CHAPTER 13

Concepts in Greek Mathematics

Reviel Netz

Just what are we studying, when we study concepts in Greek mathematics? I will end up this article arguing that the very purpose of engaging with mathematical concepts, in antiquity, could have been different from that we often assume for modern mathematics. Even concepts, even in mathematics, need to be historicized. Which means we should also look at different eras, differently. And so, most of this article will be structured by the four distinct eras of Greek mathematics: classical, Hellenistic, Imperial and Late Ancient.

Concepts, mathematics, four eras... large topics, perhaps too large for 10,000 words. It helps that I have written on some of this before – though, admittedly, in a mostly negative vein. Readers might expect to find in this chapter a discussion of questions such as 'what were the key Greek mathematical concepts' or 'how did Greek mathematical concepts evolve'. I do not discuss such questions because I believe they are wrongly put. Let me begin by briefly stating this negative position, summing up the argument in Netz (2002), (2017) and (1999).

There is a historiographical tradition, where different cultures are characterized by the different concepts they possess. Thus, for instance, it might be argued that the Greeks did not possess the concept of the real number, or of actual infinity, etc. So, it is then argued, *because* the Greeks did not possess such concepts, they also did not develop the mathematical theories that depend on them such as, for instance, the algebraic treatment of continuous magnitudes (dependent on real numbers) or set theory (depending on infinity). The historiography of *Being in Possession of a Concept* relies on an assumption of strict impossibility (if you argue that the Greeks did not develop set theory because they did not have the concept of actual infinity, then your argument is that such a development of the calculus would have been *impossible*: see the claim developed more fully in Netz 2002). This historiography is thus easily refuted. The historical pattern we find is not that a certain culture completely avoids a

https://doi.org/10.1017/9781009369596.016 Published online by Cambridge University Press

certain intellectual path. Instead, the differences have to do with the frequency of what is standardly done. Greeks did not pursue the algebraic treatment of continuous magnitudes and indeed generally did not consider geometrical magnitudes as amenable to arithmetical manipulations – but, here and there, they did. They definitely did not invent set theory – but in one occasion we find Archimedes himself engaging quite directly with attaching a quantitative value to an actual infinity (this case was discovered recently, in the Archimedes Palimpsest; a cautionary tale for historical narratives based on strict impossibility).[1] What we find, historically, is not a matter of conceptual possibility and impossibility – this is always done, this is never done – instead, we find the entrenchment of certain practices. One does not necessarily completely avoid this or that path; one simply is used to taking the one, and so usually one avoids the other, and for reasons which are best understood in concrete historical, social terms.

To delve very quickly into the epistemology underlying all of this: the arguments adduced for the existence of 'concepts' – especially in phenomenology – have to do with entities present to the mind. There is a good argument why, when discussing 'number', an individual must have access to a mental construct such as the concept 'number'. However, this is, by definition, a private construct, and inter-subjective discourse does not require that all its participants rely on the same construct. What is required, for the sake of inter-subjective discourse, is the continuity of practices shared by the members of the community. An intersubjective language does not assume a fully shared vocabulary. Different Greek mathematicians surely had different concepts more or less beneath the threshold of communal visibility: we know this because, when, occasionally, conceptual discussions emerge into the surface, we do not find a consensus. However, what Greeks did share was the practice of producing works in the Greek mathematical genre.

I suggest, then, we need to study not the underlying concepts as such (which might differ from individual to individual and which are less explanatory) and instead focus on the shared practices of conceptual work. Netz 1999 is a description of the shared practice of Greek mathematical writing. Here is what I argued regarding conceptual work. To begin with: that in many cases, Greek mathematicians just do not seem to be interested in *definition*. It is useful to start from a point made by David Fowler (1999: 222): those neat sequences of numbered definitions at the beginning of Greek mathematical texts, were in fact created by modern editors.

[1] Netz, Saito and Tchernetska 2002–3.

The ancients did not separate their introductions into neat sections (let alone number them). Instead, such introductions were, originally, discursive paragraphs. 'A point is that which has no part, and a line is breadth-less length; and the ends of lines are points'. They were part of a more general phenomenon, the more discursive introduction to the more technical text made of claims and their proofs. Once this is understood it becomes clear that an ancient mathematical author did not necessarily always put much effort into the precision of any given definition, and one understands much better the place of seemingly empty definitions ('a line is breadth-less length'), mathematically useless definitions ('a point is that which has no parts') and passages within definitions that do not define ('the ends of lines are points'). It also becomes easier to understand the important concepts that Greek mathematicians simply use without a consistent definition (for instance, tangency). But it is also intriguing, in and of itself, that Greek mathematicians do not discuss concepts separately but, instead, discourse through them as a collection of interrelated entities.

Indeed, I argued that the terminology of Greek mathematics is simply not best understood in terms of individually defined terms but in terms of a system in combined operation. Thus, the terms chosen for 'point' or 'line', separately, as well as their definition, do not matter for the practice of Greek mathematics, where both terms are most often elided. What matters, instead, is that 'point', *sēmeion*, is a neuter, while 'line', *grammē*, is a feminine. This defines a system: the neuter article followed by a diagrammatic label, alone, stands for a point; the feminine article followed by a diagrammatic label, alone, stands for a line. This principle may be generalized much further. What mattered, through the practice of Greek mathematics, was not the terms occurring in the introductory, discursive passages, but a set of expressions used throughout the corpus. Greek mathematics indeed relied on a highly regimented use of language, but the effect of regimentation was achieved not by the defined words (which anyway formed only a subset of the words used in practice) but by the reliance on a system of formulaic expressions, often several words long. Those formulaic expressions could become nested within each other and much of the work of going through a deductive proof involved the substitution of formulaic elements for each other (thus, the Greeks did verbally and formulaically, roughly, what is done in school algebra through the substitution of symbols). A formulaic expression – such as (to take the key example of proportion) '. . . is to . . . as . . . is to . . .' is not defined and is not most naturally understood as a concept; it is best to think of it as an entire practice of language, the set of used expressions nested within each other and used in reciprocal relations. Not individual words – but a system of language-use.

