \frac{\partial \mathbf{h}_0}{\partial \mathbf{x}}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{y}}\mathbf{h}_1 + \mathbf{F}_x\right) + \frac{\partial \mathbf{h}_1}{\partial \mathbf{x}}(A\mathbf{x} + \mathbf{f}), \tag{36}$$

where the functions are evaluated on $\mathbf{y} = \mathbf{h}_0(\mathbf{x})$.

In general, the equation for $\mathbf{h}_1$ cannot be solved exactly—in practice it would be solved either by iteration or by power-series expansions, in fact the same procedure as is used to find $\mathbf{h}_0$ from (31). If $\mathbf{h}_0$ is approximated with error $O(|\mathbf{x}|^p)$ then $\mathbf{h}_1$ can at best be known with error $O(|\mathbf{x}|^{p-1}|\mathbf{F}|)$. For small $|\mathbf{x}|$, the leading-order effects of the forcing are twofold: the forcing of the stable modes acts to displace the centre manifold by an amount $\mathbf{h}_1(\mathbf{x}, \mathbf{F}) \sim -B^{-1}\mathbf{F}_y$, from (36); the forcing of the marginal modes, which are those used to parametrise M, acts to change the evolution $\dot{\mathbf{x}}$ on M by an amount $\mathbf{F}_x$. This much is trivial: the power of the centre-manifold analysis is in giving the successive corrections to this leading-order description which are due to the curvature of M.

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This standard-form analysis corresponds to the solution (5) of the exact system (1). The extra sophistication of projecting along the isochronic manifolds does not arise when the dynamical system is in standard form, since the parameters used to describe both the forced and unforced centre manifolds are, by definition, the same, namely $\mathbf{x}$. The solution (5) was found from (31) and (36), which in the case of the previous section become simply

$$xh_0\frac{\partial h_0}{\partial x} + x^2 - 2h_0^2 - h_0 = 0, \tag{37}$$

and

$$x^3\frac{\partial h_1}{\partial x} - (1+2x^2)h_1 = 2xF_x - F_y. \tag{38}$$

These are readily solved, up to terms of order $e^{-1/(2x^2)}$ to give (5).

3.2 Time-dependent forcing

When $\mathbf{F}$ is small, but time-dependent, we still expect to be able to reduce the original system (33) to a simpler m-dimensional system which describes accurately its long-time behaviour. However, this long-time behaviour now takes place on an m-dimensional *time-dependent* manifold, M, given by $\mathbf{y} = \mathbf{h}_0(\mathbf{x}) + \mathbf{h}_1(\mathbf{x}, \mathbf{F}(t))$. There is no rigorous theory for "time-dependent centre manifolds", but nonetheless we expect to be able to derive expressions for $\mathbf{h}_0$ and $\mathbf{h}_1$ which capture the long-time behaviour of the original system. That this is a reasonable expectation was demonstrated in Section 2. As when $\mathbf{F}$ is constant, $\mathbf{h}_0$ satisfies (31), but now $\mathbf{h}_1$ satisfies

$$-\frac{\partial \mathbf{h}_1}{\partial \mathbf{F}}\frac{\partial \mathbf{F}}{\partial t} + B\mathbf{h}_1 + \frac{\partial \mathbf{g}}{\partial \mathbf{y}}\mathbf{h}_1 + \mathbf{F}_y = \frac{\partial \mathbf{h}_0}{\partial \mathbf{x}}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{y}}\mathbf{h}_1 + \mathbf{F}_x\right) + \frac{\partial \mathbf{h}_1}{\partial \mathbf{x}}(A\mathbf{x}+\mathbf{f}). \tag{39}$$

This can be written as

$$\left(\frac{\partial}{\partial t} - \left\{B + \frac{\partial \mathbf{g}}{\partial \mathbf{y}} - \frac{\partial \mathbf{h}_0}{\partial \mathbf{x}}\frac{\partial \mathbf{f}}{\partial \mathbf{y}} - (A\mathbf{x}+\mathbf{f})\cdot\frac{\partial}{\partial \mathbf{x}}\right\}\right)\mathbf{h}_1 = \mathbf{F}_y - \frac{\partial \mathbf{h}_0}{\partial \mathbf{x}}\mathbf{F}_x, \tag{40}$$

where $\partial \mathbf{h}_1/\partial t$ means $\partial \mathbf{h}_1/\partial \mathbf{F}\,\dot{\mathbf{F}}$. More compactly, this is

$$\left(\frac{\partial}{\partial t} - B\right)\mathbf{h}_1 = \mathbf{F}_y - \frac{\partial \mathbf{h}_0}{\partial \mathbf{x}}\mathbf{F}_x = \left[-\tfrac{\partial \mathbf{h}_0}{\partial \mathbf{x}} \quad I_n\right]\mathbf{F}, \tag{41}$$

and a typical solution is a convolution of the form

$$\mathbf{h}_1 = \int_0^\infty C(\mathbf{x}, \tau)\mathbf{F}(t-\tau)\,d\tau. \tag{42}$$

From this it follows that

$$\frac{\partial \mathbf{h}_1}{\partial t} = C(\mathbf{x}, 0)\mathbf{F}(t) + \int_0^\infty C_\tau(\mathbf{x}, \tau)\mathbf{F}(t-\tau)\,d\tau, \tag{43}$$

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and so, substituting into (41), we find

$$C(\mathbf{x}, 0)\mathbf{F}(t)+\int_0^\infty \left\{\left(\frac{\partial}{\partial\tau}-\mathcal{B}\right)C(\mathbf{x},\tau)\right\}\mathbf{F}(t-\tau)\,d\tau = \left[-\tfrac{\partial \mathbf{h}_0}{\partial \mathbf{x}} \quad I_n\right]\mathbf{F}(t). \quad (44)$$

This equation can be satisfied for arbitrary forcing if

$$C(\mathbf{x}, 0) = \left[-\tfrac{\partial \mathbf{h}_0}{\partial \mathbf{x}} \quad I_n\right], \quad (45)$$

and

$$\frac{\partial C}{\partial \tau} = \mathcal{B}C. \quad (46)$$

In general we cannot solve exactly for C, but we find approximate solutions of (46) in the form of a power series in the amplitudes, $\mathbf{x}$.

Note that the evolution equation for $\mathbf{x}$ is

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{h}_0(\mathbf{x})) + \frac{\partial \mathbf{f}}{\partial \mathbf{y}}\mathbf{h}_1(\mathbf{x}, \mathbf{F}) + \mathbf{F}_x + O(|\mathbf{F}|^2), \quad (47)$$

and this involves not only the present value of $\mathbf{F}_x(t)$, but also earlier values of $\mathbf{F}_x$ and $\mathbf{F}_y$, through $\mathbf{h}_1$: there is a memory of the history of the forcing which arises from the convolution (42).

When $|(\mathbf{x}, \mathbf{y})|$ is small, (46) reduces to

$$\frac{\partial C}{\partial \tau} \approx BC, \quad (48)$$

so that

$$C(x, \tau) \approx e^{B\tau}\left[0_m \quad -I_n\right]. \quad (49)$$

Then

$$\mathbf{h}_1 \approx -\int_0^\infty e^{B\tau}\mathbf{F}_y(t-\tau)\,d\tau. \quad (50)$$

Corrections to this expression for $\mathbf{h}_1$ may be found through iteration of (36) or by a power series expansion in $\mathbf{x}$, but already this leading-order expression for $\mathbf{h}_1$ tells us that the memory of the previous history of the forcing has time-scales which are the reciprocals of the eigenvalues of B, as is reasonable, and that the simplest approximation (which ignores the linearly damped $\mathbf{y}$ modes) gets the long-time evolution of the system (33) wrong by an amount which is of the same order as the forcing.

4. Application to a five-mode atmospheric model

In this section we examine a system which is in the standard form of the previous section, and which is of interest for numerical weather prediction.

It is well known that the bulk of the atmosphere is in approximate geostrophic equilibrium. For any atmospheric model which is not quasi-geostrophic, though, an arbitrary initial state generally gives rise to rapid

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large-amplitude gravity-wave oscillations which are not a realistic feature of the earth's atmosphere itself. A problem for numerical weather prediction is to filter out the rapidly-oscillating gravity waves from initial wind and pressure data by some initialisation process, or "balancing". If there is an *invariant* manifold on which no gravity waves develop (a "slow manifold") then balancing is achieved by projecting the initial data onto this manifold—this process would remove the gravity waves from the numerical solutions completely. It is of interest to know how this manifold, if it exists, is deformed when the system is forced, and what the evolution on the manifold is like.

