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Local Rankin–Selberg integrals for Speh representations

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Abstract

We construct analogues of Rankin–Selberg integrals for Speh representations of the general linear group over a p-adic field. The integrals are in terms of the (extended) Shalika model and are expected to be the local counterparts of (suitably regularized) global integrals involving square-integrable automorphic forms and Eisenstein series on the general linear group over a global field. We relate the local integrals to the classical ones studied by Jacquet, Piatetski-Shapiro and Shalika. We also introduce a unitary structure for Speh representation on the Shalika model, as well as various other models including Zelevinsky's degenerate Whittaker model.

Contents

1. Introduction

The theory of Rankin–Selberg integrals for $\mathrm{GL}_n \times \mathrm{GL}_{n'}$, studied by Jacquet, Piatetski-Shapiro and Shalika in a series of papers starting in the late 1970s (notably [JPSS83]), is a basic tool in the theory of automorphic forms with an abundance of applications. The theory is based on global zeta integrals (which involve Eisenstein series in the case $n' = n$) that unfold to adelic integrals of Whittaker–Fourier coefficients of cuspidal representations. By local multiplicity one, these integrals factorize into a product of local zeta integrals pertaining to generic representations and their Whittaker models.

The purpose of this paper is to study a modification of the local Rankin–Selberg integrals in the equal-rank case for a class of representations $\mathrm{Sp}(\pi, m)$ where π is an irreducible generic

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representation of GL_n over a p-adic field and $m \geqslant 1$ is an integer. If π is unitarizable (and generic), $\mathrm{Sp}(\pi, m)$ is the Langlands quotient of the parabolic induction of $\pi|\det|^{(m-1)/2} \otimes \pi|\det|^{(m-3)/2} \otimes \cdots \otimes \pi|\det|^{(1-m)/2}$. In particular, if π is discrete series, $\mathrm{Sp}(\pi, m)$ is the usual 'Speh representation'. These representations (of GL_{mn}) are not generic if $m > 1$ (i.e. they do not admit a Whittaker model). Instead, the integrals involve a different model which for simplicity we will call a Shalika model. (We caution, however, that it does not exactly coincide with the standard notion of the Shalika model in the literature.) It is known that any $\mathrm{Sp}(\pi, m)$ admits a unique Shalika model, a fact that reflects the 'smallness' of $\mathrm{Sp}(\pi, m)$. Structurally, the new integrals look very much like the classical ones and in fact they can be explicitly related. In particular, the unramified computation reduces to that of the classical Rankin–Selberg integrals [JS81] (which in turn uses Cauchy's identity and Shintani's formula for the unramified Whittaker function of GL_n [Shi76]).

Just as the Whittaker model gives rise to the so-called Kirillov model [GK75] (by restriction to the mirabolic subgroup, namely, the stabilizer of a vector in GL_n in its standard n-dimensional representation) the Shalika model gives rise to a closely related object which we call the Kirillov–Shalika model. The role of the mirabolic subgroup is now played by the joint stabilizer of m linearly independent vectors in GL_{mn}. The argument of Gelfand and Kazhdan shows that the Kirillov–Shalika model contains all functions that are compactly supported modulo the equivariance subgroup.

There are, however, some differences between the classical theory and its suggested analogue. First, in the unramified case, we are unaware of a simple closed formula for the spherical function in the Shalika model, except if $n \leqslant 2$ or if $n = 3$ and $m = 2$. A related, equally difficult, problem is the asymptotic behavior of a function in the Shalika model. Apart from the above-mentioned cases, both problems go beyond the 'comfort zone' of spherical varieties, for which the works of Sakellaridis [Sak13] and Sakellaridis and Venkatesh [SV17] provide a conceptual framework and satisfactory answers to the questions above. Moreover, at this stage it is not clear whether there is an analogue of the Bernstein–Zelevinsky theory of derivatives [BZ76, BZ77] in the case at hand. In particular, we do not know whether the restriction of $\mathrm{Sp}(\pi, m)$ to a parabolic subgroup of type $((n-1)m, m)$ is of finite length.

Another aspect of the paper is to provide an explicit, manifestly positive, unitary structure for the Speh representation in its Shalika model. (By this we mean that the positive-definiteness is 'clear and obvious' from the definition.) Once again, this is modeled on the case of generic unitarizable representations, in which Bernstein gives a unitary structure for their Whittaker models by taking the L^2-inner product of Whittaker functions restricted to the mirabolic subgroup [Ber84]. For $m > 1$ we use instead the joint stabilizer of m vectors, as before.

Along with the above-mentioned Shalika model, the representations $\mathrm{Sp}(\pi, m)$ admit various other models, for instance the degenerate Whittaker model considered by Zelevinsky (for any irreducible representation) in [Zel80]. We can think of this as a sequence of models starting from the Zelevinsky model and ending with the Shalika model. They all involve a character on a unipotent subgroup and are covered by the general construction of Mœglin and Waldspurger [MW87]. The unipotent subgroups in the sequence are decreasing. One can write down explicit isomorphisms (transition maps) between these models. This idea has been used by many authors, most recently and systematically by Gomez, Gourevitch and Sahi [GGS16, GGS17]. It also played a role in the recent work of Cai, Friedberg and Kaplan on new doubling constructions of L-functions [CFGK19, CFK18]. We write an inner product for each of these models and show that the transition maps are unitary.

As far as we know, this is the first time a purely local, manifestly positive hermitian form for a general Speh representation is explicitly given. Of course, the intertwining operator on

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the standard module whose image is the Speh representation induces its unitary structure – a fact that is true in general for any unitarizable representation on a reductive group (cf. [KZ77, § 4]). (In the case at hand we will explicitly relate this unitary structure to that on the Shalika model.) However, the semidefiniteness of the intertwining operator is far from obvious – in fact it is equivalent to unitarizability, which is known to be a difficult problem in general, as is evident from the work of Vogan and many others. Another realization of the inner product is obtained by using global theory to embed Speh representations as local constituents of automorphic forms in the discrete spectrum of GL_{mn} over the adeles [Spe83]. Finally, in the $m = 2$ case one can also realize a Speh representation in the discrete spectrum of $L^2(H\backslash \mathrm{GL}_{2n})$ where H is the symplectic group of rank n [Smi18, LO19]. However, there is no such analogue for $m > 2$.

In principle, the new local integrals are the local counterpart of certain global integrals, just as in the classical case. However, in addition to Eisenstein series, these global integrals involve automorphic forms in the discrete spectrum, rather than cusp forms, and they unfortunately do not converge (for any value of s). It should be possible (for instance by using the recent work of Zydor [Zyd19]) to carry out a regularization procedure to make sense of these integrals and to justify the unfolding procedure. However, we will not discuss this aspect in the paper. Nor we will discuss the archimedean case, for which we expect many of our results to hold without change.

The main new results of this paper are in §§ 4 and 5. The unitary structure for Speh representations (and more generally, $\mathrm{Sp}(\pi, m)$ for unitarizable generic π), on their various models, is given in Theorem 4.3. The new zeta integrals are defined in § 5. The convergence, unramified computation and local functional equations are stated in Theorem 5.1.

We now give some more details about the contents of the paper. In § 2 we first introduce some notation and recall Zelevinsky's classification of irreducible representations of the general linear group over a local non-archimedean field F. We then introduce the class of m-homogeneous representations, which includes the usual Speh representations and which is the main focus of the paper. In terms of Zelevinsky's classification, they simply correspond to multisegments consisting of segments of length m, where $m \geqslant 1$ is a fixed integer parameter. The case $m = 1$ exactly corresponds to generic representations (i.e. the classical theory). In § 3 we introduce the models pertaining to m-homogeneous representations, following Mœglin–Waldspurger. (In order to use their results, we assume from § 3 onward that F is of characteristic 0. As was pointed out to us by Dmitry Gourevitch, this assumption can be lifted. Details will appear elsewhere.) We also introduce the transition maps between the models. They are given by integrals which entail no convergence issues. Finally, we introduce the Kirillov–Shalika model which is the analogue of the classical Kirillov model for generic representations. In § 4 we introduce a family of bilinear forms on a pair of models of m-homogeneous representations. In the case where the two representations are in duality, these bilinear forms specialize to an invariant pairing, at least under some restrictions. In the unitarizable case this gives rise to a manifestly positive invariant unitary structure. The invariance is proved by induction on m using Bernstein's theorem on invariant distributions with respect to the mirabolic subgroup. In § 5 we define the local Rankin–Selberg integrals for m-homogeneous representations using their Shalika models. Applying the transition maps, we can express these integrals in terms of the Zelevinsky model. Hence, we get their rationality in q^s, the unramified computation and functional equations. In § 6 we obtain more information about the poles of the zeta functions and relate them to the above-mentioned bilinear forms and, in particular, to the invariant pairing. In § 7 we go back to the Kirillov–Shalika model and analyze in detail the case of Speh representations of GL_4 pertaining to supercuspidal representations of GL_2. We study the asymptotic behavior of a function in the Kirillov–Shalika model. At this stage, it is hard to tell whether the result is representative of the general case or merely a low-rank fluke. In § 8 we write an informal global expression, modeled after the classical Rankin–Selberg integrals, whose regularization is expected to unfold to the local integrals studied in the paper. The regularization is necessary as the integral does not converge. (It would also eliminate extraneous terms in the unfolding procedure.) However, we do not discuss the regularization procedure and only give a purely heuristic argument. Finally, in Appendix A we relate the pairing of § 4 to that induced by the intertwining operator on the standard module.

2. Preliminaries

2.1 Notation

Throughout the paper, fix a non-archimedean local field F with ring of integers $\mathcal{O}$ and absolute value $|\cdot|$. In principle it should be possible to deal with the archimedean case as well with proper adjustments, but we do not consider this case here.

From § 3 onward, F is assumed to be of characteristic 0.

If H is an algebraic group over F, we often also use H to denote $H(F)$.

We will consider complex, smooth representations of finite length of the groups $\mathrm{GL}_n(F)$, $n \geqslant 0$. We denote the set of irreducible representations of $\mathrm{GL}_n(F)$ (up to equivalence) by $\mathrm{Irr}\,\mathrm{GL}_n$ and set $\mathrm{Irr} = \bigcup_{n\geqslant 0} \mathrm{Irr}\,\mathrm{GL}_n$. We write $\mathrm{Irr}\,\mathrm{GL}_0 = \{\mathbf{1}\}$. (In contrast, the one-dimensional trivial character of GL_1 will be denoted by $\mathbf{1}_{F^*}$.) The subset of supercuspidal (respectively, square-integrable, essentially square-integrable, tempered, generic) representations will be denoted by $\mathrm{Irr}_{\mathrm{cusp}}$ (respectively, $\mathrm{Irr}_{\mathrm{sqr}}$, $\mathrm{Irr}_{\mathrm{esqr}}$, $\mathrm{Irr}_{\mathrm{tmp}}$, $\mathrm{Irr}_{\mathrm{gen}}$). Thus,

$$\mathrm{Irr}_{\mathrm{cusp}} \subset \mathrm{Irr}_{\mathrm{sqr}} \subset \mathrm{Irr}_{\mathrm{esqr}} \quad \text{and} \quad \mathrm{Irr}_{\mathrm{esqr}}, \mathrm{Irr}_{\mathrm{tmp}} \subset \mathrm{Irr}_{\mathrm{gen}}.$$

By convention $\mathbf{1} \in \mathrm{Irr}_{\mathrm{tmp}}$ but $\mathbf{1} \notin \mathrm{Irr}_{\mathrm{esqr}}$.