1 Discontinuities

Greek mathematics is not a single thing. It is always written largely within the confines of a single genre (a fact which makes possible the project of Netz 1999). The stability of genre is a feature of Greek literature as a whole (in other fields, too, genre is least susceptible to change). But underlying this stability there are important discontinuities.

Those are, precisely, discontinuities. Greek mathematics was not a seamless whole because it was not continuously pursued. While perhaps there was no time in which literally not a single individual engaged at all with mathematics, yet it is significant that most of the legacy of Greek mathematics was created in a few bursts of activity. Mathematics was concentrated in a few generational events. (Once again, this tendency of cultural practice to be concentrated in generational bursts is typical to much of Greek culture more generally and can be attributed to the lack of institutional continuity). Two of those, relatively early on, are especially important. It seems that much of the early achievement of Greek mathematics is due to a single network of mathematicians associated with names such as Archytas, Theaetetus and Eudoxus, active in the first half of the fourth century BCE (the work of this generation constitutes, among other things, much of what later gets codified in Euclid's *Elements*). Later, the great bulk of advanced Greek mathematics was created by another network of mathematicians associated with names like Archimedes and Apollonius, active late in the third century BCE (a considerable fraction of the work of this generation is still extant).

Less clearly identified in terms of generations, we can note two further clusters of mathematical practice, each with their own distinctive features. In the Imperial era, mathematical authors often seek a wider audience and perhaps the status of instructors to the elite. Mathematical authors (just like their counterparts elsewhere in Greek imperial culture) act as impresarios of Hellenism, summing up the achievement of the past in more or less eclectic fashion and providing their own stamp. This is most evident in the work of Nicomachus, but one can infer the same for Hero and Ptolemy: all extant in bulk, and all likely active during the High Imperial Era. In Late Antiquity, mathematics is mostly produced in the context of commentary and epitome (as indeed is true of Greek literary life as a whole). Almost all authors now share a broad philosophical perspective – one defined by the central presence of Plato and Aristotle – and their main identity is, indeed, often that of philosophers. Whether philosophers or not, they are all, primarily, teachers; this is why they write commentary. (Pappus, early in this era, is still decisively a mathematician and not a philosopher; Proclus is most definitely a philosopher who occasionally dabbles in mathematical commentary).

Conveniently enough, then, we identify four eras, which we can label simply as 'Classical', 'Hellenistic', 'Imperial' and 'Late Ancient'. By far the most important one, mathematically speaking, is the Hellenistic. Even though Netz 1999 did study texts from across the corpus, its central model always did remain the works of the Hellenistic era. And so one of the central thrusts of that book – to make mathematics autonomous from philosophy – was, to a large extent, an unwarranted generalization from the Hellenistic era to mathematics as a whole. This, of course, has important consequences for the question of concepts in Greek mathematics.

In the following sections I will move, gradually, from the least philosophical to the more philosophical of the mathematical eras, adding qualifications to my main thesis. I start from the Hellenistic era and then move, in conceptual rather than chronological order, to the Imperial Era; then to Late Antiquity; and finally circle back to the classical era and to Archytas, Theaetetus and Eudoxus: all authors who, significantly, communicated with Plato. We will find, finally, an interest in conceptual structure. But even that, surprisingly, will have to do less with an interest in isolated definition as such, as it will have to do with entire conceptual structures, akin to musical harmony.

2 Hellenism without Concepts

Here are the first words of Archimedes' *On Floating Bodies*:

> Let the liquid be assumed having such a nature so that the less pressed of its parts – being set equally and being contiguous – is pushed out by the more pressed, and that, further, each of its parts is pressed by the liquid above it which is along a perpendicular – unless the liquid be let into some and pressed by some other.

This is undoubtedly opaque. What is worse, this, in fact, sums up the entirety of the discursive introduction to the treatise. This introduction is followed *immediately* by a series of proofs.

The proofs themselves clear up this postulate, by putting it to practical application. It becomes clear, for instance, that 'set equally' means 'being equidistant from the center of the earth'. The very purpose of this postulate, it turns out, is to determine a condition of stability: if the equipressure at equidistance is not maintained, liquids will not be stable. We know this because the postulate is used in arguments where it is assumed that the liquid is stable; it then follows that it forms a sphere around the center of the earth and, furthermore, we can infer the position a lighter solid, immersed in a liquid, will take, on the assumption that the liquid remains stable because this condition of equipressure at equidistance must be satisfied. 'Equipressure' itself remains entirely vague but in practice it means the area of the sector of a planar section of the liquid, extending above the arc for which pressure is considered. With this opaque set of unwritten assumptions, Archimedes is capable of proving the Law of Buoyancy.

Still maddeningly complicated! But what if I tell you that, as the text moves into the second book of the treatise (where certain complicated conditions of stability are studied), an extra simplification is implicitly assumed. Instead of dealing, as was discussed in the first part of the treatise, with a spherical surface of the liquid, it is simply taken that the surface of the liquid will be flat. This perhaps can be explained as a computational simplification (it is technically crucial for the geometrical considerations of this part of the book). But it is not presented as such, instead it is simply taken on board through the process of the proofs.

On Floating Bodies is a work of rare genius, one which is relevant for our purposes in two ways. First, part of what makes it so impressive is the speed with which Archimedes obtains incredible results – not just the Law of Buoyancy but also the extremely intricate details of the conditions of stability found in the second part. It is a reasonable reading of the text – certainly, if we bring it in the context of Archimedes' project as a whole – that Archimedes *aimed* for this effect of powerful results based on minimal tools, hence the bare postulate, with its zero conceptual clarification. Doing zero work on one's conceptual foundations was quite likely, then, intentional. Second, it seems likely that there was no antecedent at all to this treatise by Archimedes. We can probably rule out the possibility that Archimedes relied on previous conceptual clarification of the terms of the argument.

