Can. J. Math., Vol. XXXIV, No. 4, 1982, pp. 910–915

ISOMETRIES OF WEIGHTED BERGMAN SPACES

CLINTON J. KOLASKI

1. Introduction. In [**2**], [**8**] and [**10**], Forelli, Rudin and Schneider described the isometries of the H^p spaces over balls and polydiscs. Koranyi and Vagi [**6**] noted that their methods could be used to describe the isometries of the H^p spaces over bounded symmetric domains. Recently Kolaski [**4**] observed that the algebraic techniques used above and Rudin's theorem on equimeasurability extended to the Bergman spaces over bounded Runge domains. In this paper we use the same general argument to characterize the onto linear isometries of the weighted Bergman spaces over balls and polydiscs, (all isometries referred to are assumed to be linear).

2. Preliminaries. Horowitz [**3**] first defined the weighted Bergman space $A^{p,\alpha}(0 < p < \infty, 0 < \alpha < \infty)$ to be the space of holomorphic functions f in the disc which satisfy

$$(1) \qquad \|f\|_{p,\alpha}{}^p = \frac{1}{\pi}\int_0^\pi \int_0^1 |f(re^{i\theta})|^p(1 - r^2)^\alpha r dr d\theta < \infty .$$

Since the Bergman kernel for the disc is given by

$$B(z, w) = [1 - \langle z, w\rangle]^{-2},$$

it was natural to define the weighted Bergman space $A^{p,r}(\Omega)$ $(0 < p < \infty)$ to be the class of holomorphic functions f on the domain $\Omega \subset \mathbf{C}^n$ for which

$$(2) \qquad \|f\|_{p,r}{}^p = \int_\Omega |f(z)|^p B_\Omega(z, z)^{-r} dm(z) < \infty$$

where B_Ω is the Bergman kernel for Ω, m denotes Lebesgue measure on $\mathbf{C}^n = R^{2n}$, and r is greater than some negative constant $K(\Omega)$ which depends on the domain Ω. For recent papers dealing with these spaces see [**1**] and [**11**].

In Section 3 we give a general theorem on the isometries of $A^p(\Omega)$ (Ω a bounded Runge domain). In Sections 4 and 5 we give a complete characterization of the isometries of $A^p(\Omega)$ onto $A^p(\Omega)$ when Ω is the ball or polydisc in $\mathbf{C}^n$.

3. A general theorem. For a given bounded domain $\Omega \subset \mathbf{C}^n$, let Aut (Ω) denote the class of biholomorphic maps of Ω onto Ω, and, for

Received April 21, 1981 and in revised form October 2, 1981.

appropriate $r > K(\Omega)$, let $A^{p,r}(\Omega)$ denote the weighted Bergman spaces defined in Section 2.

THEOREM 1. *Let Ω be a bounded Runge domain, let $0 < p < \infty$, $p \neq 2$, and let $r > K(\Omega)$.*

(i) *If $T : A^{p,r}(\Omega) \to A^{p,r}(\Omega)$ is a linear isometry and if $T1$ is denoted by g, then there is a holomorphic map Φ taking Ω onto a dense subset of Ω such that,*

$$(3) \qquad (Tf)(z) = g(z) \cdot f(\Phi(z)) \quad (z \in \Omega)$$

for all $f \in A^{p,r}(\Omega)$; and, for every bounded Borel function h on Ω,

$$(4) \qquad \int_\Omega (h \circ \Phi)|g|^p D^{-r} dm = \int_\Omega h D^{-r} dm$$

where D is the diagonalized Bergman kernel, $D(z) = B_\Omega(z, z)$, $z \in \Omega$.

(ii) *Conversely, if Φ is a holomorphic map of Ω into Ω, and if $g \in A^{p,r}(\Omega)$ satisfies* (4) *for every continuous function h on Ω, then* (3) *defines an isometry of $A^{p,r}(\Omega)$.*

(iii) *If the linear isometry T is onto $A^{p,r}(\Omega)$, then $\Phi \in$* Aut (Ω). *Conversely, if $\Phi \in$* Aut (Ω) *and if $g \in A^{p,r}(\Omega)$ is related to Φ by* (4), *then* (3) *defines an isometry of $A^{p,r}(\Omega)$ onto $A^{p,r}(\Omega)$.*

Proof. In the proof of Theorem (2.1) in [**4**], replace the measure m_Ω by the measure $d\sigma_r = D^{-r}dm$. The measure $d\nu = |g|^p d\sigma_r$ clearly satisfies $\nu \ll \sigma_r$. On the other hand, $g \not\equiv 0$ is a holomorphic function in Ω so $g(z) \neq 0$, a.e. $[m]$. Moreover, $D(z) = B_\Omega(z, z) > 0$, for all $z \in \Omega$, so $g(z) \neq 0$, a.e. $[\sigma_r]$; consequently $\sigma_r \ll \nu$. The proof of Theorem (1) may now be completed by the argument given in [**4**, Theorem 2.1].