We consider a variant of the system proposed by Lorenz & Krishnamurthy [7] as a very simple model for interacting atmospheric Rossby waves and gravity waves, with constant forcing,

$$\left.\begin{aligned} \dot{v} &= uw - buz + F_v \\ \dot{u} &= -vw + bvz + F_u \\ \dot{w} &= -uv + F_w \\ \dot{x} &= -z - ax + F_x \\ \dot{z} &= x - az + buv + F_z \end{aligned}\right\}. \tag{51}$$

The parameter a represents a damping of the fast gravity waves, (x, z), and b represents the coupling between the gravity waves and the slower Rossby waves, (v, u, w). Unlike Lorenz & Krishnamurthy, we have not directly damped the Rossby waves—without the interaction with the gravity waves, through the terms in b, the Rossby waves would be perpetual.

To relate this system with the notation of the previous section, let $\mathbf{x} = (v, u, w)^{\mathsf{T}}$, $\mathbf{y} = (x, z)^{\mathsf{T}}$,

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} -a & -1 \\ 1 & -a \end{bmatrix}, \tag{52}$$

$$\mathbf{f} = (uw - buz, -vw + bvz, -uv)^{\mathsf{T}}, \qquad \mathbf{g} = (0, buv)^{\mathsf{T}}. \tag{53}$$

Note that A indeed has three zero eigenvalues and the eigenvalues of B are $-a \pm i$, whose real parts are negative whenever there is damping, that is whenever $a > 0$. Roberts [13] has calculated the first few terms in a power-series expansion of the sub-centre manifold which the system possesses when $a = 0$ (that is when B has pure imaginary eigenvalues) and when there is no forcing.

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The analysis of the previous section leads us to the following equations to be solved for the forced centre manifold, $\mathbf{y} = \mathbf{h}_0(\mathbf{x}) + \mathbf{h}_1(\mathbf{x}, \mathbf{F})$,

$$\begin{bmatrix} -a & -1 \\ 1 & -a \end{bmatrix} \mathbf{h}_0(\mathbf{x}) + \begin{bmatrix} 0 \\ buv \end{bmatrix} = \frac{\partial \mathbf{h}_0}{\partial \mathbf{x}} \begin{bmatrix} uw - buz \\ -vw + bvz \\ -uv \end{bmatrix}, \tag{54}$$

$$\begin{bmatrix} -a & -1 \\ 1 & -a \end{bmatrix} \mathbf{h}_1(\mathbf{x}) + \begin{bmatrix} F_x \\ F_z \end{bmatrix} = \frac{\partial \mathbf{h}_0}{\partial \mathbf{x}} \left\{ \begin{bmatrix} 0 & -bu \\ 0 & bv \\ 0 & 0 \end{bmatrix} \mathbf{h}_1(\mathbf{x}) + \begin{bmatrix} F_v \\ F_u \\ F_w \end{bmatrix} \right\} + \frac{\partial \mathbf{h}_1}{\partial \mathbf{x}} \begin{bmatrix} uw - buz \\ -vw + bvz \\ -uv \end{bmatrix}. \tag{55}$$

Substituting a power-series for $\mathbf{h}_0$ and $\mathbf{h}_1$ into their governing equations, and using the computer algebra package REDUCE, we find that

$$\begin{aligned} \mathbf{h}_0 = {} & \frac{buv}{(1+a^2)} \begin{bmatrix} -1 \\ a \end{bmatrix} + \frac{bw(u^2 - v^2)}{(1+a^2)^2} \begin{bmatrix} 2a \\ 1 - a^2 \end{bmatrix} \\ & + \frac{buv}{(1+a^2)^3} \begin{bmatrix} (3a^2 - 2a^2b^2 - 1)(u^2 - v^2) + 4(3a^2 - 1)w^2 \\ -a\{(a^2 + a^2b^2 - 3 - b^2)(u^2 - v^2) + 4(a^2 - 3)w^2\} \end{bmatrix} \\ & + O(|\mathbf{x}|^5), \end{aligned} \tag{56}$$

which reduces to Roberts' result [13] for the sub-centre manifold when $a = 0$, and

$$\begin{aligned} \mathbf{h}_1 = {} & \frac{1}{(1+a^2)} \begin{bmatrix} aF_x - F_z \\ aF_z + F_x \end{bmatrix} + \frac{uF_v + vF_u}{(1+a^2)^2} \begin{bmatrix} 2ab \\ b(1 - a^2) \end{bmatrix} \\ & + \frac{1}{(1+a^2)^3} \begin{bmatrix} -(u^2 - v^2)(2ab(aF_z + F_x) + (3a^2 - 1)F_w) \\ b(u^2 - v^2)\{b(a^2 - 1)(F_x + aF_z) + a(a^2 - 3)F_w)\} \\ +2(vF_v - uF_u)(3a^2 - 1)w \\ -2abw(vF_v - uF_u)(a^2 - 3) \end{bmatrix} + O(|\mathbf{F}|\,|\mathbf{x}|^3 + |\mathbf{F}|^2). \end{aligned} \tag{57}$$

Here we can see the leading-order (that is $O(|\mathbf{F}|\,|\mathbf{x}|^0)$) displacement of the centre manifold due to the forcing to be just that determined by the geostrophic approximation $(\dot{x}, \dot{z}) \approx (0, 0)$.

Typical initial conditions for the full dynamical system (51) produce both Rossby waves and large-amplitude gravity waves, as shown by the solid line in Figure 3. The initial conditions for the integrations on $\mathcal{M}_0$ and $\mathcal{M}$ are found by appropriately projecting the initial conditions of the full system (51) [13]. The aim of the projection in this example is to balance the initial data of the full system and remove any gravity waves (these are the oscillations in Figure 3 with period of roughly six time-units). The projection onto $\mathcal{M}_0$ leads to a solution, the dotted line in Figure 3, with small-amplitude gravity waves

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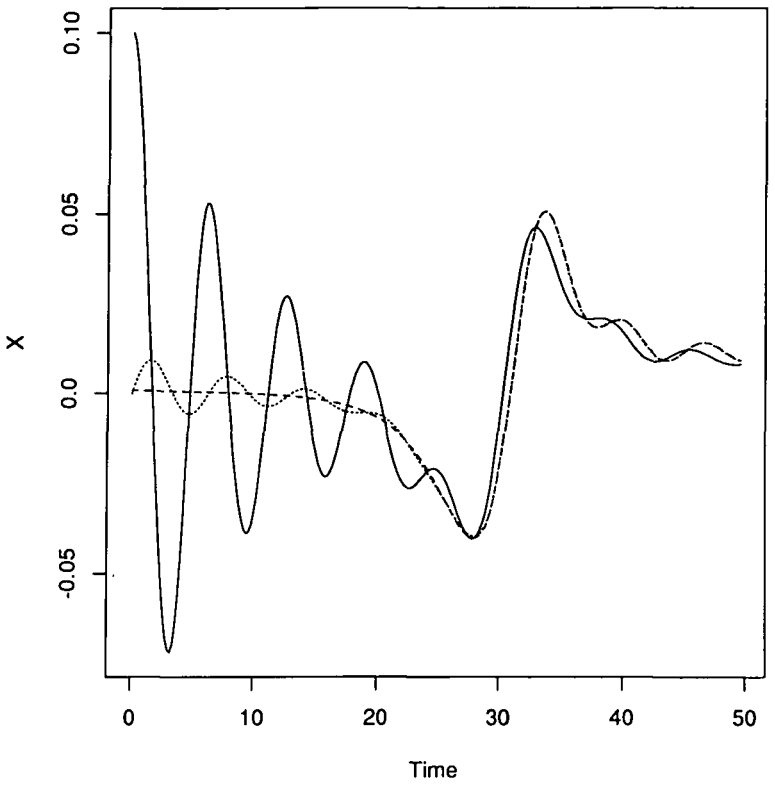

FIGURE 3. Time-series of x (one component of the gravity-wave complex (x, z)). The solid line represents evolution of x under (51) from the initial condition $(v, u, w, x, z) = (0.01, 0.01, 0.01, 0.1, 0)$, with forcing $\mathbf{F} = (0.01, 0, 0, 0.01, 0)$. Here $a = 0.1$ and $b = 1$. If the initial condition is projected according to [13] onto the *unforced* centre manifold $\mathcal{M}_0$, the the dotted line indicates the subsequent evolution of x; this projection is ineffective in balancing the initial condition. If the initial condition is projected onto the *forced* centre manifold $\mathcal{M}$ then the subsequent evolution of x follows the dashed line, which has no gravity waves (at least initially). (The gravity waves are the oscillations with a period of roughly six time-units, and the Rossby waves have a period of roughly fifty time-units.)

which decay at first, then new gravity-wave activity sets in after $t \approx 30$. The projection of initial conditions onto $\mathcal{M}$, leading to the dashed line in Figure 3, is much more successful in balancing the data: initially there is no gravity-wave activity. That small-amplitude gravity waves develop after $t \approx 30$ was used by Lorenz & Krishnamurthy [7] to argue against the global existence, for this particular 5-component system, of a slow manifold containing no gravity waves.