This last point, especially, makes this into a useful test case and if so, it is decisive: Archimedes might have cared about conceptual clarity – but he probably cared for other epistemic values (such as intellectual surprise) even more. In general, there is an inherent tension between surprise and explicitness and to the extent that a mathematical culture values surprise,

https://doi.org/10.1017/9781009369596.016 Published online by Cambridge University Press

this might mute its commitment to explicitness, including its explicitness concerning, even, axiomatic and terminological foundations. This is clearly the case in some works by Archimedes.

The above from *On Floating Bodies* is but a single example. We can point to other similar cases where Archimedes almost entirely neglects conceptual clarity (this is especially easy to show with Archimedes' more original work, for instance the study of the balance, or the many anticipations of the calculus and Archimedes' treatment of infinity). There are also counterexamples. So, for instance, in another work by Archimedes – the closest, perhaps, to an 'elementary' work by him, namely the first book on the *Sphere and the Cylinder* – the introduction includes an extended discussion of the concept of concavity. This is cleverly defined: a line is 'concave in the same direction'

> in which, if any two points whatever being taken, the straight <lines> between the <two> points either all fall on the same side of the line, or some fall on the same side, and some on the line itself, but none on the other side.

There are two indications that this is motivated, to some extent, by a concern with conceptual clarification as such. First, the definition is interestingly non-constructive and so not entirely applicable. It can be used to prove that a line is not concave in the same direction, and it can draw conclusions from the assumption that a given line is so concave; but it cannot be used to prove that a certain line is, in fact, concave in the same direction. Second, and relatedly, the definition is not in fact invoked in the treatise. Archimedes goes on to postulate that when two lines are concave in the same direction and one is contained by the other, the container is greater than the contained. Based on this, through the development of the treatise, Archimedes will use the postulate to make the judgement that this line is greater than the other, but the property of 'concavity in the same direction' will simply be assumed and will never be verified with a reference to the definition. Thus, all the definition does is to provide conceptual clarity to the statement of the postulate (the kind of clarity that Archimedes did not bother to provide in *Floating Bodies*). Once again, this may have to do with the intended, much more measured 'pace' of *Sphere and Cylinder*, Book 1. Archimedes could engage in conceptual clarification; most usually, he was less interested in this endeavour.

In general, it is rare for Hellenistic mathematicians to engage significantly with definitions. But another exception to that comes from one of the most important mathematical treatises ever written: Apollonius'

Conics. There, through the first book, a set of definitions is offered, among other things, for the three main types of conic sections. We now call them, following Apollonius, *parabola*, *hyperbola* and *ellipse*. It is almost certain that these are Apollonius' original names. Previously, these were known, respectively, as *the cut of the right-angled cone*, *the cut of the obtuse-angled cone*, and *the cut of the acute-angled cone*. The original names assume that all conic sections emerge from isosceles cones. That is: imagine an isosceles cone, being cut by a plane perpendicular to one of the sides – and you can see how the three cuts are produced (clearly, for instance, if the isosceles cone has an acute angle at its vertex, the cut perpendicular to the side will result in an ellipse). In fact, conic sections can be produced by the cut of any cone, not necessarily an isosceles one. Apollonius proceeds to show this and, perhaps related to this generalization, he also renames the sections. So much is clear. It is far less clear, however, that any of this should be seen as an engagement with conceptual foundations. The new names chosen are significant . Those take up certain quantitative properties associated with the sections. As Fried and Unguru point out (2001: 81), however, this 'does not prove that Apollonius sees in [these properties] the 'whatness' of the conic sections'. The properties are explicitly referred to as *sumptōmata*, which may well indicate properties as opposed to essences. Indeed, as Fried and Unguru document, it seems clear that Apollonius must have conceived of the conic sections as more essentially geometrical, that is, if anything, their 'essence' must be derived from the manner of their construction. The terminological move made by Apollonius was therefore in a direction *away* from a definition seeking essences. It seems likely that he redefined the conic sections simply because of the interesting geometrical observations one then gains concerning the cuts of cones. This made the previous terminology defunct and a new set of names was necessary, which was then supplied almost as an afterthought. Once again, then, this was a terminological move, with the motivation being, ultimately, internal to the geometrical exercise itself. In Archimedes' *Floating Bodies*, as in Apollonius' *Conics*, we see Greek mathematicians fully capable of conceptual analysis but more interested in the detail of geometrical proof. The goal was to create a dazzling edifice of proof, not to dig for foundations.

In Netz 2009, I argued that the main era of Hellenistic mathematics should be seen against the background of a Hellenistic culture centered on Alexandria, where the main cultural value was that of the surprising dialogue between distinct genres. Mathematics, for instance, can be reaching towards poetry (one can adduce many examples, such as Archimedes writing the *Cattle Problem* in verse form, or Eratosthenes recording his *Mesolabion* in an epigram). What mathematics did not engage with so much was philosophy. (Why this is the case is a central question discussed in Netz 2020: part 2; where I describe the overall bifurcation of Mediterranean culture in the third century BCE into the two poles of Athens, more philosophical, and Alexandria, more literary – as well as more scientific). The claim that Greek mathematics did not engage directly with philosophy was made most powerfully by Knorr in a series of studies culminating in Knorr 1986. I would like now to argue that this is especially true for Hellenistic mathematics (which is where most of our evidence comes from and so could be easily generalized by Knorr to cover all of antiquity). A non-philosophical mathematics, quite naturally, was also one that only rarely engaged with concepts.