For a given isometry T, it would be desirable to have a more explicit description of the relationship between Φ and g than that given by (4). It is improbable that such a relation can be found as the situation is extremely complicated even when $r = 0$ (see [**4**]). Thus we shall consider only the isometries which are onto. Furthermore, because we shall need an explicit description of the group Aut (Ω), we shall restrict ourselves to the cases when our domain is a ball or a polydisc. The interested reader may wish to apply the methods given here to other domains Ω for which the group Aut (Ω) is known.

4. The ball as our domain. Because our proofs are rather technical, we shall now state our setting more explicitly.

Let $\mathbf{C}^n$ denote the vector space of all ordered n-tuples $z = (z_1, \ldots, z_n)$ of complex numbers, with inner product

$$(5) \qquad \langle z, w \rangle = z_1\bar{w}_1 + \ldots + z_n\bar{w}_n$$

https://doi.org/10.4153/CJM-1982-063-5 Published online by Cambridge University Press

and norm

(6) $\quad |z| = \langle z, z\rangle^{1/2}.$

The unit ball B of $\mathbf{C}^n$ is then the set of all $z \in \mathbf{C}^n$ with $|z| < 1$. Let m_B denote Lebesgue measure on B, normalized so that $m_B(B) = 1$. As is well known, the Bergman kernel for B is given by

(7) $\quad B(z, w) = [1 - \langle z, w\rangle]^{-(n+1)} \quad (z \in B, w \in B).$

In order to simplify our computations, we define, for each $x > -1$, the measure σ_x on B by

(8) $\quad d\sigma_x(z) = (1 - |z|^2)^x dm_B(z) \quad (z \in B).$

For $0 < p < \infty$, let $A^p(\sigma_x)$ denote the subspace of $L^p(\sigma_x)$ consisting of the holomorphic functions on B. We note that $A^p(\sigma_x)$ is the closure of the analytic polynomials. By (2), (7) and (8), we see that

(9) $\quad A^p(\sigma_x) = A^{p,r}(B)$

by setting $x = (n + 1)r$. We shall determine the isometries of $A^{p,r}(B)$ by describing the isometries of $A^p(\sigma_x)$. The following lemma follows easily from [**9**, Theorem 2.2.2].

LEMMA 2. *If* $\Phi \in \operatorname{Aut}(B)$ *with* $\Phi(a) = 0$, *then*

(10) $$1 - |\Phi(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle z, a\rangle|^2} \quad (z \in B)$$

and

(11) $$1 - \langle \Phi(z), \Phi(0)\rangle = \frac{1 - |a|^2}{1 - \langle z, a\rangle} \quad (z \in B).$$

We recall [**5**] that the Hilbert space $A^2(\sigma_x)$ possesses a reproducing kernel $C_x : B \times B \to \mathbf{C}$ given by

(12) $$C_x(z, w) = \binom{n + x}{n}[1 - \langle z, w\rangle]^{-(n+1+x)}$$

where the binomial coefficient is $\Gamma(n + 1 + x)/\Gamma(n + 1)\Gamma(x + 1)$.

Define the kernel $K_x : B \times B \to (0, \infty)$ by

(13) $$K_x(z, w) = \frac{|C_x(z, w)|^2}{C_x(w, w)}.$$

We recall [**4**], if $\Phi \in \operatorname{Aut}(B)$, then

(14) $\quad |J_\Phi(z)| = K_0(z, \Phi^{-1}(0)) \quad (z \in B)$

where J_Φ denotes the Jacobean of Φ. Letting $a = \Phi^{-1}(0)$, equating (12),

(13) and (14) shows

$$(15)\quad K_x(z, a) = \binom{n+x}{n} \left[\frac{1-|a|^2}{|1-\langle z, a\rangle|^2}\right]^x |J_\Phi(z)| \quad (z \in B).$$

LEMMA 3. *If* $\Phi \in \operatorname{Aut}(B)$ *with* $\Phi(a) = 0$, *then*

$$(16)\quad \int_B (f \circ \Phi) K_x(\cdot\,, a) d\sigma_x = \binom{n+x}{n} \int_B f d\sigma_x$$

for every bounded Borel function f *on* B, *and for all* $x > -1$.