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The projection onto the unforced centre manifold, M_0, is included in the figure for comparison with the projection onto the forced centre manifold, M. It is the former projection which is implicit in [13], where it is assumed that "*the forcing of the full system is never so big that it pushes the system far from the invariant manifold* M [our M_0]" [13, p. 72]. The figure indicates clearly, as we argue in the present paper, that it is the forced centre manifold which is the appropriate summary of the long-term behaviour of (51).

Lorenz [6] proposed a scheme which successively improves an approximation to the centre manifold, and when modified slightly his scheme is equivalent to an iteration scheme for solving (54) and (55). The time derivatives of x and z are first ignored, that is they are assumed to be zero, and a first approximation to an invariant manifold, (x_1, z_1), is calculated in terms of the parameters (v, u, w) from the x and z evolution equations. Then new approximations to $\dot{x}$ and $\dot{z}$ are given by $\dot{x}_1$ and $\dot{z}_1$. This allows a second approximation to the invariant manifold, (x_2, z_2), to be calculated. This iterative procedure can be continued indefinitely to give approximations (x_n, z_n) to $\mathbf{h}_0 + \mathbf{h}_1$ accurate to arbitrarily high order. We have followed this iteration scheme as far as (x_3, z_3), and the results are consistent with (56) and (57) to the appropriate orders in $\mathbf{x}$.

5. Forcing of differential equations in general form

A dynamical system with a centre manifold generally will not be in the convenient standard form considered in Section 3. In principle, a change of variables can always remedy this; in practice such a change is frequently clumsy, and adds complexity to the process of finding M. Besides, the original variables of the problem may have a physical significance which the transformed variables do not.

For the general dynamical system, which is not in standard form,

$$\dot{\mathbf{u}} = \mathcal{L}\mathbf{u} + \mathbf{N}(\mathbf{u}), \tag{58}$$

where $\mathcal{L}$ is a linear operator with m eigenvalues with zero real part, and $\mathbf{N}$ is strictly nonlinear in the unknowns $\mathbf{u}$, the centre manifold M_0 can be parameterised by an m-vector $\mathbf{s} = (s_1, \ldots, s_m)$ of "amplitudes". Then $\mathbf{u}$ is given on M_0 by $\mathbf{u}(t) = \mathbf{V}(\mathbf{s})$, where the evolution on M_0 is given by $\dot{\mathbf{s}} = \mathbf{G}(\mathbf{s})$. To find $\mathbf{V}$ and $\mathbf{G}$, we need to solve the equation

$$\mathcal{L}\mathbf{V} + \mathbf{N}(\mathbf{V}) = \frac{\partial \mathbf{V}}{\partial \mathbf{s}}\mathbf{G}(\mathbf{s}), \tag{59}$$

together with the definitions of the amplitudes $\mathbf{s}$ [3, 12].

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The important practical advantage of this method for calculating M_0 over first having to set the system in standard form is that there is no messy algebra associated with a change of basis. Accordingly, for a physical problem the gain is that we deal throughout the analysis in physically-meaningful variables, $\mathbf{u}$, and parameters, $\mathbf{s}$.

5.1 Constant forcing

5.1.1 *The equation for the forced centre manifold.* Suppose that the system is subject to a small steady forcing, so that (58) becomes

$$\dot{\mathbf{u}} = \mathcal{L}\mathbf{u} + \mathbf{N}(\mathbf{u}) + \mathbf{F}. \tag{60}$$

The centre manifold is effectively displaced, and the evolution on it is modified by the forcing. We pose in general that the centre manifold is displaced so that

$$\mathbf{u}(t) = \mathbf{V}(\mathbf{s}) + \mathcal{W}\mathbf{F} + O(|\mathbf{F}|^2), \tag{61}$$

where the evolution is modified to be

$$\dot{\mathbf{s}} = \mathbf{G}(\mathbf{s}) + \mathcal{H}\mathbf{F} + O(|\mathbf{F}|^2). \tag{62}$$

Here $\mathcal{W}$ and $\mathcal{H}$ are linear operators which depend on $\mathbf{s}$ and act on $\mathbf{F}$. The effective displacement of M due to a forcing $\mathbf{F}$ is $\mathcal{W}\mathbf{F}$, and the corresponding modification to the evolution of the amplitudes $\mathbf{s}$ is $\mathcal{H}\mathbf{F}$. When the system is finite-dimensional, that is $\mathbf{u} \in \mathbb{R}^N$ for some N, then $\mathcal{W}$ and $\mathcal{H}$ can be written as $N \times N$ and $m \times N$ matrices, respectively, but our analysis allows them to be general linear operators whose form arises naturally from the problem at hand. By substituting (61) and (62) into (60), and then considering the terms of order $|\mathbf{F}|$, we find the equation to be satisfied by $\mathcal{W}$ and $\mathcal{H}$ is

$$\mathcal{L}\mathcal{W} + J\mathcal{W} + I = \frac{\partial \mathbf{V}}{\partial \mathbf{s}}\mathcal{H} + \mathbf{G}\cdot\nabla\mathcal{W}, \tag{63}$$

where J is the Jacobian $\frac{\partial \mathbf{N}}{\partial \mathbf{u}}|_{\mathbf{u}=\mathbf{V}}$, I is the identity operator, and we use $\mathbf{G}\cdot\nabla$ to denote $\sum_j \mathbf{G}_j \partial/\partial s_j$.

5.1.2 *Solving the equation for the forced centre manifold.* In general, equation (59) for the unforced centre manifold M_0 cannot be solved exactly, but is instead solved asymptotically. We substitute the asymptotic series

$$\mathbf{u}(t) = \mathbf{V}(\mathbf{s}) \sim \sum_{k=1}^{\infty} \mathbf{V}^{(k)}(\mathbf{s}), \tag{64}$$

$$\dot{\mathbf{s}}(t) = \mathbf{G}(\mathbf{s}) \sim \sum_{k=1}^{\infty} \mathbf{G}^{(k)}(\mathbf{s}), \tag{65}$$

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into (59), where $\mathbf{V}^{(k)}(\mathbf{s})$ and $\mathbf{G}^{(k)}(\mathbf{s})$ represent terms of order k in the amplitudes $\mathbf{s}$. Then by considering successively higher orders in $\mathbf{s}$, we may obtain $\mathbf{V}^{(k)}(\mathbf{s})$ and $\mathbf{G}^{(k)}(\mathbf{s})$ for $k = 1, 2, \ldots$; see [12]. To calculate the operators $\mathcal{W}$ and $\mathcal{H}$, we make similar expansions,

$$\mathcal{W} \sim \sum_{k=0}^{\infty} \mathcal{W}^{(k)}, \tag{66}$$

$$\mathcal{H} \sim \sum_{k=0}^{\infty} \mathcal{H}^{(k)}, \tag{67}$$

and substitute these into (63). Then we may solve at successive orders to find $\mathcal{W}^{(k)}$ and $\mathcal{H}^{(k)}$ for $k = 0, 1, 2, \ldots$ in sequence.

At the k-th stage (if $\mathbf{G}^{(1)} = \mathbf{0}$, as occurs frequently) we solve an equation of the form

$$\mathcal{L}\mathcal{W}^{(k)} = \frac{\partial \mathbf{V}^{(1)}}{\partial \mathbf{s}} \mathcal{H}^{(k)} + (\text{products of known terms of orders } 1, \ldots, k-1). \tag{68}$$

We can solve this first for $\mathcal{H}^{(k)}$ by ensuring that the right-hand side of (68) lies in the range of $\mathcal{L}$. Then we are left with an equation for $\mathcal{W}^{(k)}$ of the form

$$\mathcal{L}\mathcal{W}^{(k)} = (\text{known terms}), \tag{69}$$

which cannot be solved uniquely because the operator $\mathcal{L}$ is singular—it has m zero eigenvalues. A similar non-uniqueness arises at each order in calculating the unforced centre manifold, $\mathbf{V}^{(k)}$; there we avoid the difficulty by using the definitions of the amplitudes $\mathbf{s}$. So now we have to choose a particular parameterisation for the forced centre manifold M.