3 Teaching and Definitions

Discussing Hellenistic mathematicians, so far I have hardly mentioned Euclid. He was likely somewhat earlier than Archimedes and Apollonius, and we should thus understand him separately. The massive extant corpus transmitted under his name (likely, some of it not by Euclid himself) does an injustice to this mathematician in that it is made almost entirely of Euclid's less original works where it seems that his main aim was to compile results obtained by those before him. This is true particularly as regards the *Elements*. Indeed, it is crucial to understand the significance of the *Elements* and the very meaning of the term. This work is not some kind of ancient Zermelo Fraenkel or ancient Bourbaki, an attempt to codify the foundations of the discipline of mathematics. The meaning of 'Elements' instead is that of useful, preliminary tools. In the making of geometry, one frequently needs to assume certain basic results: congruence theorems, relations between arcs and angles, rules of proportion. The *Elements* simply surveys those basic tools in logical order. (This interpretation of the purpose of the *Elements* – now a scholarly consensus – is fundamentally due to Saito 1985). It is hard to know precisely why Euclid sat down to produce this collection. His corpus includes several similar works, some providing basic tools for other disciplines such as optics or music. (But how many of those are by the same author?) Perhaps there were previous compilations of this kind (Proclus positively asserts so[2]), perhaps not (Proclus does not seem to know anything of substance about any such

[2] *Commentary to Euclid's Elements* 1. 66.7–8; 66.20–2; 67.14–5; 66.22–3.

https://doi.org/10.1017/9781009369596.016 Published online by Cambridge University Press

previous compilation). At any rate, Euclid's project can be put alongside other mapping and surveying projects of the early third century Ptolemaic court: mapping of the stars, of the earth and of the human body; perhaps closest in spirit to Euclid's *Elements* are Callimachus' *Pinakes*, a catalogue of past literature. I suggest that Euclid's *Elements* was in the same vein – a catalogue rather than a study in foundations. With great caution, it can be used for the study of fourth-century mathematics (for which see below).

While it was perhaps intended as a catalogue, the *Elements* ended up being put to other uses as well. Advanced mathematics is almost entirely absent from the evidence of the papyri, and yet there are many dozens of papyrus fragments of a more elementary character. Those are often written and produced in an informal hand that suggests an origin in the classroom. Indeed, there are many hundreds of other papyri clearly originating in the classroom, dedicated, however, to education in literacy (tracing the alphabet, learning verses from Homer, etc.) Literacy was the mainstay of ancient Greek education, while numeracy came as a very distant second. Most pupils were probably provided no more than a very brief glance into some counting and calculation.

It is in this context that we need to understand the handful of papyri of Euclid's *Elements*. It seems that already in the Hellenistic era, but surely by the Imperial era, some teachers used Euclid's *Elements* in their elementary teaching of mathematics. What they did, however, was a significant simplification: the text was stripped of proofs, pupils instead asked to learn some of the mere statements of the *Elements* (this, indeed, is like learning verses from Homer). It is perhaps not surprising, then, that one of the papyrus fragments is precisely a list of definitions from *Elements*, Book 1 (*PMich.* Inv. 925 verso). We can easily imagine a schoolmaster, reciting the definitions and demanding that the pupils learn them by heart: a grammarian-like teaching of mathematics in the ancient, grammar-based school. (For all of this concerning ancient mathematical education, see Sidoli 2015).

This might be a relevant context for another Imperial-era work titled, simply '*Definitions*'. Our manuscripts ascribe this text to Hero of Alexandria, surely an Imperial era author, whose authorship I will take seriously. The transmitted text, however, is clearly contaminated by later additions. This very contamination is significant and points to the work's origins in the active practice of the classroom (where schoolmasters add and subtract from their texts). The very brief introduction by Hero puts this work, firmly, in an educational context – it is described as an introduction to the learning of geometry – and the rest of the treatise simply lists mathematical definitions with brief comments. (So, for instance, having quoted Euclid's definition of the point in section 1, Hero adds a few comments, of which the most extended is the distinction between the geometrical point, and the arithmetical unit). One repeated theme is that of taxonomy – thus, following the definition of a line in section 2, section 3 details the kinds of lines. This is a kitchen-sink kind of taxonomy: 'Among lines, some are straight, some are not, and of those not straight, some are called circular circumferences, some spiral-shaped, some curved'.

Education, in antiquity, was understood as introduction: the schoolmaster introduced you to the literary classics; the teacher of a specific field – to some key works in that discipline. Introductions, however brief, often acted as the treatise's own educational passageway to its reading (Mansfeld 1994) and in many technical works – including those of the exact science – we find that they begin with a statement of the scope of the discipline, in a sense its definition. For instance Ptolemy's *Geography*: 'World cartography is an imitation through drawing of the entire known part of the world together with the things that are, broadly speaking, connecting with it.' [followed immediately – as if so often the case – with a brief taxonomy:] 'It differs from regional cartography in that regional cartography, as an independent discipline, sets out the individual localities. . . while the essence of a world cartography is to show the known world as a single and continuous entity. . .'. Such introductions may be found often in various handbooks and they should be put alongside the more purely terminological discussions of works such as Hero's *Definitions.* Broadly speaking, scientific teaching is partly a meta-scientific exercise. It is natural to spend some time discussing not just the contents of the science but also its overall scope, terms, etc. While such discussions are often relatively superficial from the point of view of a professional philosopher, they are important historically, as they build up a body of writing dedicated to terminological and conceptual issues. This body was formed, then, in various educational contexts, through the Hellenistic and Roman eras.

4 Concepts in Commentaries

Technical writing in Late Antiquity tends to be second-order: it is writing, based on previous writings, in the form of compilation, epitome and commentary. The preceding discussion, concerning conceptual discussion and the ancient classroom, comes in handy. From Late Antiquity and onwards into the Middle Ages, the central model of a scholar in the field X, comes to be the *teacher* of the field X. Philosophers become teachers of philosophy (hence they write commentaries on the major philosophers), and mathematicians – teachers of mathematics (hence they compile and present past mathematics – Pappus' main project – or produce commentaries to Ptolemy – Theon's project – or to Archimedes and Apollonius' – Eutocius' project).

Commentaries, generally speaking, operate on the principle of lemma-and-commentary, going through the text in sequence: they will thus naturally linger on at least a few definitions. Further, even commentaries have their own introductory passages and these, too, might carry conceptual import. So – to pick an example – we have extant Eutocius' commentary to Archimedes' *On Balancing Planes*. (That this is by Eutocius is based on the name of author found in the manuscripts. In general I find this work of lesser quality than Eutocius' other commentaries and so I wonder if this may not be, for instance, from one of his students instead, or perhaps the words of the master, badly compiled; not much depends on this and I will refer to the author here, deferring to tradition, as 'Eutocius').