Let π be a representation of $\mathrm{GL}_n(F)$. We denote by $\pi^\vee$ the contragredient of π and by $\mathrm{soc}(\pi)$ the socle of π (the maximal semisimple subrepresentation of π). If π is non-zero, then we write $\deg \pi = n$, the degree of π. For any character ω of F^* (i.e. $\omega \in \mathrm{Irr}\,\mathrm{GL}_1$) we denote by $\pi\omega$ the representation obtained from π by twisting by the character $\omega \circ \det$. For instance, $\pi|\cdot|$ is the twist of π by $|\det|$. We also write $J_P(\pi)$ for the (normalized) Jacquet module of π with respect to a parabolic subgroup P of GL_n, defined over F. If $\tau \in \mathrm{Irr}\,\mathrm{GL}_n$, then we write $\tau \leqslant \pi$ if τ occurs as a subquotient of π, that is, if τ occurs in the Jordan–Hölder sequence of π. If τ occurs with multiplicity one in the Jordan–Hölder sequence of π, then we write $\tau \leqslant_{\mathrm{unq}} \pi$.

If $\pi_1, \dots, \pi_k$ are representations of $\mathrm{GL}_{n_1}(F), \dots, \mathrm{GL}_{n_k}(F)$ respectively, then we denote the representation parabolically induced from $\pi_1 \otimes \cdots \otimes \pi_k$ (normalized induction), with respect to the standard parabolic subgroup of block upper triangular matrices, by $\pi_1 \times \cdots \times \pi_k$ and refer to it as the product representation. We also use the notation Ind_H^G and ind_H^G to denote induction and induction with compact support (both normalized) from a subgroup H of G.

For any $\tau \in \mathrm{Irr}_{\mathrm{esqr}}$, let $\mathbf{e}(\tau)$ be the unique real number s such that the twisted representation $\tau|\cdot|^{-s}$ is unitarizable (i.e. has a unitary central character). Note that $\mathbf{e}(\tau^\vee) = -\mathbf{e}(\tau)$. Any $\pi \in \mathrm{Irr}_{\mathrm{gen}}$ can be written uniquely (up to permutation) as $\pi = \tau_1 \times \cdots \times \tau_k$ where τ_i are essentially square-integrable. Let $\mathbf{e}(\pi) = \min \mathbf{e}(\tau_i)$. (For consistency we write $\mathbf{e}(\mathbf{1}) = 0$.) Then $\mathbf{e}(\pi) + \mathbf{e}(\pi^\vee) \leqslant 0$ with equality if and only if π is essentially tempered. Moreover, π is tempered if and only if $\mathbf{e}(\pi) = \mathbf{e}(\pi^\vee) = 0$. More generally, we will say that π is 'approximately tempered' (AT) if $\mathbf{e}(\pi) + \mathbf{e}(\pi^\vee) + 1 > 0$. Equivalently, $\mathbf{e}(\tau_i) - \mathbf{e}(\tau_j) < 1$ for all i, j. It is known that every unitarizable

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$\pi \in \mathrm{Irr}_{\mathrm{gen}}$ is (AT). (This follows from the classification of the unitary dual of $\mathrm{GL}_n(F)$ by Tadić [Tad86].) We denote by $\mathrm{Irr}_{(AT)}$ the set of (AT) representations.

For any set A we denote by $\mathcal{M}(A)$ the free commutative monoid generated by A, considered as an ordered monoid. Thus, an element of $\mathcal{M}(A)$ (a multiset of A) is a finite (possibly empty) formal sum of element of A.

2.2 Zelevinsky classification

We recall the well-known results and terminology of [Zel80].

A *segment* Δ (of length $l > 0$ and center $\rho \in \mathrm{Irr}_{\mathrm{cusp}}$) is a non-empty finite subset of $\mathrm{Irr}_{\mathrm{cusp}}$ of the form

$$\Delta_\rho^{(l)} = \{\rho|\cdot|^{(1-l)/2}, \rho|\cdot|^{(3-l)/2}, \dots, \rho|\cdot|^{(l-1)/2}\}.$$

We define $\deg \Delta = l \deg \rho$ and write $e(\Delta) = \rho|\cdot|^{(l-1)/2} \in \mathrm{Irr}_{\mathrm{cusp}}$ (the endpoint of Δ),

$$\mathfrak{c}(\Delta) = \rho|\cdot|^{(1-l)/2} + \rho|\cdot|^{(3-l)/2} + \cdots + \rho|\cdot|^{(l-1)/2} \in \mathcal{M}(\mathrm{Irr}_{\mathrm{cusp}})$$

and $\Delta^\vee = \Delta_{\rho^\vee}^{(l)}$. For compatibility we also write $\Delta_\rho^{(0)} = \emptyset$. Denote by $\mathcal{SEG}$ the set of all segments. We extend deg additively to a function $\mathcal{M}(\mathcal{SEG}) \to \mathbb{Z}_{\geqslant 0}$. Similarly, we extend e and $\mathfrak{c}$ additively to functions $\mathcal{M}(\mathcal{SEG}) \to \mathcal{M}(\mathrm{Irr}_{\mathrm{cusp}})$.

For any $\Delta = \Delta_\rho^{(l)} \in \mathcal{SEG}$, let

$$Z(\Delta) = \mathrm{soc}(\rho|\cdot|^{(1-l)/2} \times \rho|\cdot|^{(3-l)/2} \times \cdots \times \rho|\cdot|^{(l-1)/2}) \in \mathrm{Irr}\,\mathrm{GL}_{\deg \Delta}.$$

(For compatibility we also set $Z(\emptyset) = \mathbf{1}$.) Then $Z(\Delta)^\vee = Z(\Delta^\vee)$. Given $\Delta_1, \Delta_2 \in \mathcal{SEG}$, we write $\Delta_2 \prec \Delta_1$ if $\Delta_i = \Delta_{\rho_i}^{(l_i)}$ with $\rho_2|\cdot|^{(1-l_2)/2+\alpha} = \rho_1|\cdot|^{(1-l_1)/2}$ for some $\alpha \in \mathbb{Z}_{>0}$ such that $l_2 - l_1 < \alpha \leqslant l_2$. If either $\Delta_2 \prec \Delta_1$ or $\Delta_1 \prec \Delta_2$, then we say that Δ_1 and Δ_2 are *linked*. The induced representation $Z(\Delta_1) \times Z(\Delta_2)$ is reducible if and only if Δ_1 and Δ_2 are linked.

The well-known classification result of Zelevinsky [Zel80, Theorem 6.5] extends the map $\Delta \mapsto Z(\Delta)$ to a degree-preserving bijection

$$\mathfrak{m} \mapsto Z(\mathfrak{m})$$

between $\mathcal{M}(\mathcal{SEG})$ and Irr. If $\mathfrak{m} = \Delta_1 + \cdots + \Delta_k$ and $\Delta_i \not\prec \Delta_j$ for any $i < j$ (which can always be arranged), then $Z(\mathfrak{m}) = \mathrm{soc}(Z(\Delta_1) \times \cdots \times Z(\Delta_k))$. An element of $\mathcal{M}(\mathcal{SEG})$ is called a multisegment. We have $Z(\mathfrak{m})^\vee = Z(\mathfrak{m}^\vee)$, where we extend $^\vee$ from $\mathcal{SEG}$ to $\mathcal{M}(\mathcal{SEG})$ additively. For any $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathcal{M}(\mathcal{SEG})$ we have $Z(\mathfrak{m}_1 + \mathfrak{m}_2) \leqslant_{\mathrm{unq}} Z(\mathfrak{m}_1) \times Z(\mathfrak{m}_2)$ [LM16, Proposition 3.5]. In particular, if $Z(\mathfrak{m}_1) \times Z(\mathfrak{m}_2)$ is irreducible, then it is equal to $Z(\mathfrak{m}_1 + \mathfrak{m}_2)$.

We note the following fact.

If every segment that occurs in $\mathfrak{m}_1$ is unlinked with every segment that occurs in $\mathfrak{m}_2$, then $Z(\mathfrak{m}_1) \times Z(\mathfrak{m}_2)$ is irreducible. (1)

By identifying an irreducible supercuspidal representation with a singleton segment we view $\mathcal{M}(\mathrm{Irr}_{\mathrm{cusp}})$ as a submonoid of $\mathcal{M}(\mathcal{SEG})$. The map Z restricts to a bijection

$$\mathcal{M}(\mathrm{Irr}_{\mathrm{cusp}}) \to \mathrm{Irr}_{\mathrm{gen}}. \tag{2}$$

An element of $\mathcal{M}(\mathrm{Irr}_{\mathrm{cusp}})$ is called a cuspidal datum. We write $\mathfrak{c}(Z(\mathfrak{m})) = \mathfrak{c}(\mathfrak{m})$. The resulting map

$$\mathfrak{c} : \mathrm{Irr} \to \mathcal{M}(\mathrm{Irr}_{\mathrm{cusp}})$$

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is the supercuspidal support (which of course can be defined without reference to the Zelevinsky classification). The restriction of $\mathfrak{c}$ to $\mathrm{Irr}_{\mathrm{gen}}$ is the inverse of (2).

For any segment $\Delta = \Delta_\rho^{(l)}$ let $\Delta^- = \Delta_{\rho|\cdot|^{-1/2}}^{(l-1)}$ denote either the segment obtained by removing the endpoint $e(\Delta)$ of Δ if $l > 1$ or the empty set otherwise.

Now let $\sigma = Z(\mathfrak{m})$, where $\mathfrak{m} = \Delta_1 + \cdots + \Delta_k$. Let

$$\mathfrak{m}^- = \Delta_1^- + \cdots + \Delta_k^- \quad \text{(disregarding empty sets).}$$

Define recursively $\mathfrak{m}^{(0)} = \mathfrak{m}$ and $\mathfrak{m}^{(k)} = (\mathfrak{m}^{(k-1)})^-$, $k > 0$, with $\mathfrak{m}^{(l)} = 0$, l minimal. Let $n_k = \deg e(\mathfrak{m}^{(k-1)})$, $k = 1, \ldots, l$, so that $n_1 + \cdots + n_l = \deg \sigma$ and let $\omega_k = Z(e(\mathfrak{m}^{(k-1)})) \in \mathrm{Irr}_{\mathrm{gen}} \mathrm{GL}_{n_k}$. Let $P = P_\sigma = M_\sigma \ltimes U_\sigma = M \ltimes U$ be the standard parabolic subgroup of type $(n_l, \ldots, n_1)$. By [Zel80, § 8.3] the Jordan–Hölder sequence of $J_P(\sigma)$ admits a unique generic irreducible representation ω of M and, moreover, $\omega \leqslant_{\mathrm{unq}} J_P(\sigma)$. Equivalently (by the uniqueness of the Whittaker model), this means that

$$\mathrm{Hom}_{N_M}(J_P(\sigma), \psi_P) = \mathrm{Hom}_N(\sigma, \psi_P) = \mathrm{Hom}_G(\sigma, \mathrm{Ind}_N^G \psi_P) \text{ is one-dimensional,} \tag{3}$$

where N is the maximal nilpotent group of upper unitriangular matrices and ψ_P is a character of N which is trivial on U and non-degenerate on $N_M = N \cap M$. (This property determines P uniquely up to association.) Moreover, $\omega = \omega_l \otimes \cdots \otimes \omega_1$ (see, for example, [MS14, Lemma 9.17]). (For an arbitrary P, $\mathrm{Hom}_N(\sigma, \psi_P)$ is finite-dimensional.) We will call the image of σ in $\mathrm{Ind}_N^G \psi_P$ the *Zelevinsky model* of σ. In general, $\sigma \not\leqslant \omega_l \times \cdots \times \omega_1$. For example, if $\sigma = Z(\{\mathbf{1}_{F^*}, |\cdot|\} + \{\mathbf{1}_{F^*}\})$, then $l = 2$, $\omega_2 = \mathbf{1}_{F^*}$, $\omega_1 = Z(\{|\cdot|\} + \{\mathbf{1}_{F^*}\})$ and $\omega_2 \times \omega_1$ is irreducible (and generic).