The amplitudes $\mathbf{s}$ are given by a definite expression $\mathbf{s} = \mathbf{S}(\mathbf{u})$. The missing m equations for $\mathcal{W}^{(k)}$ arise from considering the behaviour of $\mathbf{S}$ a distance of order $|\mathbf{F}|$ away from M_0. We see that

$$\mathbf{s} = \mathbf{S}(\mathbf{V}(\mathbf{s}) + \mathcal{W}(\mathbf{s})\mathbf{F} + O(|\mathbf{F}|^2)) \tag{70}$$

$$= \mathbf{S}(\mathbf{V}(\mathbf{s})) + \mathcal{S}\,\mathcal{W}(\mathbf{s})\mathbf{F} + O(|\mathbf{F}|^2), \tag{71}$$

where $\mathcal{S} = \frac{\partial \mathbf{S}}{\partial \mathbf{u}}|_{\mathbf{u}=\mathbf{V}}$. Since (71) must hold for all $\mathbf{F}$, and since $\mathbf{s} = \mathbf{S}(\mathbf{V}(\mathbf{s}))$, we deduce that

$$\mathcal{S}\,\mathcal{W} = 0. \tag{72}$$

The k-th order terms of this constraint, namely

$$\mathcal{S}^{(0)}\mathcal{W}^{(k)} = -\sum_{j=1}^{k} \mathcal{S}^{(j)}\mathcal{W}^{(k-j)}, \tag{73}$$

supply the missing m equations for $\mathcal{W}^{(k)}$.

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In many cases the right-hand side of (73) will vanish, often because we define $s_j = \langle \mathbf{z}_j, \mathbf{u} \rangle$, where $\{\mathbf{z}_j\}$ is a basis for the left zero-eigenspace of $\mathcal{L}$, and where $\langle \cdot, \cdot \rangle$ denotes an appropriate inner product. This is the case, for example, whenever (58) is in standard form, for then (73) takes the simple form

$$\langle \mathbf{z}_j, \mathcal{W}^{(k)} \rangle = 0 \quad \text{for } j = 1, \ldots, m. \tag{74}$$

Of course, the right-hand side of (73) need not vanish if the amplitudes have a more complicated definition. For example, in Section 2.3 we could have chosen to parameterise the centre manifold by distance from the origin or by arc-length.

5.1.3 *Parameterisation of the forced centre manifold by projection along the isochronic manifolds.* Now consider how our procedure for calculating $\mathcal{W}$ and $\mathcal{H}$ by asymptotic series compares with the method which Roberts [13] has described for calculating $\mathcal{H}$, which involves approximating the forcing $\mathbf{F}$ by a sequence of discrete impulses. The idea there is that the only significant effect of an impulse, in the long-term, is to move each point a small distance off its isochronic manifold, $\mathbf{\Pi}(\mathbf{s}_-)$, onto a different isochronic manifold, $\mathbf{\Pi}(\mathbf{s}_+)$. The result of the impulse on each point is in effect to change its value of $\mathbf{s}$ slightly. In the limit of a continuous forcing, this results in a change in the evolution of $\mathbf{s}$, which is $\mathcal{H}(\mathbf{s})\mathbf{F}$. It turns out that we can calculate $\mathcal{H}$ provided we know the m-dimensional space which is normal to the isochronic manifolds. Since m is small in general then this method requires relatively little work. Although a knowledge of $\mathcal{H}$ gives us the dynamics on the forced centre manifold $\mathcal{M}$, it does not tell us the location of $\mathcal{M}$, or how to parameterise $\mathcal{M}$: the value of $\mathcal{H}$ found by Roberts' "isochronic projection" argument only gives the correct evolution equation on $\mathcal{M}$ *provided the parameterisation of* $\mathcal{M}$ *is chosen appropriately.* We can readily provide a condition on the parameterisation for this to be the case.

Consider the parameterisation of $\mathcal{M}$ which we saw in the example of Section 2, and which in the impulse model corresponds to our specifying that a forcing (impulse) *along* $\mathbf{\Pi}(\mathbf{s})$ does not affect the values of $\mathbf{s}$, that is, $\mathbf{s}_+ = \mathbf{s}_-$. This parameterisation assigns to each point $\mathbf{u}$ off the centre manifold a value of the amplitudes $\mathbf{s}$ which corresponds to the point of intersection between $\mathcal{M}_0$ and the isochronic manifold on which $\mathbf{u}$ lies. In effect this redefines the amplitudes $\mathbf{s} = \mathbf{S}(\mathbf{u})$ for $\mathbf{u}$ off the centre manifold. Near $\mathcal{M}_0$, then, $\mathbf{S}$ approximately projects along $\mathbf{\Pi}$ to find $\mathbf{s}$, that is, we find that $\mathbf{S}(\mathbf{u})$ satisfies

$$\langle R(\mathbf{S}(\mathbf{u})), \mathbf{u} - \mathbf{V}(\mathbf{S}(\mathbf{u})) \rangle = 0, \tag{75}$$

where we consider the finite-dimensional case, and we define $R(\mathbf{s})$ to be an $m \times N$ matrix whose rows are a basis for the space normal to the projection

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manifolds. When $\mathbf{u}$ is on the forced centre manifold, $\mathbf{u} \sim \mathbf{V}(\mathbf{s}) + \mathcal{W}\mathbf{F}$ and so (75) becomes

$$\langle R(\mathbf{S}(u)),\ \mathcal{W}(\mathbf{S}(\mathbf{u}))\rangle = 0. \tag{76}$$

A comparison of this result with (72) shows that $\mathcal{S}$ is the operation of taking the inner product with R.

Now consider (63); from this equation we can show readily that Roberts' result for $\mathcal{H}$ follows naturally from the *ansatz* (61) and (62), without the need to approximate $\mathbf{F}$ by impulses. We can also readily derive a condition for the appropriate parameterisation of $\mathcal{M}$ so that the evolution equation involves Roberts' expression for $\mathcal{H}$. For simplicity, consider the finite-dimensional case, so that $\mathcal{W}$ and $\mathcal{H}$ are matrices. Then, from [13],

$$R(\mathbf{s})\{\mathcal{L} + \mathcal{J}\} + \mathbf{G}\cdot\nabla R(\mathbf{s}) = 0, \tag{77}$$

and we may choose for definiteness the initial condition

$$R^{\mathsf{T}}(\mathbf{0}) = \left(\frac{\partial \mathbf{V}}{\partial \mathbf{s}}\right)\Bigg|_{\mathbf{s}=\mathbf{0}}. \tag{78}$$

It follows then that

$$\begin{aligned} 0 &= [R(\mathbf{s})\{\mathcal{L} + \mathcal{J}\} + \mathbf{G}\cdot\nabla R(\mathbf{s})]\mathcal{W}(\mathbf{s}) \\ &= R(\mathbf{s})[\{\mathcal{L} + \mathcal{J} - \mathbf{G}\cdot\nabla\}\mathcal{W}(s)] + \mathbf{G}\cdot\nabla\{R(\mathbf{s})\mathcal{W}(\mathbf{s})\} \\ &= R(\mathbf{s})\left[-I_N + \frac{\partial \mathbf{V}}{\partial \mathbf{s}}\mathcal{H}(\mathbf{s})\right] + \mathbf{G}\cdot\nabla\{R(\mathbf{s})\mathcal{W}(\mathbf{s})\}, \end{aligned} \tag{79}$$

where I_N is the $N \times N$ identity matrix. Therefore, provided the choice of $\mathcal{W}$ is such that

$$\mathbf{G}\cdot\nabla\{R(\mathbf{s})\mathcal{W}(\mathbf{s})\} = 0, \tag{80}$$

then

$$\mathcal{H}(\mathbf{s}) = \left(R(\mathbf{s})\frac{\partial \mathbf{V}}{\partial \mathbf{s}}\right)^{-1} R(\mathbf{s}), \tag{81}$$

where we know that, for small $|\mathbf{s}|$ at least, $R(\mathbf{s})\partial\mathbf{V}/\partial\mathbf{s}$ is invertible. (Compare this with [13], equation (7.7), where our $\mathcal{H}\mathbf{F}$ is Roberts' $\mathbf{F}^{\mathcal{M}}$.)