While not at the same level of original tour de force as *Floating Bodies*, Archimedes does use, in *Balancing Planes*, many undefined terms. The most essential is that of the center of the weight (in Archimedes' Doric, *kentron tou bareos*). Eutocius (or perhaps ps.-Eutocius?) approaches this term indirectly. First, he discusses the term *rhopē*, (hard to translate: 'pull-down', perhaps), which is not directly used by Archimedes (although Archimedes does use the verb *isorropein*, 'to be pulled-down equally' or 'to balance'). This discussion of *rhopē* is made of very brief quotations from Plato, Aristotle and Ptolemy. Following that, Eutocius makes an important contribution which could possibly be original. He asserts that (Heiberg 1915: 264.10–3): 'Archimedes, in this book, calls the center of the *rhopē* of a plane figure <the point>, hung from which, it < = the plane figure> remains parallel to the horizon'. To this Eutocius adds the detail on how to expand this to multiple figures and then gives an example. Following that, he notes that Geminus observed correctly that Archimedes calls axioms 'postulates' and then adds that the claims postulated by Archimedes (e.g., that equal weights balance at equal distances) are clear enough (hence, Eutocius skips commentary to all but one, which happens to be *geometrically* more complicated).

Here we see Eutocius engaging in some conceptual clarification, though of a very modest kind. He mostly relies on citing previous authors; he ignores many crucial concepts while, at the same time, noting relatively

https://doi.org/10.1017/9781009369596.016 Published online by Cambridge University Press

trivial distinctions of sheer nomenclature, such as 'axiom' and 'postulate'. And what he does assert of significance – the definition of center of the weight – is strange at several levels. It does not, in fact, employ Archimedes' actual language (is Eutocius cribbing from somewhere else, independent of Archimedes?). It is presented without any motivation or argument. And it is in some obvious ways unsatisfactory. Once again – just as we saw with the conic sections – the stability of a plane figure when hung from its center of the weight is an important property of the center of the weight but it is at the very least questionable whether this should be considered its 'essence'. Not that such a definition is easy! From a mathematical point of view, the center of the weight is perhaps best understood as the mean position of all the points in a geometrical magnitude, which is very hard to express in Greek terms; one can argue that Archimedes' proofs add up to an implicit, geometric definition. Which only reminds us: there was a lot of ground that Eutocius could have covered here, of great conceptual significance. This is not what Eutocius seems to do.

Proclus, in his commentary to Euclid's *Elements* Book 1, is a much more thorough commentator and also a much more professional philosopher. Unlike Eutocius, he does go through the definitions by order and comments on them, sometimes quite extensively. Here, indeed, we find several serious engagements with the question of conceptual foundations. The manner in which Proclus gets there is interesting as well. When discussing 'lines', for instance, Proclus quotes Euclid's definition, and then adds on two others (without attribution): 'the flowing of a point', 'magnitude extended in one direction'. When discussing 'plane angle', Proclus observes that Euclid's definition takes this to be a relation, whereas others (unnamed) put it under quality, others yet, under quantity. The ensuing conceptual discussion then aims to reconcile the various approaches. This is, indeed, philosophical, conceptual analysis of real value. So, for instance: Proclus simply accepts the definition of 'magnitude extended in one direction' which is perhaps indeed not different at all from Euclid's 'breadthless length'; and points out that 'the flowing of a point' is merely a generative cause, not the essence of the line. There are a number of such passages in Proclus' commentary, but it should be stressed that they are not many. What is perhaps significant is that some of those discussions – following upon Euclid's lead – engage with the manner in which definitions inter-relate: so, that of a line with that of a point. But even so, it should be made clear: most definitions are followed simply by a commentary that notes various historical facts concerning them. For instance, following the definition of the diameter of the circle, there comes the famous historical note that Thales was the first to prove that the diameter bisects the circle; following Euclid's taxonomy of rectilinear figures, there comes a notice of Posidonius' alternative taxonomy. This is already a little reminiscent of Hero's *Definitions*, where the passage concerning the taxonomy of lines simply includes a somewhat unstructured list of types produced in the literature. The most basic operation of the commentator, the maker-of-books-out-of-past-books, is to consult the authorities and to cite relevant facts from them. It seems that even Proclus' richer conceptual discussions should be understood, in part, along such lines.

Still, there are few parallels in the extant literature to the depth of Proclus' conceptual engagement. It is perhaps useful to put Proclus' treatise next to another work from the same philosophical school, *The Handbook of 'The Arithmetical Introduction'* by Domninus of Larissa. This is a very minimal rendering of the contents of Nicomachus' book (the original *The Arithmetical Introduction*) and it is often made of definitions, albeit interspersed with examples (so, 1.2–3: 'some numbers can be divided into two equal parts, such as 4 and 6, other numbers do not allow for this, like 3, 5, 7, 9... Those numbers which can be divided into two equal parts are called even, while those which do not allow for this are called odd.') Such treatises are not significantly different from Hero's *Definitions* and what we see here is simply that, in some cases, the teaching of mathematics moved, in Late Antiquity, into the philosophical school, without changing much in its pedagogy. The typical thing about such surveys is that they become, primarily, a taxonomy of a certain field: the kinds of quadrilaterals, or the kinds of numbers, presented in logical order.

Some teaching of mathematics, of course, was pursued (at least relatively early on) by authors who identified themselves more closely as mathematicians and not as philosophers. Of these, Pappus is our most important witness. Indeed, certainly as compared to previous extant works in Greek mathematics, the work of Pappus is very rich in second-order, meta-mathematical discussions. It is thus noteworthy how little those discussions involve conceptual issues.