2.3 Ladder representations

A multisegment $\mathfrak{m}$ is called a (strict) ladder if it can be written as $\mathfrak{m} = \Delta_1 + \cdots + \Delta_k$ where $\Delta_{i+1} \prec \Delta_i$ for all $i = 1, \ldots, k-1$. The corresponding irreducible representation $Z(\mathfrak{m})$ is called a ladder representation.

LEMMA 2.1 [LM16]. *The following two statements hold.*

- (i) *[LM16, Lemma 6.17] Let $\pi_1, \ldots, \pi_k$ be ladder representations. Then $\pi_1 \times \cdots \times \pi_k$ is irreducible if and only if $\pi_i \times \pi_j$ is irreducible for all i, j.*[1]
- (ii) *[LM16, Lemma 6.21] Suppose that $Z(\mathfrak{m}_1)$ and $Z(\mathfrak{m}_2)$ are two ladder representations and that each segment of $\mathfrak{m}_1$ also occurs in $\mathfrak{m}_2$. Then $Z(\mathfrak{m}_1) \times Z(\mathfrak{m}_2)$ is irreducible.*

The Jacquet module of ladder representations was described in [KL12]. The following lemma is an immediate consequence.

LEMMA 2.2 [KL12]. *Let $\pi = Z(\mathfrak{m})$ be a ladder representation and P a maximal parabolic subgroup. The following statements hold.*

- (i) *$J_P(\pi)$ is a direct sum of irreducible representations of the form $\tau \otimes \omega$ where both τ and ω are products of ladder representations.*
- (ii) *If $\tau \otimes \omega \leqslant J_P(\pi)$ and $\omega \notin \mathrm{Irr}_{\mathrm{gen}}$, then there exists $\rho \in \mathrm{Irr}_{\mathrm{cusp}}$ such that $\rho \leqslant \mathfrak{c}(\omega)$, $\rho|\cdot| \leqslant e(\mathfrak{m})$ but $\rho \not\leqslant e(\mathfrak{m})$.*

[1] In fact, this holds for any $\pi_1, \ldots, \pi_k \in \mathrm{Irr}$ by using a result of Hernandez [Her10] and the quantum Schur–Weyl duality [CP96].

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(iii) *If* $\tau \otimes \omega \leqslant J_P(\pi)$ *with* $\omega \in \mathrm{Irr}_{\mathrm{gen}}$, *then* $\mathfrak{c}(\omega) \leqslant e(\mathfrak{m})$. *Moreover, if* $\rho \in \mathrm{Irr}_{\mathrm{cusp}}$ *is such that* $\rho \leqslant e(\mathfrak{m})$ *and* $\rho|\cdot| \leqslant \mathfrak{c}(\omega)$, *then* $\rho \leqslant \mathfrak{c}(\omega)$.

(iv) *If* $\tau \otimes \omega \leqslant J_P(\pi)$ *and* $\rho \in \mathrm{Irr}_{\mathrm{cusp}}$ *is such that* $\rho|\cdot| \not\leqslant \mathfrak{c}(\omega)$, *then* ρ *occurs in* $\mathfrak{c}(\omega)$ *with multiplicity at most one.*

Strictly speaking, the results of [KL12] are stated in terms of the Langlands classification. However, they are also valid in the form above (for the Zelevinsky classification) by either repeating the arguments, or using the Zelevinsky involution.

2.4 m-homogeneous representations[2]

From now on let $m, n \geqslant 1$ be integers and $G = \mathrm{GL}_{mn}$. We say that $\sigma \in \mathrm{Irr}\, G$ is *m-homogeneous* if $\sigma = Z(\Delta_1 + \cdots + \Delta_k)$ where each Δ_i is of length m. (If $m = 1$ this simply means that σ is generic.) We denote by $\mathrm{Irr}_{m\text{-hmgns}}\, G$ the set of irreducible m-homogeneous representations of G. For any $\pi = Z(\{\rho_1\} + \cdots + \{\rho_k\}) \in \mathrm{Irr}_{\mathrm{gen}}$, define

$$\mathrm{Sp}(\pi, m) = Z(\Delta_{\rho_1}^{(m)} + \cdots + \Delta_{\rho_k}^{(m)}) \in \mathrm{Irr}\,.$$

The following result is clear.

Lemma 2.3. *The map* $\pi \mapsto \mathrm{Sp}(\pi, m)$ *defines a bijection between* $\mathrm{Irr}_{\mathrm{gen}}\, \mathrm{GL}_n$ *and* $\mathrm{Irr}_{m\text{-hmgns}}\, G$. *We have* $\mathrm{Sp}(\pi, m)^\vee = \mathrm{Sp}(\pi^\vee, m)$ *for any* $\pi \in \mathrm{Irr}_{\mathrm{gen}}$.

Remark 2.4. The notion of m-homogeneous representations is very close to the concept of 'representations of type (n, m)' introduced in [CFGK19] and studied further in [CFK18]. The difference is that we only consider irreducible representations and emphasize the roles of the Mœglin–Waldspurger models.

Remark 2.5. If $\pi \in \mathrm{Irr}_{\mathrm{sqr}}\, \mathrm{GL}_n$, then $\mathrm{Sp}(\pi, m)$ is known as a 'Speh representation'. (Strictly speaking, these representations were introduced by Speh in the archimedean case.)

Remark 2.6. In general, if π is unramified (and generic), then $\mathrm{Sp}(\pi, m)$ is not necessarily unramified if $m > 1$. More precisely, if $\pi = Z(\{\rho_1\} + \cdots + \{\rho_k\})$ is unramified (so that ρ_i are unramified characters of F^* and $\rho_i \neq \rho_j|\cdot|$ for all i, j), then $\mathrm{Sp}(\pi, m)$ is unramified if and only if $\Delta_{\rho_1}^{(m)}, \dots, \Delta_{\rho_k}^{(m)}$ are mutually unlinked. For instance, this is the case if π is (AT).

Suppose that $\sigma = \mathrm{Sp}(\pi, m)$ with $\pi \in \mathrm{Irr}_{\mathrm{gen}}\, \mathrm{GL}_n$. Then, in the notation of (2), $P_\sigma = P_{m,n} = M \ltimes U$ is the standard parabolic subgroup of G of type $(\overbrace{n, \dots, n}^{m})$, consisting of the block upper triangular matrices with blocks of size $n \times n$. Thus, $M \simeq \overbrace{\mathrm{GL}_n \times \cdots \times \mathrm{GL}_n}^{m}$.

Just as in the case $m = 1$, there are simple building blocks for m-homogeneous representations.

Proposition 2.7. *Let* $\sigma = \mathrm{Sp}(\pi, m) \in \mathrm{Irr}\, G$ *be m-homogeneous. Then there exist* $\pi_1, \dots, \pi_t \in \mathrm{Irr}_{\mathrm{gen}}$ *such that the following statements hold.*

(i) $\sigma_i := \mathrm{Sp}(\pi_i, m)$ *is a ladder representation for all* i.

(ii) $\mathrm{Sp}(\pi, l) = \mathrm{Sp}(\pi_1, l) \times \cdots \times \mathrm{Sp}(\pi_t, l)$ *for all* $l = 1, \dots, m$. *In particular,* $\pi = \pi_1 \times \cdots \times \pi_t$ *and* $\sigma = \sigma_1 \times \cdots \times \sigma_t$.

[2] This notion should not be confused with Zelevinsky's notion of homogenous representations [Zel80].

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Moreover, let Q be the maximal standard parabolic subgroup of type $((m-1)n, n)$ and denote by $J_Q(\sigma)_{;\mathfrak{d}}$ the direct summand of $J_Q(\sigma)$ pertaining to the supercuspidal data $\mathfrak{d} \in \mathcal{M}(\mathrm{Irr}_{\mathrm{cusp}})$ in the second (GL_n) factor. Then

$$J_Q(\sigma)_{;\mathfrak{c}(\pi)|\cdot|^{(m-1)/2}} = \mathrm{Sp}(\pi, m-1)|\cdot|^{-1/2} \otimes \pi|\cdot|^{(m-1)/2}.$$

Remark 2.8. For $m = 1$, π_i is essentially a discrete series. This is not the case for $m > 1$ in general.

Proof. Write $\pi = Z(\sum_{i \in I}\{\rho_i\})$ with $\rho_i \in \mathrm{Irr}_{\mathrm{cusp}}$ and let $l \geqslant 1$. We say that a subset J of I is an l-chain if it can be written, necessarily uniquely, as $J = \{i_1, \dots, i_r\}$ where for all $j = 1, \dots, r-1$ we have $\rho_{i_j} = \rho_{i_{j+1}}|\cdot|^{\alpha_j}$ with $\alpha_j \in \{1, \dots, l\}$. (For example, for a 1-chain, $\rho_{i_r}, \dots, \rho_{i_1}$ is a segment.) Clearly, J is an l-chain if and only if $Z(\sum_{j \in J}\Delta_{\rho_j}^{(l)})$ is a ladder representation.

We say that two partitions of I are equivalent if one can be obtained from the other by applying a permutation τ of I such that $\rho_{\tau(i)} = \rho_i$ for all i. It is easy to see that for any $l \geqslant 1$ there exists a partition $\mathcal{P}^{(l)}(I)$ of I consisting of l-chains, such that for any $J, J' \in \mathcal{P}^{(l)}(I)$ at least one of the following conditions holds:

(i) $\{\rho_j : j \in J\} \subset \{\rho_j : j \in J'\}$;

(ii) $\{\rho_j : j \in J'\} \subset \{\rho_j : j \in J\}$;

(iii) for every $j \in J$ and $j' \in J'$ the segments $\Delta_{\rho_j}^{(l)}$ and $\Delta_{\rho_{j'}}^{(l)}$ are unlinked.

Moreover, $\mathcal{P}^{(l)}(I)$ is unique up to equivalence. Indeed, $\mathcal{P}^{(l)}(I)$ can be defined inductively by taking a maximal l-chain J of I (with respect to inclusion) together with the partition $\mathcal{P}^{(l)}(I \setminus J)$. It follows from this description that if $l \leqslant m$, then up to equivalence, $\mathcal{P}^{(l)}(J) = \{J' \in \mathcal{P}^{(l)}(I) : J' \subset J\}$ for any $J \in \mathcal{P}^{(m)}(I)$ and, in particular, $\mathcal{P}^{(l)}(I)$ is a refinement of $\mathcal{P}^{(m)}(I)$.

For any $J \subset I$, let $\pi_J = Z(\sum_{j \in J}\{\rho_j\}) \in \mathrm{Irr}_{\mathrm{gen}}$ and $\sigma_J = \mathrm{Sp}(\pi_J, m)$. Then σ_J is an (m-homogeneous) ladder representation for any $J \in \mathcal{P}^{(m)}(I)$. It follows from the defining property of $\mathcal{P}^{(m)}(I)$, (1) and Lemma 2.1 that $\times_{J \in \mathcal{P}^{(m)}(I)} \sigma_J$ is irreducible, hence equals σ. Likewise, for any $l \leqslant m$, we have $\mathrm{Sp}(\pi, l) = \times_{J' \in \mathcal{P}^{(l)}(I)} \mathrm{Sp}(\pi_{J'}, l)$. Since we may assume that $\mathrm{Sp}(\pi_J, l) = \times_{J' \in \mathcal{P}^{(l)}(I) : J' \subset J} \mathrm{Sp}(\pi_{J'}, l)$ for all $J \in \mathcal{P}^{(m)}(I)$, we infer that

$$\mathrm{Sp}(\pi, l) = \mathop{\times}_{J \in \mathcal{P}^{(m)}(I)} \mathrm{Sp}(\pi_J, l).$$

In particular, $\pi = \times_{J \in \mathcal{P}^{(m)}(I)} \pi_J$.