Now we see that the parameterisation of $\mathcal{M}$ defined by projection along the isochronic manifolds—which in (72) has $\mathcal{S}$ denoting the inner product with R—results in $\mathcal{W}(\mathbf{s})$ such that $R(\mathbf{s})\mathcal{W}(\mathbf{s}) = 0$, that is, such that (80) is trivially satisfied. Therefore this parameterisation of $\mathcal{M}$ is the appropriate one for Roberts' evolution equation. From (81) we see that $\mathcal{H}$ annihilates any component of the forcing tangent to the projection manifold; as we would expect, such a forcing makes no contribution to the evolution of $\mathbf{s}$ for this particular parameterisation. Also when $R(\mathbf{s})\mathcal{W}(\mathbf{s}) = 0$, the matrix $[-I_N + \partial\mathbf{V}/\partial\mathbf{s}\mathcal{H}(\mathbf{s})]$ is seen from (79) to be of the form $K(\mathbf{s})L(\mathbf{s})$, where $K(\mathbf{s})$ is an $N \times (N-m)$ matrix whose columns span the projection space, and $L(\mathbf{s})$

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is some $(N-m)\times N$ matrix. To see what $L(\mathbf{s})$ is like, we note that (79) and (81) imply that

$$\left[-I_N+\frac{\partial \mathbf{V}}{\partial \mathbf{s}}\mathcal{H}(\mathbf{s})\right]\frac{\partial \mathbf{V}}{\partial \mathbf{s}}=0, \tag{82}$$

so $K(\mathbf{s})L(\mathbf{s})\partial \mathbf{V}/\partial \mathbf{s}=0$, and see that the rows of $L(\mathbf{s})$ must span the space orthogonal to the centre manifold. Our results for the exact system of Section 2 illustrate this—we find

$$\begin{aligned}-I_N+\frac{\partial \mathbf{V}}{\partial \mathbf{s}}\mathcal{H} &= \begin{bmatrix} 2s^2 & -s \\ 2s(1+2s^2) & -(1+2s^2)\end{bmatrix} \\ &= \begin{bmatrix} s \\ 1+2s^2\end{bmatrix}\begin{bmatrix} 2s & -1\end{bmatrix}.\end{aligned} \tag{83}$$

Here the first matrix in the product gives the direction of the projection manifold (compare with [13, p. 56]) and the second matrix gives the normal to the unforced centre manifold, $(x, y)=(s, s^2)$.

We can see on "physical" grounds that (81) is reasonable. Consider a forcing which is instantaneously along the direction of increasing s_j on $\mathcal{M}$. Then $\mathbf{F}=F\partial \mathbf{V}/\partial s_j$ say, and so from (81), $\mathcal{H}\mathbf{F}=F\mathbf{e}_j$, where $\{\mathbf{e}_j\}$ is the standard basis for $\mathbb{R}^m$. this just says that due to this particular forcing $\dot{s}_j$ is increased by an amount F, which is what we would expect for such a forcing.

Let us return now to the power-series calculation of $\mathcal{W}$ and $\mathcal{H}$. When the parameterisation of $\mathcal{M}$ is by projection along the isochronic manifolds, the condition (72) implies

$$\langle R(\mathbf{s}), \mathcal{W}\rangle = 0, \tag{84}$$

so that, as we have anticipated above, (80) is satisfied, and so $\mathcal{H}$ is given by (81). From this result, it is apparent that

$$\mathcal{H}\mathcal{W}=0. \tag{85}$$

Thus the appropriate set of m equations to adjoin to (69) to specify $\mathcal{W}^{(k)}$ uniquely is the k-th order part of (85), that is

$$\mathcal{H}^{(0)}\mathcal{W}^{(k)}=-\sum_{j=1}^{k}\mathcal{H}^{(j)}\mathcal{W}^{(k-j)}. \tag{86}$$

In general, in projecting along the isochronic manifolds the right-hand side of (86) is nonzero.

5.1.4 *Relation between various choices for* $\mathcal{W}$. We have seen that (81) is satisfied provided $\mathbf{G}\cdot\nabla\{R\mathcal{W}\}=0$, and we have considered above the special case that $R\mathcal{W}=0$, but the question arises—what if $R(\mathbf{s})\mathcal{W}(s)$ is constant, but nonzero? Without going into details, this corresponds to a degeneracy which

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allows us to re-parameterise the unforced centre manifold M_0 by altering the definition of the parameters $\mathbf{s}$ by an amount of order $|\mathbf{F}|$.

There are two "natural" choices for W which we have seen in our examples: firstly, that which keeps the same parameters for the forced problem as for the unforced (this is the case, for example, with any system in standard form); and secondly, that which projects along the isochronic manifolds. If we wish to carry out this second re-parameterisation then even if we only want to know the effect of a *particular* forcing, $\mathbf{F}_*$ say, we must still calculate the *operator* H, and not just the quantity $H\mathbf{F}_*$. This corresponds to the necessity of knowing the tangent space of the isochronic manifolds in order to project $\mathbf{F}_*$.

Equation (85), in effect, specifies that the column space of W be the isochronic tangent plane. Here lies a point of potential confusion, when we talk about the isochronic manifolds "of the problem", since the tangent space, $\mathbf{\Pi}_0$, of the isochronic manifolds of the unforced problem and this tangent space, $\mathbf{\Pi}_\mathbf{F}$, of the forced problem have slopes which differ by $O(|\mathbf{F}|)$. Since M_0 and M are a distance of the order $|\mathbf{F}|$ apart, the projection between M_0 and M along $\mathbf{\Pi}_0$ differs from the projection along $\mathbf{\Pi}_\mathbf{F}$ by an amount of the order $|\mathbf{F}|^2$. Our scheme only includes corrections to the unforced results up to $O(|\mathbf{F}|)$ and so these differences can consistently be neglected. In fact, our scheme for H and W finds $\mathbf{\Pi}_0$ rather than $\mathbf{\Pi}_\mathbf{F}$ as the column space of W, since the calculations are "based" on M_0.

The isochronic manifolds were originally introduced to find which initial point $\mathbf{s}_0$ on M corresponds to an arbitrary initial condition $\mathbf{u}_0$ of the full system, in the sense that their evolutions are exponentially close as $t \to \infty$. Consider now an unforced system. The projection of initial conditions along the isochronic manifolds, by the definition of these manifolds, is *exact*. This (nonlinear) projection, however, is as complicated to accomplish as the full problem is to solve, and so an approximation is made whereby the projection is made along the tangent space, $\mathbf{\Pi}_0$, to the isochronic manifolds at M. Thus if $\mathbf{u}_0$ is a distance of order l from M, then the error in locating $\mathbf{s}_0$ introduced by projecting along the tangent space $\mathbf{\Pi}_0$ rather than exactly along the curved isochronic manifold itself is of the order l^2, due to the neglected curvature of the isochronic manifold. In the forced system, when (nonlinear) projection is along the isochronic manifolds, again, the projection is *exact*, by definition. If the projection is instead made along the tangent space $\mathbf{\Pi}_\mathbf{F}$ there are introduced errors of the order l^2. Finally, if the projection is made along $\mathbf{\Pi}_0$ then the errors in locating $\mathbf{s}_0$ are of order $(l^2, l|\mathbf{F}|, |\mathbf{F}|^2)$. This last reduction in accuracy, over projection along $\mathbf{\Pi}_\mathbf{F}$, may be considered a worthwhile compromise (to project nonlinearly along the isochronic manifolds exactly is assumed out of the question as the full problem is assumed

to be intractable) since the calculation of $\mathbf{\Pi}_0$ is considerably easier than that of $\mathbf{\Pi}_\mathbf{F}$. This is because we need an "initial condition" for calculating either $\mathbf{\Pi}$, that is, its direction at some point on the centre manifold. Then we can integrate along the centre manifold to find $\mathbf{\Pi}$ everywhere on $\mathcal{M}$. The initial condition for $\mathbf{\Pi}_0$ comes naturally from a linear stability analysis of the origin $\mathbf{u} = \mathbf{0}$, whereas there is no simple initial condition for $\mathbf{\Pi}_\mathbf{F}$. For example, from (68) when $k = 0$ we find

$$\mathcal{L}\mathcal{W}^{(0)} = \frac{\partial \mathbf{V}^{(1)}}{\partial \mathbf{s}}\mathcal{H}^{(0)} - I\,. \tag{87}$$

If Z is a matrix composed of the left zero-eigenvectors of $\mathcal{L}$ then

$$\langle Z\,,\, \mathcal{L}\mathcal{W}^{(0)}\rangle = \left\langle Z\,,\, \frac{\partial \mathbf{V}^{(1)}}{\partial \mathbf{s}}\mathcal{H}^{(0)}\right\rangle - Z\,, \tag{88}$$

from which it follows that

$$\mathcal{H}^{(0)} = \left\langle Z\,,\, \frac{\partial \mathbf{V}^{(1)}}{\partial \mathbf{s}}\right\rangle^{-1} Z\,. \tag{89}$$

Now from (86) it follows that

$$\langle Z\,,\, \mathcal{W}^{(0)}\rangle = 0\,. \tag{90}$$

Thus, as a linear stability analysis predicts, the isochronic manifold at the origin has no component in the tangent space of $\mathcal{M}$—it lies tangent to the stable manifold of the origin.