Pappus' major work, the *Collection*, is made of eight discrete books (of which the first, as well as the beginning of the second, are entirely lost). The third book is a critique of a proposed solution to a famous problem, which brings Pappus to discuss in general terms the question of admissible solutions to mathematical problems. Book Four is a compilation of mathematical tour de forces, but as Sefrin-Weis (2010) shows, it is structured according to the principle of ascending order of difficulty, implying a ladder of problems; the book ultimately discusses, explicitly, the classification of problems according to type. (See further Cuomo 2000: 186 ff.) Book 7 is a collection of studies related to geometrical analysis and so famously includes a brief passage on the meaning of the term 'analysis' (Pappus VII, Hultsch 1877: 634–36). Finally, in Book 8 – a collection of results in mathematical mechanics – Pappus has an explicit priamel concerning his refusal to discuss concepts (Hultsch 1878: 1030.1–6): he will not discuss the meaning of the heavy or the light, why things move up and down, and what up and down even mean – because, he says, Ptolemy already discussed it (a discussion which now appears to be lost). Instead, Pappus says, he will discuss the center of the weight. This is the closest we come in Pappus to a conceptual discussion and what we find is an account close to Eutocius' (Ibid.: 1030.11–13): the center of the weight of a body is 'a point internal to it so that, hung in thought from that point, the weight remains stable and keeps its initial position, and does not turn over as it moves'. There is some thought put into this account. (for instance, the clause 'does not turn over as it moves' refers, I believe, to a case such as a plane, oriented vertically, so that, at that position, it will remain stable hung from any of its points – but will not remain stable if moved, unless it is hung, in fact, from its center of the weight). It is in fact sufficiently more expansive than Eutocius' account to make me suspect that the two, indeed, could have been independent. It was an obvious lacuna in Archimedes' treatment, so that any future commentator would naturally be led to offer a conceptual account. But at any rate, the more fundamental point is that this is, in fact, as close to conceptual as either Eutocius or Pappus ever come. Even in this discursive, meta-mathematical moment of commentary, ancient mathematicians do not engage extensively with conceptual clarification.

The reason is clear enough. Whatever else it is, the commentary tradition is immersive in tradition. Constructing one's attitude to the past is the very point of the exercise, as emphasized by Cuomo 2000; everything is based on the reference to past texts. There was a fair amount to work with – grammarian-like accumulation of lists of definitions, some philosophical discussions of concepts, even a few hints in the extant corpus of Hellenistic mathematics – but largely speaking, the corpus of mathematics available to Late Antiquity was not oriented towards conceptual discussions and so, even now, mathematics remains focused on the first-order act of individual proofs. What we do see with meta-mathematical engagement, again, is more closely attuned to the reality of mathematical practice: we find a discussion of methods – how to solve problems? What is analysis? – and above all an interest in the classification of tools. This emphasis on classification is significant: as noted above, even the brief quotation from Domninus, for instance, is not really about the definition of 'odd' and 'even' in isolation but instead a classification of types of number, an element within a taxonomy. In all of this, we go back quite simply to the original works: already in Euclid, the discursive passages setting out the terms of geometry and arithmetic are often arranged not in the terms of isolated concepts, standing on their own, but instead in the terms of a system based on a classification and an internal order ('a point is that which has no part; line is breadthless length; and the ends of line are points'). Let us bear this point in mind as we peer – more speculatively – into the earliest, and most important, period of conceptual discussion in Greek mathematics: the fourth century.

5 Mathematics alongside Philosophy?

I suggested that we think of early Greek mathematics in terms of two networks: the first active mostly early in the fourth century BCE; the second active mostly late in the third. Our knowledge of the two is not alike. We have a very significant extant corpus of the work made by the later generation but only a few fragments from the earlier (which we may supplement by the evidence that Euclid's compilation presents for the kind of mathematics likely developed in previous generations). I suspect that this asymmetry goes back to Late Antiquity: authors such as Pappus had much more access to third-century works – which they cite extensively – and no longer to fourth-century works – of which their knowledge is mediated.

In one late, tantalizing hint Pappus' commentary to Euclid's *Elements* 10 states (Thomson 1930: 63) that 'it was … Theaetetus who distinguished the powers which are commensurable in length from those which are incommensurable, and who divided the more generally known irrational lines according to the different means, assigning the medial lines to geometry, the binomial to arithmetic, and the apotome to harmony, as is stated by Eudemus, the Peripatetic'. The reference to Eudemus – Aristotle's pupil and author of a history of geometry, and therefore in a position to know but also, not the kind of author whose fame attracted apocryphal writings – makes it likely that there is substance to this account. This is tantalizing because, indeed, the extant book X involves an entire system of classification and definitions of types of irrationals. Furthermore, and what makes this particularly intriguing from our point of view, is that Plato, in all likelihood, obliquely referred to an achievement of this type in the *Theaetetus*. There, famously (147d–148b), in response to Socrates' posing a *ti estin* question, Theaetetus brings up an example that just came up in the classroom. He recounts how Theodorus the teacher provided isolated proofs that certain lines were incommensurable, whereupon Theaetetus and (he modestly adds) his classmate both, together, generalized the results with the aid of bringing in more general concepts, the proof then provided at the appropriate level of generality.

It may be that Plato is simply inserting here his own *ti estin* concerns, but it is also possible that a hallmark of Theaetetus' studies into irrationality was the introduction of new concepts, with proofs concerning those. And indeed, the construction of an entire system of concepts is at the heart of the extant Book 10 of the *Elements*.

The quotation from Pappus associates Theaetetus' types of irrationals with types of means. This may be anachronistic, but there is plausible enough evidence that Archytas engaged with just such a classification (fr. 2: Huffman 2005: 162–81). The texts themselves clearly suggest, in this case, that Archytas' contribution was precisely in conceptual arrangement. Prior to him, the word 'mean' was already used in different senses, referring sometimes to arithmetic, sometime to geometric, sometime to what was later called 'harmonic' mean. It appears that Archytas' contribution was to analyze the various uses, add a few more, and define and name the lot. If so, this will be a fairly clear case of conceptual analysis – aimed at constructing a system.

Finally, let us bring a scholion to Euclid's *Elements* 5, stating that the results of this book are due to Eudoxus[3]. This makes Eudoxus the author of the general proportion theory extant through Euclid's *Elements*. And here, for once, we come upon a study in *foundations*. Book 5 defines proportion; there is no doubt that previous mathematicians have relied on its results, previously, by taking them for granted. If so, the goal is to take established results and to find their proper foundations – finally, a Bourbaki! (Also, this is very close to Archytas, redefining already established means). The key definition (usually given as number 5) is astonishing:

> magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

[3] Euclid's *Elements*, Scholia, Heiberg 1886: 280.