By [KL12], we have

$$\mathrm{Sp}(\pi_J, m-1)|\cdot|^{-1/2} \otimes \pi_J|\cdot|^{(m-1)/2} \leqslant_{\mathrm{unq}} J_{((m-1)n_J, n_J)}(\sigma_J)$$

for all $J \in \mathcal{P}^{(m)}(I)$ where $n_J = \deg \pi_J$. Therefore, by the geometric lemma of Bernstein and Zelevinsky [BZ77],

$$\begin{aligned}\mathrm{Sp}(\pi, m-1)|\cdot|^{-1/2} \otimes \pi|\cdot|^{(m-1)/2} &= \times_{J \in \mathcal{P}^{(m)}(I)} \mathrm{Sp}(\pi_J, m-1)|\cdot|^{-1/2} \otimes \times_{J \in \mathcal{P}^{(m)}(I)} \pi_J|\cdot|^{(m-1)/2} \\ &\leqslant J_Q(\sigma).\end{aligned}$$

On the other hand, suppose that $\tau_J, \omega_J \in \mathrm{Irr}$ with $\tau_J \otimes \omega_J \leqslant J_P(\sigma_J)$, $J \in \mathcal{P}^{(m)}(I)$ and

$$\sum_{J \in \mathcal{P}^{(m)}(I)} \mathfrak{c}(\omega_J) = \mathfrak{c}(\pi)|\cdot|^{(m-1)/2}. \tag{4}$$

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We claim that this is possible only if $\tau_J = \mathrm{Sp}(\pi_J, m-1)|\cdot|^{-1/2}$ and $\omega_J = \pi_J|\cdot|^{(m-1)/2}$ for all J. We prove it by induction on $\deg\pi$ using the geometric lemma. The base of the induction is trivial. For the induction step, it is enough to prove that if J is a maximal m-chain, then $\omega_J = \pi_J|\cdot|^{(m-1)/2}$. We use Lemma 2.2. By part (ii), if J is a maximal m-chain, then ω_J is generic. For otherwise, since $\mathfrak{c}(\omega_J) \leqslant \mathfrak{c}(\pi)|\cdot|^{(m-1)/2}$, there would exist $i \in I$ such that $\rho_i \notin \{\rho_j : j \in J\}$ but $\rho_i|\cdot| \in \{\rho_j : j \in J\}$ in contradiction to the maximality of J. On the other hand, by part (iv), if $\rho|\cdot| \nleqslant \mathfrak{c}(\pi)|\cdot|^{(m-1)/2}$, then ρ can occur in $\mathfrak{c}(\omega_J)$ at most once for any $J \in \mathcal{P}^{(m)}(I)$. It follows from (4) that if $\rho|\cdot| \nleqslant \mathfrak{c}(\pi)|\cdot|^{(m-1)/2}$, then $\rho \leqslant \mathfrak{c}(\omega_J)$ if and only if $\rho = \rho_j|\cdot|^{(m-1)/2}$ for some $j \in J$. By part (iii), it now follows that if J is a maximal m-chain, then $\mathfrak{c}(\omega_J) = \sum_{j\in J}\{\rho_j|\cdot|^{(m-1)/2}\}$ and hence $\omega_J = \pi_J|\cdot|^{(m-1)/2}$ (since ω_J is generic) as required.

This concludes the proof of the proposition. □

Remark 2.9. It can be shown that up to permutation, $\sigma_1, \dots, \sigma_t$ are the unique ladder representations such that $\sigma = \sigma_1 \times \dots \times \sigma_t$. We will not need to use this fact.

By Frobenius reciprocity and [LM16, Corollary 4.10], we have the following corollary.

Corollary 2.10. *For any* $\pi \in \mathrm{Irr}_{\mathrm{gen}}\,\mathrm{GL}_n$,

$$\mathrm{Sp}(\pi, m) = \mathrm{soc}(\mathrm{Sp}(\pi, m-1)|\cdot|^{-1/2} \times \pi|\cdot|^{(m-1)/2}) \leqslant_{\mathrm{unq}} \mathrm{Sp}(\pi, m-1)|\cdot|^{-1/2} \times \pi|\cdot|^{(m-1)/2}.$$

By induction on m, we get the following result.

Corollary 2.11. *For any* $\pi \in \mathrm{Irr}_{\mathrm{gen}}\,\mathrm{GL}_n$, $\mathrm{Sp}(\pi, m)$ *is a subrepresentation of*

$$\Pi := \pi|\cdot|^{(1-m)/2} \times \pi|\cdot|^{(3-m)/2} \times \dots \times \pi|\cdot|^{(m-1)/2}.$$

Equivalently (by passing to the contragredient), $\mathrm{Sp}(\pi, m)$ *is a quotient of*

$$\tilde{\Pi} := \pi|\cdot|^{(m-1)/2} \times \pi|\cdot|^{(m-3)/2} \times \dots \times \pi|\cdot|^{(1-m)/2}.$$

Remark 2.12. If π is (AT), then $\mathrm{Sp}(\pi, m)$ is the Langlands quotient of $\tilde{\Pi}$, In particular, in this case $\mathrm{Sp}(\pi, m)$ is the image of the standard intertwining operator from $\tilde{\Pi}$ to Π and $\mathrm{Sp}(\pi, m) = \mathrm{soc}(\Pi) \leqslant_{\mathrm{unq}} \Pi$. However, in general for $m > 2$ and $\pi \in \mathrm{Irr}_{\mathrm{gen}}\,\mathrm{GL}_n$ it is not true that $\mathrm{Sp}(\pi, m) \leqslant_{\mathrm{unq}} \Pi$. For instance, if $\pi = |\cdot| \times |\cdot|^{-1} \in \mathrm{Irr}_{\mathrm{gen}}\,\mathrm{GL}_2$, then $\mathrm{Sp}(\pi, 3)$ occurs with multiplicity two in the Jordan–Hölder sequence of $\pi|\cdot|^{-1} \times \pi \times \pi|\cdot|$. Note that in this case we still have $\mathrm{Sp}(\pi, m) = \mathrm{soc}(\Pi)$ but we do not know whether this holds in general, that is, whether $\mathrm{soc}(\Pi)$ is always irreducible.

3. The models

3.1 Definition of models

Throughout this section, fix $\pi \in \mathrm{Irr}_{\mathrm{gen}}\,\mathrm{GL}_n$, let $\sigma = \mathrm{Sp}(\pi, m) \in \mathrm{Irr}\,G$ and let $P = P_\sigma = P_{m,n} = M \ltimes U$ be the standard parabolic subgroup of G of type $\overbrace{(n, \dots, n)}^{m}$. Let $\bar{U} = {}^tU$ be the opposite of U. Fix a non-trivial character ψ of F. Let Ψ be the function on G given by

$$\Psi(g) = \psi\Bigg(\sum_{1\leqslant i<nm: n\nmid i} g_{i,i+1}\Bigg).$$

We denote the restriction of Ψ to a subset A of G by Ψ_A. Let $N = N_{mn}$ (respectively, $\bar{N} = {}^tN$) be the group of upper (respectively, lower) unitriangular matrices in G. Then Ψ_N is a (degenerate) character on N that is trivial on U and non-degenerate on $N_M := N \cap M$. Recall that by (3), $\operatorname{Hom}_G(\sigma, \operatorname{Ind}_N^G \Psi_N)$ is one-dimensional.

Denote by $\mathcal{W}^{\Psi_N}(\sigma)$ the image of σ in $\operatorname{Ind}_N^G \Psi_N$, that is, the Zelevinsky model of σ. By Corollaries 2.10 and 2.11, for any $W_{\mathrm{Ze}} \in \mathcal{W}^{\Psi_N}(\sigma)$, we have

$$(|\det|^{(1-n)/2} \otimes |\det|^{(m-1)(n-1)/2}) W_{\mathrm{Ze}}|_{\mathrm{GL}_{(m-1)n} \times \mathrm{GL}_n} \in \mathcal{W}^{\Psi_{N \cap (\mathrm{GL}_{(m-1)n} \times \mathrm{GL}_n)}}(\mathrm{Sp}(\pi, m-1) \otimes \pi), \tag{5a}$$

$$\delta_P^{-1/2} \delta' W_{\mathrm{Ze}}|_M \in \mathcal{W}^{\Psi_{N_M}}(\pi^{\otimes m}), \tag{5b}$$

where $\pi^{\otimes m} = \overbrace{\pi \otimes \cdots \otimes \pi}^{m}$, δ_P is the modulus character of P and $\delta' = \delta_P^{1/2n}$ is the character of M given by

$$\delta'(\operatorname{diag}(g_1, \ldots, g_m)) = |\det g_1|^{(m-1)/2} |\det g_2|^{(m-3)/2} \cdots |\det g_m|^{(1-m)/2}.$$

The model $\mathcal{W}^{\Psi_N}(\sigma)$ is a particular case of more general models considered in [MW87] (for any reductive group). Let us recall the setup. Let $\mathfrak{g} = \operatorname{Mat}_{nm,nm}$ be the Lie algebra of G over F. For any cocharacter φ of the diagonal torus T, let $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j^{\varphi}$ be the corresponding grading

$$\mathfrak{g}_j^{\varphi} = \{X \in \mathfrak{g} : \operatorname{Ad}(\varphi(s))X = s^j X\}$$

and let $\mathfrak{g}_{\geqslant j}^{\varphi} = \bigoplus_{k \geqslant j} \mathfrak{g}_k^{\varphi}$, $j \in \mathbb{Z}$, be the corresponding filtration. Let P_φ be the semistandard parabolic subgroup such that $\operatorname{Lie} P_\varphi = \mathfrak{g}_{\geqslant 0}^{\varphi}$. Then $P_\varphi = M_\varphi \ltimes U_\varphi$ where M_φ is the centralizer of φ, $\operatorname{Lie} M_\varphi = \mathfrak{g}_0^{\varphi}$ and $\operatorname{Lie} U_\varphi = \mathfrak{g}_{>0}^{\varphi}$. Concretely, if $\varphi(s) = \operatorname{diag}(s^{\lambda_1^\varphi}, \ldots, s^{\lambda_{mn}^\varphi})$ where $(\lambda_1^\varphi, \ldots, \lambda_{mn}^\varphi) \in \mathbb{Z}^{mn}$, then

$$\begin{aligned} P_\varphi &= \{g \in G : g_{i,j} = 0 \text{ if } \lambda_i < \lambda_j\}, \\ M_\varphi &= \{g \in G : g_{i,j} = 0 \text{ if } \lambda_i \neq \lambda_j\}, \\ U_\varphi &= \{g \in G : g_{i,j} = \delta_{i,j} \text{ if } \lambda_i \leqslant \lambda_j\}. \end{aligned}$$

Consider the nilpotent $nm \times nm$ matrix $J_{m,n}$ consisting of m lower triangular Jordan blocks of size $n \times n$ each. We say that φ is of type (m, n) if $\operatorname{Ad}(\varphi(s))J_{m,n} = s^{-1} J_{m,n}$, or equivalently, if $\lambda_i^\varphi - \lambda_{i+1}^\varphi = 1$ for all i not dividing n. If φ is of type (m, n), then Ψ_{U_φ} is a character of U_φ. By [MW87] (in particular, § II.2) we obtain the following theorem.[3]

Theorem 3.1 [MW87]. *Suppose that φ is of type (m, n). Then the space*

$$\operatorname{Hom}_{U_\varphi}(\sigma, \Psi_{U_\varphi}) = \operatorname{Hom}_G(\sigma, \operatorname{Ind}_{U_\varphi}^G \Psi_{U_\varphi})$$

is one-dimensional.

In the setting of [MW87] the data pertaining to Theorem 3.1 is the pair $(\varphi^2, J_{m,n})$. (The more general context of [MW87] applies to cocharacters which are not necessarily even. However, we will not discuss them here.)