5.2 Time-dependent forcing

For definiteness, consider a finite-dimensional system, so that $\mathcal{L}$, I, $\mathcal{W}$ and $\mathcal{H}$ are matrices. The equation to be solved for $\mathcal{W}$ and $\mathcal{H}$ is, to $O(|\mathbf{F}|)$,

$$\frac{\partial}{\partial t}\mathcal{W} - \{\mathcal{L}\mathcal{W} + J\mathcal{W} - \mathbf{G}\cdot\nabla\mathcal{W}\} = I - \frac{\partial \mathbf{V}}{\partial \mathbf{s}}\mathcal{H}\,. \tag{91}$$

This can be written as

$$\left(\frac{\partial}{\partial t} - \mathcal{T}\right)\mathcal{W} = I - \frac{\partial \mathbf{V}}{\partial \mathbf{s}}\mathcal{H}\,, \tag{92}$$

where the operator $\mathcal{T}$ is defined by

$$\mathcal{T} = \mathcal{L}\mathcal{W} + J\mathcal{W} - \mathbf{G}\cdot\nabla\,. \tag{93}$$

We try a solution of the form

$$\mathcal{W}\mathbf{F} = C(\mathbf{s}\,,\, t)\star\mathbf{F}(t) = \int_0^\infty C(\mathbf{s}\,,\, \tau)\mathbf{F}(t-\tau)\,d\tau\,. \tag{94}$$

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Then, in a similar way to the standard-form results, we can satisfy the governing equation for C by choosing

$$C(\mathbf{s}, 0) = I - (\partial \mathbf{V}/\partial \mathbf{s})\mathcal{H}, \tag{95}$$

and

$$\partial C/\partial \tau = \mathcal{T} C. \tag{96}$$

Here it is implicit in the initial condition (95) for C that the operator $\mathcal{H}$ involves only the current value of the forcing, and not previous values. That is, we have chosen the parameterisation of $\mathcal{M}$ to be such that *the evolution of* $\mathbf{s}$ *depends only on the instantaneous value of the forcing.* In general an exact solution for C cannot be found; instead an approximate solution is constructed by power series in the amplitudes, $\mathbf{s}$.

Formally, the solution for $\mathcal{W}$ is

$$\mathcal{W}\mathbf{F} = \int_0^\infty e^{\tau\mathcal{T}} \left(I - \frac{\partial \mathbf{V}}{\partial \mathbf{s}} \mathcal{H} \right) \mathbf{F}(t-\tau)\, d\tau. \tag{97}$$

The terms which act on $\mathbf{F}$ inside the integral in (97) have the following interpretation. Firstly, the term $I - \frac{\partial \mathbf{V}}{\partial \mathbf{s}}\mathcal{H}$ acts to eliminate the component of the forcing along $\mathcal{M}$ (see (82)). Such a component does not alter the position of $\mathcal{M}$, although it does alter the evolution on $\mathcal{M}$. Secondly, the term $e^{\tau\mathcal{T}}$ weights the integrand so that the most recent values of $\mathbf{F}$ contribute most to the instantaneous position of $\mathcal{M}$. The memory of the earlier forcing decays exponentially.

When $|\mathbf{s}|$ is small, we can easily obtain a first approximation to the expansion (97) for $\mathcal{W}\mathbf{F}$, valid near the origin in $\mathbf{s}$-space. The simplest case, and a typical one, is when $\mathbf{G}^{(1)} = \mathbf{0}$. Then

$$\partial C/\partial \tau \approx \mathcal{L} C, \tag{98}$$

so that

$$C(\mathbf{s}, \tau) \approx e^{\mathcal{L}\tau}(I - \mathcal{V}\mathcal{H}^{(0)}) = e^{\mathcal{L}\tau}(I - \mathcal{V}\{\mathcal{V}^{\mathsf{T}}\mathcal{V}\}^{-1}\mathcal{V}^{\mathsf{T}}), \tag{99}$$

where $\mathcal{V}$ is the Jacobian matrix $\frac{\partial \mathbf{V}^{(1)}}{\partial \mathbf{s}}$. Then

$$\mathcal{W}\mathbf{F} \approx \int_0^\infty e^{\mathcal{L}\tau}(I - \mathcal{V}\{\mathcal{V}^{\mathsf{T}}\mathcal{V}\}^{-1}\mathcal{V}^{\mathsf{T}})\mathbf{F}(t-\tau)\, d\tau. \tag{100}$$

The terms which act on $\mathbf{F}$ inside the integral here have the following straightforward interpretation. Firstly, the term $I - \mathcal{V}\{\mathcal{V}^{\mathsf{T}}\mathcal{V}\}^{-1}\mathcal{V}^{\mathsf{T}}$ eliminates the component of the forcing along the tangent plane to $\mathcal{M}$ at the origin. (Such a component may be written as $\mathcal{V}\mathbf{q}$ for some vector $\mathbf{q}$, but then $(I - \mathcal{V}\{\mathcal{V}^{\mathsf{T}}\mathcal{V}\}^{-1}\mathcal{V}^{\mathsf{T}})\mathcal{V}\mathbf{q} = I\mathcal{V}\mathbf{q} - \mathcal{V}\mathbf{q} = \mathbf{0}$.) Secondly, the term $e^{\mathcal{L}\tau}$ indicates that the memory of the present position of $\mathcal{M}$ on the previous values

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of the forcing decays exponentially, to a first approximation, with the decay rates of the damped modes (that is, the eigenvalues of $\mathcal{L}$).

What effect does the assumption that the evolution of $\mathbf{s}$ depends only on the instantaneous value of the forcing have on the re-parameterisation? Firstly, we note that

$$\begin{aligned} \mathcal{H}C(\mathbf{s}, 0) &= \mathcal{H}\{I - \partial\mathbf{V}/\partial\mathbf{s}\mathcal{H}\} \\ &= \mathcal{H} - \mathcal{H}(\partial\mathbf{V}/\partial\mathbf{s})\mathcal{H} \\ &= 0, \end{aligned} \tag{101}$$

from (81). Secondly, if we premultiply (96) by $R(\mathbf{s})$, and use the definition of $\mathcal{T}$ we obtain

$$-\mathbf{G}\cdot\nabla\{RC\} + (R\{\mathcal{L} + J\} + \mathbf{G}\cdot\nabla R)C = \frac{\partial RC}{\partial\tau}. \tag{102}$$

The term in parentheses vanishes (see (77)) and so

$$\left(\frac{\partial}{\partial\tau} + \mathbf{G}\cdot\nabla\right)(RC) = 0. \tag{103}$$

But from (81) and (101) it follows that

$$R(\mathbf{s})C(\mathbf{s}, 0) = 0, \tag{104}$$

and so,

$$R(\mathbf{s})C(\mathbf{s}, \tau) = 0 \quad \text{for all } \tau. \tag{105}$$

Using (81) again, it follows that

$$\mathcal{H}C(\mathbf{s}, \tau) = 0 \quad \text{for all } \tau, \tag{106}$$

and therefore

$$\mathcal{H}\mathcal{W} = 0. \tag{107}$$

The result is then that an evolution equation for $\mathbf{s}$ which depends only on the instantaneous value of the forcing can be derived by choosing the parameterisation which involves projection along the isochronic manifolds. This justifies such a parameterisation of the forced centre manifold, which at first sight may seem unnecessarily complicated.