Proof. If $\Phi \in \operatorname{Aut}(B)$ with $\Phi(a) = 0$, then (8) and (15) show

$$\begin{aligned}(17)\quad & \int_B (f \circ \Phi) K_x(\cdot\,, a) d\sigma_x \\ &= \binom{n+x}{n} (1-|a|^2)^x \int_B (f \circ \Phi)(z) \left[\frac{1-|z|^2}{|1-\langle z, a\rangle|^2}\right]^x |J_\Phi(z)| dm_B(z) \\ &= \binom{n+x}{n} (1-|a|^2)^x \int_B f(z) \left[\frac{1-|\Phi^{-1}(z)|^2}{|1-\langle \Phi^{-1}(z), a\rangle|^2}\right]^x dm_B(z) \\ &= \binom{n+x}{n} \int_B f(z)(1-|z|^2)^x dm_B(z).\end{aligned}$$

The last equality follows from Lemma (2) applied to Φ^{-1}, and the fact that $|\Phi(0)|^2 = |a|^2$. This completes the proof of Lemma (3).

THEOREM 4. *Let* $x > -1$ *and let* $0 < p < \infty$, $p \neq 2$.

(i) *If* T *is a linear isometry of* $A^p(\sigma_x)$ *onto* $A^p(\sigma_x)$, *then there is a* $\Phi \in \operatorname{Aut}(B)$ *such that*

$$(18)\quad (Tf)(z) = g(z) \cdot (f \circ \Phi)(z) \quad (z \in B)$$

for all $f \in A^p(\sigma_x)$. *Moreover*, g *is related to* Φ *by*

$$(19)\quad g(z) = \theta \left[\frac{1-|a|^2}{(1-\langle z, a\rangle)^2}\right]^{(n+1+x)/p} \quad (z \in B)$$

where $\theta \in \mathbf{C}$, $|\theta| = 1$ *and* $\Phi(a) = 0$.

(ii) *Conversely, if* $\Phi \in \operatorname{Aut}(B)$ *and if* g *is related to* Φ *by* (19), *then* (18) *defines an isometry of* $A^p(\sigma_x)$ *onto* $A^p(\sigma_x)$.

Proof. To see that (ii) holds, let $\Phi \in \operatorname{Aut}(B)$ and assume g is defined by (19). Then (12) and (13) imply

$$(20)\quad |g(z)|^p = \binom{n+x}{n}^{-1} K_x(z, a) \quad (z \in B)$$

where $\Phi(a) = 0$. Lemma (3) now shows that (4) holds for the pair (Φ, g). Thus (ii) follows from Theorem (1) part (iii).

The argument given in the proof of Theorem (3.2) of [**4**] may now be modified to prove (i).

5. The polydisc as our domain. As in Section 4, we shall determine the isometries of $A^{p,r}(U^n)$ by describing the isometries of $A^p(\sigma_X)$ for an appropriate measure σ_X defined on the polydisc U^n.

Let m_U denote normalized Lebesgue measure on the unit disc U in $\mathbf{C}^1$. For $1 \leqq i \leqq n$, let $x_i > -1$ and let σ_i be the measure on the disc U given by

$$(21) \quad d\sigma_i(\omega) = (1 - |\omega|^2)^{x_i} dm_U(\omega).$$

For a given choice of x_i with $x_i > -1$ $(1 \leqq i \leqq n)$, let $X = (x_1, \ldots, x_n)$, let $\sigma_X = \sigma_1 \times \ldots \times \sigma_n$ be the product measure on the polydisc U^n and, for $0 < p < \infty$, let $A^p(\sigma_X)$ be the subspace of $L^p(\sigma_X)$ consisting of the holomorphic functions on U^n.

Since the Bergman kernel B_n for U^n is given by

$$(22) \quad B_n(z, w) = \prod_{i=1}^{n} (1 - z_i\bar{w}_i)^{-2} \quad (z \in U^n, w \in U^n)$$

it is clear that

$$(23) \quad A^p(\sigma_X) = A^{p,r}(U^n)$$

by setting $x_1 = \ldots = x_n = 2r$. Thus our class of spaces $A^p(\sigma_X)$ properly contains the class of weighted Bergman spaces $A^{p,r}(U^n)$.