[3] This is the only place in the paper where we use that F is of characteristic 0, but as mentioned in the introduction, this assumption can be removed. (Note that the assumption on the residual characteristic in [MW87] was removed in [Var14].)

We denote by $\mathcal{W}^{\Psi_{U_\varphi}}(\sigma)$ the image of σ in $\operatorname{Ind}_{U_\varphi}^G \Psi_{U_\varphi}$. It consists of functions that are left equivariant (with respect to some character) under the centralizer of Ψ_{U_φ} in P_φ.

Clearly, any φ of type (m,n) is determined by the m-tuple $(\lambda_n^\varphi, \lambda_{2n}^\varphi, \dots, \lambda_{mn}^\varphi)$. We consider $(m-1)n+1$ cocharacters $\varphi_0, \dots, \varphi_{(m-1)n}$ of T of type (m,n) such that $\lambda_{nk}^{\varphi_i} - \lambda_{n(k+1)}^{\varphi_i} = \max(0, nk-i)$, $k = 1, \dots, m-1$. (Up to a cocharacter of the center of G, the cocharacter $\varphi_{(m-1)n}^2$ corresponds to the SL_2-triple pertaining to $J_{m,n}$.) For simplicity we write $P_i = P_{\varphi_i}$, $M_i = M_{\varphi_i}$, $U_i = U_{\varphi_i}$. If $i = nd + r$ where $d = \lfloor i/n \rfloor$ and $0 \leqslant r < n$, then U_i consists of the matrices whose $n \times n$ blocks $A_{j,k}$ satisfy:

(i) $A_{j,j}$ is upper unitriangular for all $j = 1, \dots, m$;
(ii) $A_{j,k}$ is strictly upper triangular if $j \neq k$ and $j, k \leqslant d+1$;
(iii) for any $k < d+2$, $(A_{d+2,k})_{a,b} = 0$ if $b - a \leqslant n - r$ and $(A_{k,d+2})_{a,b} = 0$ if $a - b \geqslant n - r$;
(iv) $A_{j,k} = 0$ if $j > k$ and $j > d+2$.

(There is no constraint on $A_{j,k}$ if $j < k$ and $d+2 < k$.)

In particular, $U_0 = N$ while $U_{(m-1)n}$ consists of the matrices whose difference from the identity matrix is strictly upper triangular in each $n \times n$ block. Also, $U_{i+1} \cap N \subset U_i \cap N$ and $U_i \cap \bar{N} \subset U_{i+1} \cap \bar{N}$ for all i.

For brevity we write $P' = P_{(m-1)n}$, $M' = M_{(m-1)n} \simeq \overbrace{\mathrm{GL}_m \times \cdots \times \mathrm{GL}_m}^{n}$, $U' = U_{(m-1)n}$. In analogy with the case $m = 2$ we will refer to $\mathcal{W}^{\Psi_{U'}}(\sigma)$ as the *Shalika model* of σ. We caution, however, that in the literature, this terminology usually refers to the image of $\tau \in \operatorname{Irr} \mathrm{GL}_{2n}$ (possibly generic) in $\operatorname{Ind}_S^{\mathrm{GL}_{2n}} \psi_S$ under a non-trivial intertwining operator, if it exists (in which case it is unique up to a scalar [JR96]), where S is the Shalika group

$$S = \left\{ \begin{pmatrix} g & \\ & g \end{pmatrix} \begin{pmatrix} I_n & X \\ & I_n \end{pmatrix} : g \in \mathrm{GL}_n, X \in \mathrm{Mat}_{n,n} \right\}$$

and ψ_S is the character on S given by $\psi(\operatorname{tr} X)$. In the case at hand, any $W_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}}(\sigma)$ automatically satisfies an equivariance property under the centralizer of $\Psi_{U'}$ in P' (which is conjugate to S in the case $m = 2$), which justifies our terminology. In general, even for $m = 2$, $\operatorname{Hom}_{U'}(\tau, \Psi_{U'})$ is infinite-dimensional for $\tau \in \operatorname{Irr} G$.

Letting G act on right on the vector space F^{mn} of row vectors with standard basis $e_1, \dots, e_{mn}$, P' is the stabilizer of the flag

$$(\operatorname{span}\{e_{nj-k} : j = 1, \dots, m,\ k = 0, \dots, i-1\})_{i=0,\dots,n}.$$

We denote by $\kappa : \overbrace{\mathrm{GL}_m \times \cdots \times \mathrm{GL}_m}^{n} \to M'$ the isomorphism such that the ith copy of GL_m acts on $\operatorname{span}\{e_{nj+i} : j = 0, \dots, m-1\}$.

If X is a matrix over F, then we write $\|X\|$ for the maximum of the absolute value of its entries.

Lemma 3.2. *Suppose that $W_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}}(\sigma)$. Then there exists $C > 0$ with the following property. Suppose that $g \in G$ with $W_{\mathrm{Sh}}(g) \neq 0$. Write $g = u'l'k$ where $u' \in U'$, $l' = \kappa(g_1, \dots, g_n) \in M'$ and $k \in G(\mathcal{O})$. Then $\|g_{i+1}^{-1} g_i\| \leqslant C$ for all $i < n$.*

Proof. It is enough to consider the case $g = \kappa(g_1, \dots, g_n) \in M'$. Assume that $W_{\mathrm{Sh}}(g) \neq 0$. Fix $1 \leqslant i < n$. For any $X \in \mathrm{Mat}_{m,m}(F)$, let $Y \in U'$ be the matrix such that $Y_{nj+i,nk+i+1} = X_{j+1,k+1}$ for all $0 \leqslant j, k < m$ and all other non-diagonal entries of Y are zero. Then $W_{\mathrm{Sh}}(gY) =$

$\psi(\operatorname{tr} g_i X g_{i+1}^{-1}) W_{\mathrm{Sh}}(g)$. It follows that there exists $C_1 > 0$ depending only on W_{Sh} such that if $W_{\mathrm{Sh}}(g) \neq 0$, then $\psi(\operatorname{tr} g_i X g_{i+1}^{-1}) = \psi(\operatorname{tr} g_{i+1}^{-1} g_i X) = 1$ for all $X \in \mathrm{Mat}_{m,m}(F)$ with $\|X\| \leqslant C_1$. The lemma follows. □

3.2 Model transition part I

We denote by M_i^{Ψ} (respectively, $P_i^{\Psi} = M_i^{\Psi} \ltimes U_i$) the stabilizer of Ψ_{U_i} in M_i (respectively, P_i). (Note that M_i determines i so this notation is unambiguous.) We also write $M'^{\Psi} = M^{\Psi}_{(m-1)n}$ and $P'^{\Psi} = P^{\Psi}_{(m-1)n} = M'^{\Psi} \ltimes U'$. Note that P'^{Ψ} is unimodular. Explicitly, M'^{Ψ} is the image under κ of GL_m diagonally embedded in $\overbrace{\mathrm{GL}_m \times \cdots \times \mathrm{GL}_m}^{n}$. It consists of the matrices in G whose $n \times n$ blocks are all scalar matrices. Let $\iota : \mathrm{GL}_m \to M'^{\Psi}$ be the resulting identification.

In general, write $i = nd + r$, $0 \leqslant r < n$. Then the reductive part of M_i^{Ψ} is the image under ι of the subgroup

$$\{\mathrm{diag}(l, t_{d+2}, \ldots, t_m) : l \in \mathrm{GL}_{d+1}, t_{d+2}, \ldots, t_m \in F^*\}.$$

The unipotent radical of M_i^{Ψ} consists of the matrices whose $n \times n$ blocks $A_{j,k}$ satisfy the following requirements.

(i) $A_{j,j} = I_n$ for all j.

(ii) If $j \neq k$, then $A_{j,k} = 0$ unless $k = d+2$ and $j < k$, in which case $(A_{j,k})_{a,b} = 0$ unless $a - b = n - r$. Moreover, all the entries of $A_{j,k}$ on the diagonal $a - b = n - r$ coincide.

This group is trivial if i is divisible by n, and is of dimension $d+1$ otherwise.

Lemma 3.3. *Let $0 \leqslant i < (m-1)n$. Then the following statements hold.*

(i) *The commutator $[U_i, U_{i+1}]$ is contained in $U_i \cap U_{i+1}$. Thus, $U_i \cdot U_{i+1}$ is a subgroup of G which contains U_i and U_{i+1} as normal subgroups and the quotients $U_i U_{i+1}/U_i \simeq U_{i+1}/U_i \cap U_{i+1}$ and $U_i U_{i+1}/U_{i+1} \simeq U_i/U_i \cap U_{i+1}$ are abelian. Moreover,*

$$U_{i+1} = (M_i \cap U_{i+1}) \ltimes (U_i \cap U_{i+1}) \quad \text{and} \quad U_i = (M_{i+1} \cap U_i) \ltimes (U_i \cap U_{i+1}). \tag{6}$$

(ii) *We have a short exact sequence*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{i+1}^{\Psi} \cap U_i & \longrightarrow & U_i/U_i \cap U_{i+1} & \xrightarrow{\mathfrak{c}_i} & PD(U_{i+1}/U_i \cap U_{i+1}) & \longrightarrow & 0 \\ & & & & \| & & \| & & \\ & & & & M_{i+1} \cap U_i & \xrightarrow{\mathfrak{c}_i} & PD(M_i \cap U_{i+1}) & & \end{array}$$

where $\mathfrak{c}_i$ denotes the map $u \mapsto \Psi([\cdot, u])$ and PD denotes the Pontryagin dual. Dually,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_{i+1}/U_i \cap U_{i+1} & \xrightarrow{\mathfrak{c}'_i} & PD(U_i/U_i \cap U_{i+1}) & \longrightarrow & PD(M_{i+1}^{\Psi} \cap U_i) & \longrightarrow & 0 \\ & & \| & & \| & & & & \\ & & M_i \cap U_{i+1} & \xrightarrow{\mathfrak{c}'_i} & PD(M_{i+1} \cap U_i) & & & & \end{array}$$

where $\mathfrak{c}'_i$ is defined by the same formula as $\mathfrak{c}_i$.

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Proof. For any j,k, we have $\lambda_j^{\varphi_i}-\lambda_k^{\varphi_i}-(\lambda_j^{\varphi_{i+1}}-\lambda_k^{\varphi_{i+1}})\in\{-1,0,1\}$. It follows that

$$\mathfrak{g}_{\geqslant j+1}^{\varphi_{i+1}}\subset\mathfrak{g}_{\geqslant j}^{\varphi_i}\subset\mathfrak{g}_{\geqslant j-1}^{\varphi_{i+1}}\quad\text{for all } j. \tag{7}$$

Therefore, $U_i\subset P_{i+1}$ and $U_{i+1}\subset P_i$. Hence, U_i and U_{i+1} normalize each other, so that $U_i\cdot U_{i+1}$ is a subgroup of G that contains U_i and U_{i+1} as normal subgroups. The equalities (6) are now clear since $T\subset M_i, M_{i+1}$. By (7), we have $\operatorname{Lie} M_{i+1}\cap U_i=\mathfrak{g}_{>0}^{\varphi_i}\cap\mathfrak{g}_0^{\varphi_{i+1}}\subset\mathfrak{g}_1^{\varphi_i}$. It follows that $M_{i+1}\cap U_i$ is abelian since $[\mathfrak{g}_1^{\varphi_i},\mathfrak{g}_1^{\varphi_i}]\subset\mathfrak{g}_2^{\varphi_i}$. Similarly, $M_i\cap U_{i+1}$ is abelian. The rest of the lemma follows easily from the fact that $U_{i+1}\cap M_i^{\Psi}=1$. □

Remark 3.4. If $i=nd+r$, $0\leqslant r<n$, then $U_i\cap M_{i+1}^{\Psi}$ consists of the upper unitriangular matrices whose $n\times n$ blocks $A_{j,k}$ satisfy the following requirements.