It turns out that we can write the solution for C for the example of Section 2 in the form $C = \overline{C}\gamma$ where both $\overline{C}$ and γ are 2×2 matrices, with $\overline{C} = \overline{C}(s)$ and $\gamma = \gamma(s, \tau)$. We find that

$$\overline{C} = \begin{bmatrix} s^3 e^{-1/(2s^2)} & s^3/2 \\ s^2(1 + 2s^2)e^{-1/(2s^2)} & s^4 \end{bmatrix}, \tag{108}$$

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and that γ is in fact some function of $\phi = \tau - 1/(2s^2)$. Imposing the initial condition (95) on C fixes the functional dependence of γ on its argument, and gives

$$\gamma(\phi) = e^{-\phi}\begin{bmatrix} -2\sqrt{-2\phi} & -2\phi \\ 0 & 0 \end{bmatrix}. \tag{109}$$

Therefore

$$C(\mathbf{s}, \tau) = e^{-\tau}\begin{bmatrix} -2s^2\sqrt{1-2s^2\tau} & s(1-2s^2\tau) \\ -2s(1+2s^2)\sqrt{1-2s^2\tau} & (1+2s^2)(1-2s^2\tau) \end{bmatrix}. \tag{110}$$

Hence (29). If (80) is not satisfied then the evolution of the parameters $\mathbf{s}$ which define $\mathcal{M}$ will depend not only on the present value of the forcing, but also on previous values.

6. Application to the Kuramoto-Sivashinsky equation

Here we consider the forced Kuramoto-Sivashinsky equation,

$$u_t + u_{xx} + \alpha u_{xxxx} + uu_x = f(x, t) \quad \text{for } x \in (0, \pi), \tag{111}$$

with boundary conditions that $u = 0$ at $x = 0$ and at $x = \pi$; see [1, 4, 5, 11]. This equation is used to model laminar flame fronts, the solidification fronts of dendrites in dilute binary alloys, the instabilities between viscous fluids, and the evolution of the concentrations of reacting chemical species. We use the results described in the previous section to find the centre manifold of the forced system (111) near the first pitchfork bifurcation (of the unforced system) at $\alpha = 1$.

6.1 The unforced centre manifold

Consider the evolution equation (111) when $f(x, t) \equiv 0$, that is, when there is no forcing. The equation is of the form (58), with

$$\mathcal{L} = -\frac{\partial^2}{\partial x^2} - \frac{\partial^4}{\partial x^4}, \qquad N(u) = -u\frac{\partial u}{\partial x} + \epsilon\frac{\partial^4 u}{\partial x^4}, \tag{112}$$

where

$$\alpha = 1 - \epsilon.$$

Note that formally we adjoin the equation $\dot{\epsilon} = 0$ to (111) and then consider that the parameter ϵ is really a variable on the same footing as the true variable u, so that the product $\epsilon\partial^4 u/\partial x^4$ is indeed nonlinear in the variable (ϵ, u). This is the same standard trick used to analyse bifurcations [2] that we used in Section 3.1. The linear operator $\mathcal{L}$ has a single zero eigenvalue, corresponding to the eigenmode $u = \sin x$, and therefore there is a single amplitude, $s(t)$. For convenience we define s to be the component of $\sin x$

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in the Fourier series for u. We find that for small ϵ the centre manifold is given by

$$u \sim s \sin x - \left(\frac{1}{24} + \frac{7}{144}\epsilon\right) s^2 \sin 2x + \frac{1}{1152} s^3 \sin 3x + O(|(\epsilon, s)|^4), \quad (113)$$

on which the system evolves according to

$$\dot{s} = \epsilon s - \frac{1}{48} s^3 + O(|(\epsilon, s)|^4). \quad (114)$$

The pitchfork bifurcation when $\epsilon = 0$ can be clearly seen in this Landau equation.

6.2 Constant forcing

Applying the analysis of the previous section we find that the first few modifications to the centre manifold due to the forcing $f(x)$ are

$$\mathcal{H}^{(0)}(f) = \langle \sin x, f \rangle, \quad (115)$$

$$\mathcal{H}^{(1)}(f) = (s/2) \left\langle \sin 2x, \mathcal{W}^{(0)}(f) \right\rangle, \quad (116)$$

$$\mathcal{L}\mathcal{W}^{(0)}(f) = \langle \sin x, f \rangle \sin x - f, \quad (117)$$

$$\mathcal{L}\mathcal{W}^{(1)}(f) = \frac{\partial}{\partial x}[s \sin x \ \mathcal{W}^{(0)}(f)] - \epsilon \frac{\partial^4}{\partial x^4} \mathcal{W}^{(0)}(f) - (s/2) \sin x \left\langle \sin 2x, \mathcal{W}^{(0)}(f) \right\rangle - (s/12) \sin 2x \langle \sin x, f \rangle, \quad (118)$$

where $\langle a(x), b(x) \rangle$ denotes the inner product

$$\frac{2}{\pi} \int_0^\pi a(x) b(x) \, dx. \quad (119)$$

To find $\mathcal{W}$ we need to invert the operator $\mathcal{L}$. In choosing an inverse we select a parameterisation for the centre manifold, $\mathcal{M}$. For example we choose the parameter, s, to remain as the component of $\sin x$ in the solution, u, and so we choose $\mathcal{L}^{-1}$ such that

$$\mathcal{L}(\mathcal{L}^{-1}(a(x))) = a(x) - \langle \sin x, a(x) \rangle \sin x \quad \text{and} \quad \left\langle \sin x, \mathcal{L}^{-1}(a(x)) \right\rangle = 0. \quad (120)$$

To leading order (as in general) the component of the forcing $f(x)$ in the range of the linear operator $\mathcal{L}$ acts to shift the position of the centre manifold of the system (111), while the component of $f(x)$ in the kernel of $\mathcal{L}$ modifies the evolution on $\mathcal{M}$. The power of our analysis, however, is that it gives higher-order corrections to the model forcing of the low-dimensional approximation (114). For example, the evolution of s is governed by

$$\dot{s} = \epsilon s - s^3/48 + \langle \sin x, f \rangle + (s/2) \left\langle \sin 2x, \mathcal{W}^{(0)}(f) \right\rangle + O(|(\epsilon, s)|^4, |(\epsilon, s)|^2 |f|), \quad (121)$$

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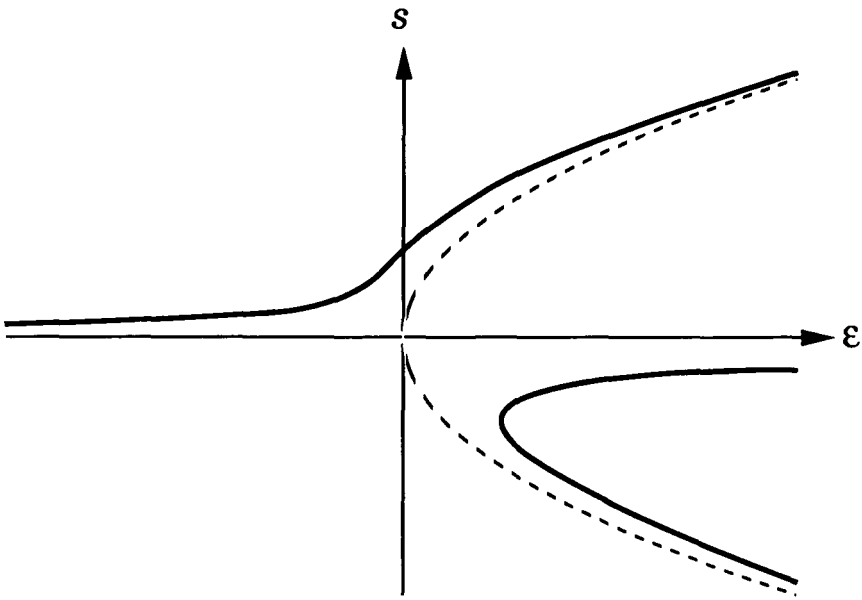

FIGURE 4(a). The solid line is the position of the equilibria of (123) for $a_0 > 0$ and $a_1 = 0$ and shows how the pitchfork bifurcation which occurs in the unforced Kuramoto-Sivashinsky equation, dashed line, is broken by the forcing.

which can be written as

$$\dot{s} = a_0 + (\epsilon + a_1/2)\, s - s^3/48 + O(|(\epsilon\,,\, s)|^4\,,\, |(\epsilon\,,\, s)|^2|f|)\,, \tag{122}$$

where $a_0 = \langle \sin x\,,\, f\rangle$ and $a_1 = \langle \sin 2x\,,\, \mathcal{W}^{(0)}(f)\rangle$. If $a_0 \neq 0$ then the pitchfork bifurcation is broken as shown in Figure 4(a). This is easily seen by inspection of (111): if we ignore the nonlinear term and balance u_t with f, then the evolution of the $\sin x$ component of u is subject to a constant forcing which is due to the $\sin x$ component of f. Suppose instead that $a_0 = 0$, that is, the forcing has no component like $\sin x$, then we cannot easily determine the effect of the forcing on solutions of (111) by simple inspection. However, the evolution equation (122) for s derived from the centre-manifold analysis tells us that the pitchfork bifurcation remains, but occurs when $\epsilon = -a_1/2$ as shown in Figure 4(b).