This is so abstract that it can only be legible to a modern reader with the aid of notation. The claim is that for four magnitudes, a, b, c and d, they are in proportion

a:b::c:d
if and only if for any integers m, n
$ma >=< nb \longleftrightarrow mc >=< nd$

As often noted, this is essentially equivalent to Dedekind's definition of real numbers. It may be that Eudoxus' goal was to find a definition that accounted for rational as well as irrational ratios. It seems likely enough that Aristotle, in a key passage, had this very definition in mind – and if so, took it slightly differently (*An. post.* 1.5 74a17–23):

> This – that if terms are in proportion, they also alternate – could apply to things inasmuch as they are numbers, or lines, or solids, or times – as it was used to be proved in the past, separately for each, while being quite capable of being proved by a single proof for all. But because all of these things (numbers, lengths, times, solids) are not any singly named thing, and since they differ in species from each other, they were taken separately. But now this is proved in general.

It seems evident that Aristotle refers to something such as Eudoxus' treatment of proportion. It is unlikely that there were in fact pre-Eudoxean theories of proportion for separate domains such as solids or times (though it seems quite possible that there were, for the case of number, which is indeed somewhat distinct and is still treated separately in Euclid's Elements). The point is rather that, conceptually, Eudoxus may have identified a correct level of generality with his new definition – and that at least one reader, namely Aristotle, saw this as the very purpose of the definition.

This, in fact, strikes me as a reasonable interpretation. Book 5 of Euclid's *Elements* proves results such as, indeed, alternation: if a:b::c:d, then a:c::b:d. This is the kind of result that I can definitely imagine working mathematicians simply taking for granted. Specifically, in the practice of Greek mathematics, an important set of results involves proportion *in*equality: for instance, that from a:b>c:d, one can deduce a:c>b:d. This is harder, and no longer intuitive – and yet Book 5 as it now stands, does not prove such inequalities and is instead almost entirely confined to the very basic, the level that requires proof only if one is truly interested in seeking the foundations of a theory that is otherwise clear enough.

Even the definition is not exactly a practical mathematical tool. Just as we saw with Archimedes' concavity: this definition is non-constructive.

There is no finite, doable procedure that can establish that four terms are in proportion. This is because a single counterexample will establish that the four terms are not in proportion and so, to verify that they are, one needs to check *all* equimultiples – all the possible m's and n's, all infinitely many of them, twice over – in order to conclude that a, b, c and d are in fact in proportion.

And so, finally, it does seem reasonable that this is all motivated by a more philosophical question: essentially, Euclid's *Elements* Book 5 could ultimately derive from a treatise dedicated to the question 'what is proportion'. The definition is not the tool, then, to derive the results. The results are there to verify the definition which, constructive or not, seems to carve out, with supreme analytical precision, something essential about the nature of proportion – proportion, as it were, is a kind of extension of the concept of equality.

6 A Couple of Historical Observations.

First of all, we have just made a comparison between Eudoxus' proportion, and Archimedes' concavity. It makes sense to bring Eudoxus into the discussion of Archimedes' *Sphere and Cylinder*, Book 1, because *Archimedes already did*. Archimedes' main achievement in this treatise is a measurement of the volume of the sphere in terms of that of the cylinder and the cone. In his introduction, Archimedes proudly proposes that this should be considered comparable to Eudoxus' measurement of the cone in terms of the cylinder. (This is indeed yet another achievement by Eudoxus, now extant as a book of Euclid's *Elements*, in this case number 12). It is reasonable enough that *Sphere and Cylinder*, Book 1 is conceived, among other things, as Archimedes' competitive homage to Eudoxus' *Elements* 12. Could the definition of concavity – brilliant, non-constructive and somewhat redundant – be understood as Archimedes' homage to the definition at the heart of Eudoxus' *Elements* 5?

This is, of course, extremely speculative. A more basic historical observation is that we have mentioned Plato, alongside Theaetetus; Aristotle, alongside Eudoxus, and there is no question that this is justified. All three major mathematicians mentioned above – Archytas, Theaetetus and Eudoxus – are closely connected to Plato[4]. We have to speculate, but

[4] Archytas: Huffman 2005: 32–43. Theaetetus is known primarily via Plato's tribute to him. Eudoxus had a significant impact as a philosopher (see e.g. Warren 2009), adding some credibility to the implication of Diogenes Laertius' biography that Eudoxus did encounter Plato in Athens (7.86–8.

the historical context is suggestive. Mathematics was emerging (when Archytas wrote, right at the beginning of the fourth century, could he even envisage an audience of readers interested in pure mathematics?). Philosophy was booming, with perhaps dozens of active philosophical authors across the Aegean, with a prolific network in and around Athens responding to the memory of Socrates' charismatic presence. It is entirely plausible that the mathematicians active in this generation often saw themselves as philosophers and at any rate envisaged a philosophically minded audience. It thus seems likely enough that this period was also the most active in terms of the conceptual elaboration of mathematics – an observation that we must make now, however, based on no more than indirect indications.

7 Concepts in Harmony

The main theme of this survey is variety: a significant engagement, perhaps, with conceptual discussion, in the classical era; its avoidance, apparently, in the Hellenistic era; throughout (but perhaps more so in the Imperial era) the drilling of mathematical vocabulary as part of elementary education – which is at least superficially akin to conceptual engagement. In Late Antiquity, the tradition of commentary is often prepared to approach mathematics in a more conceptual manner. Yet it is above all a traditional enterprise, and, at this point, the earliest generation of Greek mathematics is probably already lost. Even the philosophically trained Late Ancient Philosophers, then, are mostly bound by the practices of Hellenistic mathematics, expanded mostly through the traditions of later mathematical education.

This is then not only a story of variety but also a story of philosophical loss. A substantial corpus of Greek mathematical works does survive – in fact, it is possible that few other ancient genres survive as well. But the survival is chronologically uneven and the complete loss of the earliest Greek mathematics must mean that we have also lost some of the most interesting mathematics, philosophically. A speculation, and yet not a very wild one, for after all, we have lost Archytas and Eudoxus!