(i) $A_{j,j}=I_n$ for all j.

(ii) If $j<k$, then $A_{j,k}=0$ unless $k=d+2$, in which case $(A_{j,k})_{a,b}=0$ unless $a-b=n-r-1$ and all entries of $A_{j,k}$ along the diagonal $a-b=n-r-1$ are identical.

This group is of dimension $d+1$. (It coincides with the unipotent radical of M_{i+1}^{Ψ} unless $i+1$ is divisible by n.)

In the rest of the section we endow various unipotent subgroups of G with Haar measures. Thanks to the choice of basis $e_1,\dots,e_{mn}$, the Lie algebra of any of these unipotent groups has a natural basis as a vector space over F. Our convention will be to take the measure corresponding to the product measure where the Haar measure on F is the one which is self-dual with respect to ψ.

The following is a special case of [GGS16] (see also [GGS17]). For future reference and in order to be self-contained we provide the (elementary) proof. We refer the reader to [GGS16, GGS17] for a more thorough discussion about interplay between models.

Proposition 3.5. *For any* $i=0,\dots,(m-1)n-1$, *the map*

$$\begin{aligned}W_i\mapsto\int_{U_i\cap U_{i+1}\backslash U_{i+1}}W_i(u'\cdot)\Psi_{U_{i+1}}(u')^{-1}\,du'&=\int_{U_i\cap U'\backslash U_{i+1}\cap U'}W_i(u'\cdot)\Psi_{U_{i+1}}(u')^{-1}\,du'\\&=\int_{U_i\cap\bar N\backslash U_{i+1}\cap\bar N}W_i(u'\cdot)\,du'\end{aligned}\tag{8}$$

defines an isomorphism $\mathcal{T}_i=\mathcal{T}_i^{\psi}:\mathcal{W}^{\Psi_{U_i}}(\sigma)\to\mathcal{W}^{\Psi_{U_{i+1}}}(\sigma)$. *Its inverse is given by*

$$W_{i+1}\mapsto\int_{U_i\cap P_{i+1}^{\Psi}\backslash U_i}W_{i+1}(u\cdot)\Psi_{U_i}(u)^{-1}\,du. \tag{9}$$

In both cases the integrands are compactly supported.

Proof. For any $W_i\in\operatorname{Ind}_{U_i}^G\Psi_{U_i}$, $u\in U_i$ and $u'\in U_{i+1}$, we have $W_i(u'u)=\mathfrak{c}_i(u)(u')\Psi_{U_i}(u)W_i(u')$. It follows from Lemma 3.3 and the smoothness of W_i that $W_i|_{U_{i+1}}$ is compactly supported modulo $U_i\cap U_{i+1}$ and that, for any $u\in U_i$,

$$\Psi_{U_i}(u)^{-1}\mathcal{T}_iW_i(u)\text{ is the Fourier transform of the function } W_i\Psi_{U_{i+1}}^{-1}|_{U_{i+1}/U_i\cap U_{i+1}}\text{ at }\mathfrak{c}_i(u). \tag{10}$$

Recall that any $W_{i+1}\in\mathcal{W}^{\Psi_{U_{i+1}}}(\sigma)$ is left-M_{i+1}^{Ψ}-equivariant under a character and, in particular, is left-invariant under any unipotent subgroup of M_{i+1}^{Ψ}. Also, $U_i\cap P_{i+1}^{\Psi}=(U_i\cap M_{i+1}^{\Psi})\ltimes(U_i\cap U_{i+1})$. By similar reasoning as before, $W_{i+1}|_{U_i}$ is compactly supported modulo $U_i\cap P_{i+1}^{\Psi}$. By Lemma 3.3 and Fourier inversion, the map (9) defines a G-equivariant left inverse to $\mathcal{T}_i$. Since the spaces are irreducible, it is also a right inverse. □

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Remark 3.6. Suppose that σ is unramified, ψ has conductor $\mathcal{O}$ and $W_i \in \mathcal{W}^{\Psi_{U_i}}(\sigma)$ is the unramified vector such that $W_i(e) = 1$. Then $\mathcal{T}_i W_i(e) = 1$. This follows immediately from the proof of Proposition 3.5.

We write

$$\mathcal{T} = \mathcal{T}^{\psi} = \mathcal{T}_{(m-1)n-1} \circ \cdots \circ \mathcal{T}_0 : \mathcal{W}^{\Psi_N}(\sigma) \to \mathcal{W}^{\Psi_{U'}}(\sigma).$$

This operator was considered in [CFK18, § 2.4].

3.3 Model transition part II

We now introduce a subgroup of G that will play an important role in what follows. Let $D = D_{m,n}$ be the joint stabilizer of the vectors e_{jn}, $j = 1, \dots, m$, in G and let $N_D = N \cap D \supset N_M$. Note that $U' \subset D$. (In the case $m = 1$, D is the standard mirabolic subgroup.)

The following lemma is straightforward.

LEMMA 3.7. *We have $M_{i+1} \cap U_i = (M_{i+1} \cap U_i \cap D) \times (U_i \cap M^{\Psi}_{i+1})$. Hence, the restriction of $\mathfrak{c}_i$ to $D \cap U_i / D \cap U_i \cap U_{i+1} \simeq D \cap M_{i+1} \cap U_i$ is an isomorphism. Dually, $\mathfrak{c}'_i$ defines an isomorphism between $U_{i+1}/U_i \cap U_{i+1} \simeq M_i \cap U_{i+1}$ and the Pontryagin dual of $D \cap U_i / D \cap U_i \cap U_{i+1} \simeq D \cap M_{i+1} \cap U_i \simeq U_i \cap M_{i+1} / U_i \cap M^{\Psi}_{i+1}$.*

Hence, we can rewrite (9) as

$$W_{i+1} \mapsto \int_{D\cap U_i \cap U_{i+1}\backslash D\cap U_i} W_{i+1}(u\cdot)\Psi_{U_i}(u)^{-1}\, du = \int_{N_D\cap U_{i+1}\backslash N_D\cap U_i} W_{i+1}(u\cdot)\Psi_{U_i}(u)^{-1}\, du.$$

LEMMA 3.8. *Any $W_{\mathrm{Ze}} \in \mathrm{Ind}_N^G \Psi_N$ is compactly supported on $D \cap \bar{N}$. Hence,*

$$\mathcal{T} W_{\mathrm{Ze}} = \int_{U'\cap N\backslash U'} W_{\mathrm{Ze}}(u'\cdot)\Psi_{U'}(u')^{-1}\, du' = \int_{U'\cap \bar{N}} W_{\mathrm{Ze}}(u'\cdot)\, du' = \int_{U'\cap \bar{U}} W_{\mathrm{Ze}}(u'\cdot)\, du'$$

where the integrand is compactly supported.

Proof. Let $g = ank \in G$ with $a = \mathrm{diag}(a_1, \dots, a_{nm})$, $n \in N$ and $k \in G(\mathcal{O})$. It is well known and easy to prove that if $g \in \bar{N}$, then $\|g\| \leqslant \max_{i=1,\dots,mn} |\prod_{j=i}^{nm} a_j|$. On the other hand, it is also easy to see that if $g \in D$, then $|a_{jn}| \leqslant 1$ for $j = 1, \dots, m$. Thus, if $g \in D$ and $W_{\mathrm{Ze}}(g) \neq 0$, then by the support condition for Whittaker functions we get $|a_i| \leqslant C_1$ for all i where C_1 depends only on W_{Ze}. By the above, if moreover $g \in \bar{N}$, then $\|g\|$ is bounded in terms of W_{Ze} as required. □

Recall that any $W_i \in \mathcal{W}^{\Psi_{U_i}}(\sigma)$ is M_i^{Ψ}-equivariant with respect to some character χ_i of M_i^{Ψ} (depending only on σ). As in [CFGK19, CFK18] we can easily explicate this character.

LEMMA 3.9 (Cf. [CFGK19, Proposition 24], [CFK18, § 2.6]). *For any $i = nd + r$, $0 \leqslant r < n$, $l \in \mathrm{GL}_{d+1}$ and $t_{d+2}, \dots, t_m \in F^*$, we have*

$$\chi_i(\iota(\mathrm{diag}(l, t_{d+2}, \dots, t_m))) = \omega_\pi(t_{d+2}\dots t_m \det l)|\det l|^{-\binom{r}{2}+\binom{n}{2}(m-d-1)}|t_{d+2}|^{\binom{r}{2}(d+1)} \prod_{j=d+2}^{m} |t_j|^{n(n-1)((m+1)/2-j)}$$

where ω_π is the central character of π. In particular, for any $W_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}}(\sigma)$,

$$W_{\mathrm{Sh}}(\iota(l)g) = \omega_\pi(\det l) W_{\mathrm{Sh}}(g) \quad \forall l \in \mathrm{GL}_m,\ g \in G.$$

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Proof. It is enough to evaluate χ_i on an element $\iota(t)$ where $t = \mathrm{diag}(t_1, \dots, t_m)$ is in the diagonal torus of GL_m. Note that $\iota(t)$ lies in the center Z_M of M. Writing $W_i = \int_{\bar{N}\cap U_i} W_{\mathrm{Ze}}(u\cdot)\, du$ with $W_{\mathrm{Ze}} \in \mathcal{W}^{\Psi_N}(\sigma)$ (the integrand is compactly supported by Lemma 3.8), the required relation follows from the equality

$$\delta_P^{1/2}\delta'^{-1}(\iota(t)) = \prod_{j=1}^{m} |t_j|^{n(n-1)((m+1)/2-j)}$$
$$= \left(\prod_{j=1}^{d+1} |t_j|^{-\binom{r}{2}+\binom{n}{2}(m-d-1)}\right)\left(\prod_{j=d+2}^{m} |t_j|^{n(n-1)((m+1)/2-j)}\right)|t_{d+2}|^{\binom{r}{2}(d+1)}\delta_{Z_M\ltimes(U_i\cap\bar{N})}(\iota(t))^{-1}.$$

□

Remark 3.10. Let $\tilde{w}_m = \begin{pmatrix} & & (-1)^{m-1} \\ & \iddots & \\ 1 & & \end{pmatrix} \in \mathrm{SL}_m$ (alternating signs on the non-principal diagonal) and $\tilde{w}_{m,n} = \iota(\tilde{w}_m)$. By Lemma 3.9, we have

$$W_{\mathrm{Sh}}(\tilde{w}_{m,n}g) = W_{\mathrm{Sh}}(g)$$

for any $W_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}}(\sigma)$.

Lemma 3.11. *The inverse of $\mathcal{T}$ is given by*

$$W_{\mathrm{Sh}} \mapsto \int_{N\cap U'\backslash N_D} W_{\mathrm{Sh}}(u\cdot)\Psi_N(u)^{-1}\, du \tag{11}$$

where the integrand is compactly supported.

Proof. From Proposition 3.5 we only need to check that the integrand is compactly supported. Assume that $W_{\mathrm{Sh}} = \mathcal{T}W_{\mathrm{Ze}}$. By Remark 3.10, the integral equals

$$\int_{N\cap U'\backslash N_D} W_{\mathrm{Sh}}(\tilde{w}_{m,n}u\cdot)\Psi_N(u)^{-1}\, du = \int_{U\cap U'\backslash U_D}\left(\int_{U'\cap\bar{U}} W_{\mathrm{Ze}}(v\tilde{w}_{m,n}u\cdot)\Psi_{U'}(v)^{-1}\Psi_N(u)^{-1}\, dv\right)du$$

where $U_D = U \cap D$. The latter double integral is

$$\int_{\bar{U}\cap U'\backslash\bar{U}_D}\left(\int_{U'\cap\bar{U}} W_{\mathrm{Ze}}(v\bar{u}\tilde{w}_{m,n}\cdot)\Psi_{U'}(v)^{-1}\Psi_N(\tilde{w}_{m,n}^{-1}\bar{u}\tilde{w}_{m,n})^{-1}\, dv\right)d\bar{u}.$$

By Lemma 3.8 the integrand is compactly supported in v, $\bar{u}$. Thus, the integrand on the right-hand side of (11) is compactly supported. □

3.4 Kirillov–Shalika model

The following lemma is an analogue of [GK75, Proposition 2].