Given $X = (x_1, \ldots, x_n)$, define the kernels C_{x_i} and K_{x_i} on $U \times U$ by setting $n = 1$ in (12) and (13), respectively. If we define the kernel K_X on $U^n \times U^n$ by

$$(24) \quad K_X(z, w) = \prod_{i=1}^{n} K_{x_i}(z_i, w_i) = \prod_{i=1}^{n} (1 + x_i)\left[\frac{1 - w_i\bar{w}_i}{|1 - z_i\bar{w}_i|^2}\right]^{(2+x_i)}$$

then Lemma (3) shows:

LEMMA 5. *If* $\psi_i \in \mathrm{Aut}\,(U)$ *with* $\psi_i(a_i) = 0$ $(1 \leqq i \leqq n)$, *then*

$$(25) \quad \int_U (f \circ \psi_i) K_{x_i}(\cdot\,, a_i) d\sigma_i = (1 + x_i)\int_U f d\sigma_i$$

for every bounded Borel function f on U.

We recall [**7**], if $\Phi \in \mathrm{Aut}\,(U^n)$, then

$$(26) \quad \Phi(z_1, \ldots, z_n) = (\phi_1(z_{i_1}), \ldots, \phi_n(z_{i_n}))$$

where $\phi_1, \ldots, \phi_n$ are biholomorphic maps of U onto U, and $(i_1, \ldots, i_n)$ is a permutation of $(1, \ldots, n)$.

It follows easily from (26) and Lemma (5) that:

https://doi.org/10.4153/CJM-1982-063-5 Published online by Cambridge University Press

LEMMA 6. *If* $\Phi \in \operatorname{Aut}(U^n)$ *with* $\Phi(a) = 0$, *then*

$$(27) \quad \int_{U^n} (f \circ \Phi) K_X(\cdot, a) d\sigma_X = [(1 + x_1) \ldots (1 + x_n)] \cdot \int_{U^n} df\sigma_X$$

for every bounded Borel function f *on* U^n.

For a given fixed $X = (x_1, \ldots, x_n)$ and a given $\Phi \in \operatorname{Aut}(U^n)$ (described by (26)) we define the function $g = g_{X,\Phi}$ on U^n by

$$(28) \quad g(z_1, \ldots, z_n) = \theta[\psi_1(z_{i1}) \ldots \psi_n(z_{in})]^{1/p}$$

where

$$(29) \quad \psi_i(\omega) = \left[\frac{1 - a_i\bar{a}_i}{(1 - \bar{a}_i\omega)^2}\right]^{(2+x_i)} \quad (\omega \in U, 1 \leqq i \leqq n)$$

and $\theta \in \mathbf{C}$, $|\theta| = 1$, $\Phi(a_1, \ldots, a_n) = 0$.

By (24), (28) and (29)

$$(30) \quad |g|^p = K_X(\cdot, a)$$

with $a = (a_1, \ldots, a_n)$. Using (30) and Lemma (6), a slight modification of the proof of Theorem (4) shows:

THEOREM 6. *If one replaces* σ_x *by* σ_X, *and* B *by* U^n *in Theorem* (4), *then the resulting statement is true when* g *is related to* Φ (*and* X) *by* (28).

REFERENCES

1. R. R. Coifman and R. Rochberg, *Representation theorems for holomorphic and harmonic functions*, to appear.
2. F. Forelli, *A theorem on isometries and the application of it to the isometries of* $H^p(S)$ *for* $2 < p < \infty$, Can. J. Math. *25* (1973), 284–289.
3. C. Horowitz, *Zeros of functions in the Bergman spaces*, Duke Math. J. *41* (1974), 693–710.
4. C. Kolaski, *Isometries of Bergman spaces over bounded Runge domains*, Can. J. Math. *33* (1981), 1157–1164.
5. ——— *A new look at a theorem of Forelli and Rudin*, Indiana Univ. Math. J. *28* (1979), 495–499.
6. A. Koranyi and S. Vagi, *Isometries of* H^p *spaces of bounded symmetric domains*, Can. J. Math. *28* (1976), 334–340.
7. W. Rudin, *Function theory in polydiscs* (Benjamin, 1969).
8. ——— L^p*-isometries and equimeasurability*, Indiana Univ. Math. J. *25* (1976), 215–228.
9. ——— *Function theory in the unit ball of* $\mathbf{C}^n$ (Springer-Verlag, 1980).
10. R. Schneider, *Iosmetries of* $H^p(U^n)$, Can. J. Math. *25* (1973), 92–95.
11. M. H. Taibleson and G. Weiss, *The molecular characterization of certain Hardy spaces*, to appear.

The University of Texas at San Antonio,
San Antonio, Texas