This means that we need to guess not just the extent of the engagement of early mathematicians with the question of mathematical concepts but also its very nature. I have briefly alluded to the very fact that Archytas may

All of this is Diogenean anecdote – but the prior probability is that such encounters did in fact take place. Why shouldn't they?)

have discussed the concept of 'mean', Theaetetus may have discussed the concept of 'irrational', Eudoxus may have discussed the concept of 'proportion', but what did such discussion even include, and what motivated it? Here we are becoming truly speculative; but we cannot complete this discussion without a venture into this question.

In fact, some kind of pattern does seem to emerge, even through the fog of our limited evidence. Means, irrationals, proportion... The concepts under discussion all belong to a well-defined domain. This domain was certainly characteristic of the mathematicians of the early fourth century. It appears that one of the leading motivations for Archytas' very interest in science and mathematics was the (Pythagorean?) theory of musical harmony as simultaneously physical and mathematical. This theory – that harmony can be explained in terms of numerical relations – is first unambiguously attested in Philolaus (fr. 6a) and was essential not only to Archytas (fr. 1, testimonies 16–19) but also, of course, to Plato and so, through him, to an entire philosophical generation. The mathematical elaboration of music is extensively pursued in the fourth century BCE and is then largely neglected throughout the Hellenistic period (revived, however, in the Imperial era). It seems to me that mathematical music was a hallmark of philosophical mathematics and so the non-philosophical mathematicians of the Hellenistic era neglected it (the one exception – the one Hellenistic author somewhat interested in both music as well as mathematics – was the idiosyncratic Eratostenes who, tellingly, was also the rare Hellenistic mathematician-philosopher). This is because mathematical music is, in and of itself, a philosophical statement. It presents the metaphysical position, according to which underlying the physical universe there is another, more abstract order, one understood in pure conceptual terms. The world is, in fact, a system of concepts.

This recalls, after all, some of the hints of philosophical engagement with mathematical concepts from later in the tradition. We recall Hero's *Definitions*, often engaged not just with isolated definitions but also with the taxonomy of a field. Even closer, we recall Domninus' *Handbook*: what he offers, primarily, is a taxonomy of number. We recall Proclus' more extensive engagement with conceptual understanding – and this in fact involves the question of the relation between the concepts of point and line – extended into the relation between those, and surface and solid – a question which has a long philosophical pedigree, going back to the early Academy (Glasner 1992). The key point, then, is that even when Euclid – surely following some fourth-century predecessors – discusses individual concepts such as 'points' or 'line', and of course when Proclus returns to analyze such concepts, what is at stake is not the nature of the individual concepts but rather the elaboration of a certain order composed out of them.

I started out from Fowler's observation, that the very structuring of Greek mathematical definitions, in modern editions, as isolated, separately numbered statements, is misleading. The introductions, in the original mathematical works, were instead organized as discursive passages. Which is, in fact, quite reasonable, in view of this last observation. It is for a good reason that Euclid does not first define point, then line, each separately; and instead he makes a single overarching set of statements about both points, lines, and the system they constitute. This may well be because one of the fundamental ways to engage with concepts, for him, for his sources, and for his ultimate readers, was in terms of the structure and order that such concepts give rise to.

Throughout this article, I have studied the question of the extent to which this or that mathematical author is concerned with a particular exercise: striving for conceptual clarity. This is an endeavor which modern philosophers can easily conceive of because, in fact, it is at the heart of our current discipline. Analytic philosophy is fundamentally, well, analytic: it takes individual concepts and tries to capture our intuitions concerning them. Historically, analytic philosophy owes a great deal to the tradition of logical atomism – that of Russell and the early Wittgenstein. The metaphysics of logical atomism anticipates the methodology of analytic philosophy. The universe is the sum total of facts, each subsisting as an isolated logical entity… a metaphysics, emerging out of a particular tradition, seeking the foundations of mathematics through conceptual analysis. This is what we think of, then, as we consider concepts in mathematics!

And quite obviously, such was not Plato's world. His world was a well ordered whole, which was precisely why mathematics mattered: because the world was, specifically, harmonic. If so, for authors immersed in Plato's philosophy, the very purpose, and practice, of engaging with concepts, could have been different from those authors – such as ourselves – immersed in Wittgenstein's.

This chapter may have felt, at times, negative and disappointing. I do emphasize that many mathematicians – especially in the Hellenistic era – may have deliberately avoided philosophy; and that a large part of the tradition of engaging with terminology, in antiquity, was just that – terminological, and mostly sustained through the educational process. And yet the negative claims I make concerning the residue that does remain – the few indications of conceptual discussions in very early, or very late mathematical writings – should not be seen as disappointing. We do find little by way of conceptual *analysis*, but this is perhaps because the basic attitude to concepts in mathematics among philosophically minded readers, was not analytical. It was, we may say, harmonic. We need to envisage an intellectual environment where the significance of concepts is not in their individual meaning, but rather in the overall order to which they give rise.

Indeed, as I surveyed my past comments in Netz 2002/2017 and Netz 1999, concerning the relatively marginal position of conceptual analysis for Greek mathematical practice, I emphasized that what matters, for working mathematicians, is not so much the isolated definition, as entire systems: technical language, as a whole; the web of practice. This is surely a debatable theoretical position, and it represents my own biases in sociology or in metaphysics. Perhaps I am wrong. But the point is that this is a possible position: that what matters is not isolated atoms, but interrelated systems. Obviously, the interest in harmony-like conceptual structures, typical of the Platonic tradition, is only one metaphysics among many, where structures matter more and atoms less. Indeed, to a modern mind, the specific conceptual hierarchies of Plato and of his followers may seem far-fetched. But the basic position – not atoms, but structures – remains reasonable enough and at least one that modern philosophers ought to debate, in their discussion of conceptual structure. And if so, this may suggest that, after all, there might be something surprising, and worthy of study, about our subject matter: concepts in Greek mathematics.