6.3 Time-dependent forcing

When f is time-dependent, we pose that

$$\mathcal{W}(f(x\,,\, t)) = \int_0^\infty C(s\,,\, \tau) f(x\,,\, t-\tau)\, d\tau\,, \tag{123}$$

and since $\mathcal{W}^{(0)} = O(s^0)$, then

$$\mathcal{W}^{(0)}(f(x\,,\, t)) = \int_0^\infty C^{(0)}(\tau) f(x\,,\, t-\tau)\, d\tau\,. \tag{124}$$

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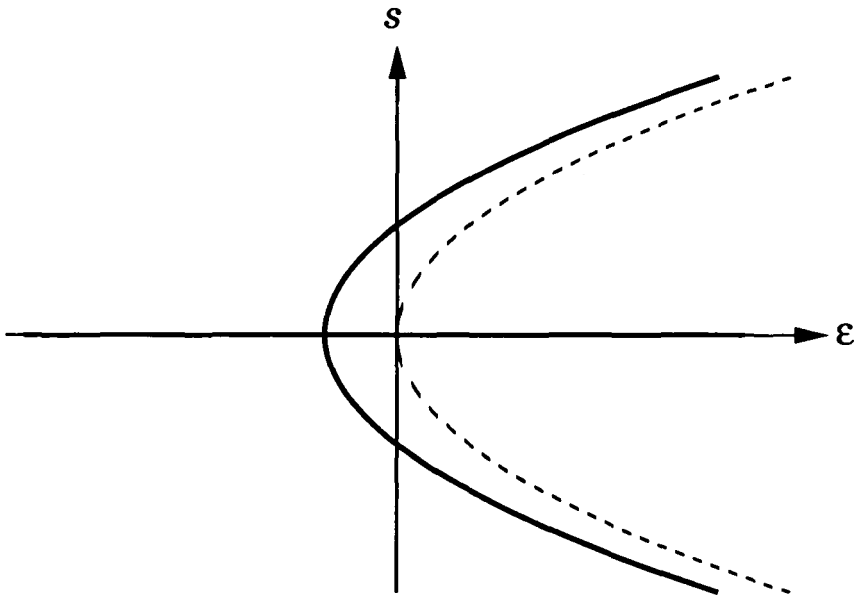

FIGURE 4(b). The solid line is the position of the equilibria of (123) for $a_0 = 0$ and $a_1 > 0$ and shows that the pitchfork bifurcation which occurs in the unforced Kuramoto-Sivashinsky equation, dashed line, may occur at a different value of the critical parameter under forcing.

Note that in determining $\mathcal{W}^{(0)}$ (but not $\mathcal{W}^{(1)}, \mathcal{W}^{(2)}, \ldots$) the choice of parameterisation for $\mathcal{M}$ is irrelevant.

The operator $\mathcal{C}^{(0)}$ satisfies the initial condition

$$\mathcal{C}^{(0)}(0)f(x, t-\tau) = f(x, t-\tau) - \sin x \langle \sin x, f(x, t-\tau) \rangle \tag{125}$$

and the governing equation

$$\frac{\partial \mathcal{C}^{(0)}}{\partial \tau} = \mathcal{L}\mathcal{C}^{(0)}(\tau). \tag{126}$$

Formally then, the solution for $\mathcal{W}^{(0)}$ is

$$\mathcal{W}^{(0)}(f) = \int_0^\infty e^{\mathcal{L}\tau}\{f(x, t-\tau) - \langle \sin x, f(x, t-\tau) \rangle \sin x\}\, d\tau, \tag{127}$$

which is equivalent to the expression

$$\mathcal{W}^{(0)}(f) = \sum_{n=2}^{\infty} \sin nx \left\langle \sin nx, \int_0^\infty e^{-n^2(n^2-1)\tau} f(x, t-\tau)\, d\tau \right\rangle. \tag{128}$$

This expression illustrates more concretely the comments made generally after (100). Firstly, to leading order there is no contribution to the position of $\mathcal{M}$ from the $\sin x$ component of f. Secondly, the memory of previous values of the forcing decays exponentially on time-scales which are precisely

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those of the decay of the damped modes (and which are different for the different components of f). The leading contribution to the evolution of s due to the time-dependent forcing $f(x, t)$ is, as when there is constant forcing $\mathcal{H}^{(0)}(f) = \langle \sin x, f \rangle$. The subsequent corrections $\mathcal{H}^{(1)}, \mathcal{H}^{(2)}, \ldots$ will depend on the choice of inverse for $\mathcal{L}$, that is, on the choice of parameterisation for $\mathcal{M}$. In particular, if we choose s to be the $\sin x$ component of u, then $\mathcal{H}(f)$ involves not only the present value of f, but also previous values.

7. Conclusion

We have extended the centre-manifold formalism to dynamical systems which are subject to some forcing, and we have described a way to calculate both the position of the forced centre manifold, $\mathcal{M}$, and the evolution thereon for such systems. In this way we can capture all the interesting long-term behaviour of a (suitable) forced high-dimensional system in a low-dimensional model.

The need to incorporate correctly the forcing of the full system into the centre-manifold model is well illustrated by (our modification of) Lorenz' 5-mode atmospheric model. The balancing of the initial data which is necessary in numerical weather prediction to remove unphysical gravity waves can only be done correctly in this model if proper account is taken of the effects of the forcing on the centre manifold (Figure 3). A second issue is that the initial conditions of the full system (which do not in general lie in $\mathcal{M}$) must be projected appropriately onto the centre manifold [13]. This projection must be along the isochronic manifolds.

We have only considered systems for which $\mathbf{G}^{(1)} = \mathbf{0}$, since this is often true. If not, then the algebra becomes more complicated, but the same general strategy for finding the forced centre manifold applies. In fact, our analysis could be extended to calculate more general invariant manifolds [14] under forcing, not just centre manifolds.

When a dynamical system in standard form is analysed as in Section 3, the same parameters (namely the amplitudes of the marginal modes) are necessarily used to describe both the unforced and forced centre manifolds ($\mathcal{M}_0$ and $\mathcal{M}$ respectively). Clearly, this is desirable if these amplitudes have some physical significance, for then the centre-manifold model describes the evolution of physical quantities, and the position of $\mathcal{M}$ gives the stable modes as functions of the same physical quantities. For systems in general form, as in Section 5, we may similarly choose the parameters, $\mathbf{s}$, to be the amplitudes of the marginal modes. In this case, however, we can instead choose the

parameters for M to be different from those for M_0. A useful alternative choice for $\mathbf{s}$ is that which identifies points on M_0 and M which lie on the same isochronic manifolds (Section 5.1.3 and [13]). Although this second choice for $\mathbf{s}$ is more complicated than the first, it has the nice property that the evolution of such $\mathbf{s}$ under time-dependent forcing depends only on the current value of the forcing, and not on previous values. Of course, if the original parameters have some physical significance then we lose much in dealing with this less physically-meaningful second choice of parameters.

When the forcing is time-dependent, the long-term evolution of solutions takes place on some time-dependent manifold. In principle this manifold can be found, and we have written down the equations to be solved for it. The practicalities of finding $M(t)$ are rather difficult (Sections 3.2 and 5.2), although approximate solutions may be found as power series in the amplitudes, as for the case of steady forcing. Indeed, we have found and interpreted the leading-order term in this power series for $M(t)$ for the example of the forced Kuramoto-Sivashinsky equation in Section 6. This example illustrates how the position of $M(t)$ "remembers" the previous history of the forcing, $\mathbf{F}(t-\tau)$, with a weight which decays in time like the most slowly decaying transient mode.

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