Lemma 3.12. *Any non-zero D-invariant subspace of $\mathrm{Ind}_{U'}^{D}\Psi_{U'}$ contains $\mathrm{ind}_{U'}^{D}\Psi_{U'}$. In particular, $\mathrm{ind}_{U'}^{D}\Psi_{U'}$ is irreducible.*

The proof of [GK75, pp. 110–111] (for the case $m = 1$) applies verbatim. One only needs to observe that the unipotent radical V of D is abelian, the stabilizer of the character Ψ_V under the action of D modulo V is isomorphic to $D_{m,n-1}$ and the map $p \mapsto \Psi_V(p^{-1}\cdot p)$ defines an open map from D to the Pontryagin dual of V.

Let Q be the stabilizer of $\mathrm{span}\{e_{ni} : i = 1, \dots, m\}$ in G – a maximal parabolic subgroup of G of type $((n-1)m, m)$. Thus, $Q = D \rtimes M'^{\Psi}$ and $\delta_Q|_{M'^{\Psi}} \equiv 1$.

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Corollary 3.13. *For any m-homogeneous $\sigma \in \operatorname{Irr} G$, the image of the restriction map*

$$W_{\mathrm{Sh}} \mapsto W_{\mathrm{Sh}}|_D, \quad \mathcal{W}^{\Psi_{U'}}(\sigma) \to \operatorname{Ind}_{U'}^{D} \Psi_{U'} \tag{12}$$

contains $\operatorname{ind}_{U'}^{D} \Psi_{U'}$. Equivalently, by Lemmas 2.3 and 3.9, if $\sigma = \operatorname{Sp}(\pi, m)$, then the image $\mathcal{K}^{\psi}(\sigma)$ of the restriction map

$$W_{\mathrm{Sh}} \mapsto W_{\mathrm{Sh}}|_Q, \quad \mathcal{W}^{\Psi_{U'}}(\sigma) \to \operatorname{Ind}_{P'^{\Psi}}^{Q} \omega_\pi^{\Psi}$$

contains $\operatorname{ind}_{P'^{\Psi}}^{Q} \omega_\pi^{\Psi}$ where ω_π^{Ψ} is the character of P'^{Ψ} such that $\omega_\pi^{\Psi}|_{U'} = \Psi_{U'}$ and $\omega_\pi^{\Psi} \circ \iota = \omega_\pi \circ \det$.

We will call $\mathcal{K}^{\psi}(\sigma)$ the Kirillov–Shalika model of σ.

Lemma 3.14. *For any $i = 0, \dots, (m-1)n - 1$, the map*

$$\tilde{\mathcal{T}}_i : W_i \mapsto \int_{U_i \cap U' \backslash U_{i+1} \cap U'} W_i(u' \cdot) \Psi_{U_{i+1}}(u')^{-1} \, du' = \int_{U_i \cap \bar{N} \backslash U_{i+1} \cap \bar{N}} W_i(u' \cdot) \, du'$$

is an isomorphism between $\operatorname{Ind}_{D \cap U_i}^{D} \Psi_{U_i}$ and $\operatorname{Ind}_{D \cap U_{i+1}}^{D} \Psi_{U_{i+1}}$, whose inverse is given by

$$W_{i+1} \mapsto \int_{D \cap U_i \cap U_{i+1} \backslash D \cap U_i} W_{i+1}(u \cdot) \Psi_{U_i}(u)^{-1} \, du = \int_{N_D \cap U_{i+1} \backslash N_D \cap U_i} W_{i+1}(u \cdot) \Psi_{U_i}(u)^{-1} \, du.$$

Moreover,

$$\tilde{\mathcal{T}}_i(\operatorname{ind}_{D \cap U_i}^{D} \Psi_{U_i}) = \operatorname{ind}_{D \cap U_{i+1}}^{D} \Psi_{U_{i+1}}.$$

Finally, the D-module $\operatorname{ind}_{D \cap U_i}^{D} \Psi_{U_i}$ is irreducible.

Proof. Let $W_i \in \operatorname{Ind}_{D \cap U_i}^{D} \Psi_{U_i}$. As in the proof of Proposition 3.5, by Lemma 3.7 the function

$$u \in D \cap U_i \cap U_{i+1} \backslash D \cap U_i \mapsto \Psi_{U_i}(u)^{-1} \mathcal{T}_i W_i(u)$$

is the Fourier transform of the function $W_i \Psi_{U_{i+1}}^{-1}|_{U_i \cap U' \backslash U_{i+1} \cap U'}$ at $\mathfrak{c}_i(u)$. The first claim follows by Fourier inversion.

Suppose that $W_i \in \operatorname{ind}_{D \cap U_i}^{D} \Psi_{U_i}$. From the definition (and since $U' \subset D$), $\tilde{\mathcal{T}}_i W_i$ is supported on $(D \cap U_i \cdot U_{i+1})\Omega$, where Ω is a compact subset of D. Fix $g \in \Omega$. It follows from the above that the function $\tilde{\mathcal{T}}_i W_i(g)$ is compactly supported modulo $D \cap U_i \cap U_{i+1}$. Hence, $\tilde{\mathcal{T}}_i W_i$ is compactly supported modulo $D \cap U_{i+1}$.

The last part now follows from the fact that $\operatorname{ind}_{U'}^{D} \Psi_{U'}$ is irreducible. □

From Lemma 3.14, Proposition 3.5 and Corollary 3.13 we obtain the following result.

Corollary 3.15. *For any m-homogeneous $\sigma \in \operatorname{Irr} G$ and for any $i = 0, \dots, (m-1)n$, the image of the restriction map*

$$W_i \mapsto W_i|_D, \quad \mathcal{W}^{\Psi_{U_i}}(\sigma) \to \operatorname{Ind}_{U_i \cap D}^{D} \Psi_{U_i} \tag{13}$$

contains $\operatorname{ind}_{U_i \cap D}^{D} \Psi_{U_i}$.

Once again, in analogy with the case $m = 1$ (conjectured in [GK75], proved in [BZ76, BZ77]) it is natural to make the following conjecture.

Conjecture 3.16. For any m-homogeneous $\sigma \in \operatorname{Irr} G$, the restriction map (12) (or equivalently, (13)) is injective.

We will prove a special case in Corollary 4.4 below.

We do not know whether, in general, the restriction of σ to Q is of finite length. (See Proposition 7.1 for a very special case.) Recall that in the case $m = 1$ this is known (for any $\pi \in \operatorname{Irr}$, not necessarily generic) using the theory of derivatives of Bernstein and Zelevinsky [BZ76, BZ77]. It would be very interesting to have an analogous theory for $m > 1$.

4. Unitary structure

We take the unnormalized Tamagawa measure on GL_r with respect to ψ, that is, the Haar measure associated to the standard gauge form $(\bigwedge_{i,j=1,\dots,r} dg_{i,j})/(\det g^r)$ on GL_r and the self-dual Haar measure on F with respect to ψ. Following our convention on Haar measures for unipotent groups (see § 3.2), we obtain a (right) Haar measure on the F-points of any algebraic group whose reductive part is a product of general linear groups. This will cover all algebraic groups considered here.

Throughout this section let $\pi, \pi' \in \operatorname{Irr}_{\mathrm{gen}} \mathrm{GL}_n$ and let $\sigma = \operatorname{Sp}(\pi, m)$ and $\sigma' = \operatorname{Sp}(\pi', m)$. We will work with the models considered in the previous section.

For any $0 \leqslant i \leqslant (m-1)n$ and $s \in \mathbb{C}$, we define a bilinear form on $\mathcal{W}^{\Psi_{U_i}}(\sigma) \times \mathcal{W}^{\Psi_{U_i}^{-1}}(\sigma')$ by

$$\mathcal{B}_i(W_i, W_i', s) = \int_{D \cap U_i \backslash D} W_i(g) W_i'(g) |\det g|^s \, dg$$

(assuming convergence). In particular, for $W_{\mathrm{Ze}} \in \mathcal{W}^{\Psi_N}(\sigma)$, $W'_{\mathrm{Ze}} \in \mathcal{W}^{\Psi_N^{-1}}(\sigma')$,

$$\mathcal{B}_0(W_{\mathrm{Ze}}, W'_{\mathrm{Ze}}, s) = \int_{N_D \backslash D} W_{\mathrm{Ze}}(g) W'_{\mathrm{Ze}}(g) |\det g|^s \, dg, \tag{14}$$

and for $W_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}}(\sigma)$, $W'_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}^{-1}}(\sigma')$,

$$\mathcal{B}_{\mathrm{Sh}}(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, s) := \mathcal{B}_{(m-1)n}(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, s) = \int_{U' \backslash D} W_{\mathrm{Sh}}(g) W'_{\mathrm{Sh}}(g) |\det g|^s \, dg.$$

It follows from Lemma 3.2 that $|\det|$ is bounded above on the support of $W_{\mathrm{Sh}}|_D$. Hence,

$$\begin{gathered}\text{if } \mathcal{B}_{\mathrm{Sh}}(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, s) \text{ converges absolutely at } s_0 \in \mathbb{R}, \\ \text{then it converges absolutely for any } s \text{ with } \operatorname{Re} s \geqslant s_0.\end{gathered} \tag{15}$$

A similar statement holds for any $\mathcal{B}_i$, although we will not use it.

We also write $\mathcal{B}_i(W_i, W_i') = \mathcal{B}(W_i, W_i', 0)$ assuming the latter is well defined (either as a convergent integral, or by analytic continuation), in which case it is D-invariant.

In general, we do not know whether $\mathcal{B}_i(\cdot, \cdot)$ is always defined. (See § 6 and in particular Example 6.5 for further discussion.)

Proposition 4.1. *The integral defining $\mathcal{B}_i(W_i, W_i', s)$ converges for $\operatorname{Re} s + \mathbf{e}(\pi) + \mathbf{e}(\pi') + 1 > 0$. Moreover, for all $0 \leqslant i < (m-1)n$, $W_i \in \mathcal{W}^{\Psi_{U_i}}(\sigma)$, $W_i' \in \mathcal{W}^{\Psi_{U_i}^{-1}}(\sigma')$, we have*

$$\mathcal{B}_{i+1}(\mathcal{T}_i^{\psi} W_i, \mathcal{T}_i^{\psi^{-1}} W_i', s) = \mathcal{B}_i(W_i, W_i', s). \tag{16}$$

Finally, there exist $W_i \in \mathcal{W}^{\Psi_{U_i}}(\sigma)$ and $W_i' \in \mathcal{W}^{\Psi_{U_i}^{-1}}(\sigma')$ such that $\mathcal{B}_i(W_i, W_i', s) \equiv 1$ for all $s \in \mathbb{C}$.

Remark 4.2. In Proposition 6.2 below we prove that $\mathcal{B}_i(W_i, W_i', s)$ admits meromorphic continuation in s to a rational function in q^s.

Proof. First note that the last statement follows from Corollary 3.15.

Next, we show the convergence of the integral defining $\mathcal{B}_0$. Upon twisting π and π' by $|\cdot|^{(s+\mathbf{e}(\pi')-\mathbf{e}(\pi))/2}$ and $|\cdot|^{(s+\mathbf{e}(\pi)-\mathbf{e}(\pi'))/2}$ respectively and using the inequality $|xy| \leqslant (|x|^2+|y^2|)/2$, we may assume without loss of generality that $\pi' = \overline{\pi}$, $W'_{\mathrm{Ze}} = \overline{W_{\mathrm{Ze}}}$ and $s = 0$. Thus, we need to show the convergence of

$$\int_{N_D\backslash D} |W_{\mathrm{Ze}}(g)|^2\, dg$$

provided that $\mathbf{e}(\pi) > -\frac{1}{2}$. In fact, we show a slightly stronger assertion, namely the convergence of

$$\int_{D\backslash G}\int_{N_D\backslash D} |W_{\mathrm{Ze}}(lg)|^2 dl \Phi(\eta g)|\det g|^m\, dg \tag{17}$$

for any $0 \leqslant \Phi \in \mathcal{S}(\mathrm{Mat}_{m,nm}(F))$ where $\eta \in \mathrm{Mat}_{m,nm}(F)$ is the matrix whose ith row is e_{ni}, $i = 1, \ldots, m$. Note that the stabilizer of η under the right G-action on $\mathrm{Mat}_{m,nm}(F)$ is D. Since the modulus character of D is $|\det|^m$, (17) is formally well defined and can be rewritten as

$$\begin{aligned}
&\int_{N_D\backslash G} |W_{\mathrm{Ze}}(g)|^2\Phi(\eta g)|\det g|^m\, dg\\
&\quad= \int_{P\backslash G}\int_{N\backslash P} |W_{\mathrm{Ze}}(lg)|^2 \int_{N_D\backslash N} \Phi(\eta u l g)\, du\ |\det l|^m \delta_P(l)^{-1}\, dl\ |\det g|^m\, dg\\
&\quad= \int_{P\backslash G}\int_{N_M\backslash M} |W_{\mathrm{Ze}}(lg)|^2 \int_{U_D\backslash U} \Phi(\eta u l g)\, du\ |\det l|^m \delta_P(l)^{-1}\, dl\ |\det g|^m\, dg.
\end{aligned} \tag{18}$$

We may identify the vector space $\mathrm{Mat}_{m,nm}(F)$ with $\mathrm{Mat}_{m,m}(F^n)$. Observe that, for any $l = \mathrm{diag}(g_1, \ldots, g_m) \in M$, $g \in G$, we have

$$|\det l|^{(m-1)/2} \int_{U_D\backslash U} \Phi(\eta u l g)\, du = \tilde{\Phi}_g(e_n g_1, \ldots, e_n g_m)\delta'(l) \tag{19}$$

where $\tilde{\Phi}_g \in \mathcal{S}((F^n)^m)$ is the function

$$\tilde{\Phi}_g(v_1, \ldots, v_m) = \int \Phi(Xg)\, dX, \quad v_1, \ldots, v_m \in F^n, \tag{20}$$

where the integral is taken over the $n\binom{m}{2}$-dimensional affine space of upper triangular F^n-valued $m \times m$-matrices whose diagonal entries are $v_1, \ldots, v_m$. Thus, (18) is equal to

$$\int_{P\backslash G}\int_{(N_n\backslash \mathrm{GL}_n)^m} |\delta_P^{-1/2}\delta'(l)W_{\mathrm{Ze}}(lg)|^2\tilde{\Phi}_g(e_n g_1, \ldots, e_n g_m)\prod_{i=1}^m |\det g_i|^i\ dg_1 \cdots dg_m\ |\det g|^m\, dg$$

where $l = \mathrm{diag}(g_1, \ldots, g_m) \in M$. Thus, by (5b) the inner integral is a finite linear combination of products of Rankin–Selberg integrals for $\pi \times \bar{\pi}$ at i, $i = 1, \ldots, m$. The assumption that $\mathbf{e}(\pi) > -\frac{1}{2}$ guarantees that these Rankin–Selberg integrals converge. Since the outer integral is a finite sum, we obtain the convergence of (17).

Now let $0 \leqslant i < (m-1)n$, $W_i \in \mathcal{W}^{\Psi_{U_i}}(\sigma)$, $W_i' \in \mathcal{W}^{\Psi_{U_i}^{-1}}(\sigma')$. Recall that $W_i|_{U_{i+1}}$ (respectively, $\mathcal{T}_i^{\psi} W_i|_{U_i\cap D}$) is compactly supported modulo $U_i \cap U_{i+1}$ (respectively, $U_i \cap U_{i+1} \cap D$).

Moreover, by the unitarity of the Fourier transform and the argument of Proposition 3.5 (cf. (10)) we have

$$\int_{D\cap U_{i+1}\backslash D\cap U_iU_{i+1}} \mathcal{T}_i^{\psi} W_i(u)\mathcal{T}_i^{\psi^{-1}} W_i'(u)\,du = \int_{N_D\cap U_{i+1}\backslash N_D\cap U_i} \mathcal{T}_i^{\psi} W_i(u)\mathcal{T}_i^{\psi^{-1}} W_i'(u)\,du = \int_{U_i\backslash U_iU_{i+1}} W_i(u)W_i'(u)\,du = \int_{D\cap U_i\backslash D\cap U_iU_{i+1}} W_i(u)W_i'(u)\,du \tag{21}$$

where the integrals are absolutely convergent. (We can also write the integrals as

$$\int_{U_i\cap U'\backslash U_{i+1}\cap U'} W_i(u)W_i'(u)\,du = \int_{U_i\cap \bar{N}\backslash U_{i+1}\cap \bar{N}} W_i(u)W_i'(u)\,du.)$$

It follows that if at least one of integrals

$$\int_{D\cap U_i\backslash D} |W_i(g)|^2 + |W_i'(g)|^2\,dg \quad \text{or} \quad \int_{D\cap U_{i+1}\backslash D} |\mathcal{T}_i W_i(g)|^2 + |\mathcal{T}_i W_i'(g)|^2\,dg$$

converges, then so does the other and

$$\mathcal{B}_{i+1}(\mathcal{T}_i^{\psi} W_i, \mathcal{T}_i^{\psi^{-1}} W_i') = \int_{D\cap U_iU_{i+1}\backslash D} \int_{D\cap U_{i+1}\backslash D\cap U_iU_{i+1}} \mathcal{T}_i^{\psi} W_i(ug)\mathcal{T}_i^{\psi^{-1}} W_i'(ug)\,du\,dg = \int_{D\cap U_iU_{i+1}\backslash D} \int_{D\cap U_i\backslash D\cap U_iU_{i+1}} W_i(ug)W_i'(ug)\,du\,dg = \mathcal{B}_i(W_i, W_i').$$

We can now conclude the convergence for all i and the identity (16) since they clearly reduce to the case $i = 0$. □

Theorem 4.3. *Suppose that $\pi' = \pi^\vee$ (or equivalently, $\sigma' = \sigma^\vee$) and π is (AT) (see § 2.1). Then $\mathcal{B}_i(W_i, W_i')$ is a well-defined G-invariant pairing on $\mathcal{W}^{\Psi_{U_i}}(\sigma) \times \mathcal{W}^{\Psi_{U_i}^{-1}}(\sigma^\vee)$. In particular, if $\pi \in \operatorname{Irr}_{\mathrm{gen}} \mathrm{GL}_n$ is unitarizable, then $\mathcal{B}_i$ gives a unitary structure on $\mathcal{W}^{\Psi_{U_i}}(\sigma)$.*

Proof. By Proposition 4.1 $\mathcal{B}_i(\cdot,\cdot)$ is well defined and not identically zero. To show invariance it suffices to consider $i = 0$. We use induction on m. The case $m = 1$ (in which D is the standard mirabolic subgroup) is well known and follows from Bernstein's theorem [Ber84]. For the induction step, let $m > 1$ and let Q' be the subgroup of the standard maximal parabolic subgroup of G of type $((m-1)n, n)$ consisting of the matrices whose lower right $n \times n$ corner is upper unitriangular. Write

$$\mathcal{B}_0(W_{\mathrm{Ze}}, W_{\mathrm{Ze}}') = \int_{D\cap Q'\backslash D} \int_{N_D\backslash D\cap Q'} W_{\mathrm{Ze}}(qg)W_{\mathrm{Ze}}'(qg)|\det q|^{1-n}\,dq\,dg = \int_{D\cap Q'\backslash D} \int_{D_{m-1,n}\cap N\backslash D_{m-1,n}} W_{\mathrm{Ze}}(qg)W_{\mathrm{Ze}}'(qg)|\det q|^{1-n}\,dq\,dg.$$

Here we consider $\mathrm{GL}_{(m-1)n}$ (and hence, $D_{m-1,n}$) as a subgroup of G. (Note that $\delta_D = |\det|^m$ while $\delta_{D\cap Q'} = |\det|^{n+m-1}$.) By (5a) and the induction hypothesis, the inner integral is left-(Q', $|\det|^{n-1}$)-equivariant in g. Hence, we can replace the domain of outer integration by $Q'\backslash D_{1,mn}$ where $D_{1,mn}$ is the standard mirabolic subgroup of G (the stabilizer of e_{mn}). (Note that $\delta_{D_{1,mn}} = |\det|$ and $\delta_{Q'} = |\det|^n$.) It follows that $\mathcal{B}_0(\cdot,\cdot)$ is $D_{1,mn}$-invariant. By Bernstein's theorem, it is G-invariant as required. □

We immediately deduce a special case of Conjecture 3.16.

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Corollary 4.4. *Conjecture 3.16 holds for any* $\pi \in \mathrm{Irr}_{(AT)} \mathrm{GL}_n$. *In particular, it holds for any unitarizable* $\pi \in \mathrm{Irr}_{\mathrm{gen}} \mathrm{GL}_n$.

Remark 4.5. By analytic continuation it is easy to prove Conjecture 3.16 for π of the form $\pi = \tau_1|\cdot|^{\lambda_1} \times \cdots \times \tau_k|\cdot|^{\lambda_k}$ where $\tau_i \in \mathrm{Irr}_{\mathrm{sqr}}$ are fixed and $(q^{\lambda_1}, \dots, q^{\lambda_k})$ is in general position.

In view of Theorem 4.3 and Bernstein's theorem, it is natural to make following related conjecture.

Conjecture 4.6. For any m-homogeneous $\sigma \in \mathrm{Irr}\, G$, every D-invariant bilinear form on $\sigma \times \sigma^\vee$ is G-invariant.

Perhaps even more is true.

Conjecture 4.7. For any m-homogeneous $\sigma, \sigma' \in \mathrm{Irr}\, G$, there is a unique up to scalar D-invariant bilinear form on $\sigma \times \sigma'$.

(We do not know whether this is known even in the case $m = 1$.)

5. Local zeta integrals

Throughout this section let π, $\pi' \in \mathrm{Irr}_{\mathrm{gen}} \mathrm{GL}_n$ and $\sigma = \mathrm{Sp}(\pi, m), \sigma' = \mathrm{Sp}(\pi', m) \in \mathrm{Irr}\, G$. Let $L(s, \pi \times \pi')$ and $\gamma(s, \pi \times \pi', \psi)$ be the local factors defined by Jacquet, Piatetski-Shapiro and Shalika [JPSS83]. (See § 5.2 below.)

5.1 Statement of the result

We write an analogue of the Rankin–Selberg integral for $\sigma \times \sigma'$ on the Shalika model as follows. Recall that $\eta \in \mathrm{Mat}_{m,nm}(F)$ is the matrix whose ith row is e_{ni}, $i = 1, \dots, m$, so that D is the stabilizer of η in G. For any $W_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}}(\sigma)$, $W'_{\mathrm{Sh}} \in \mathcal{W}^{\Psi^{-1}_{U'}}(\sigma')$, $\Phi \in \mathcal{S}(\mathrm{Mat}_{m,nm}(F))$, consider

$$Z(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, \Phi, s) = \int_{U' \backslash G} W_{\mathrm{Sh}}(g) W'_{\mathrm{Sh}}(g) \Phi(\eta g) |\det g|^s \, dg.$$

This expression was already considered in some form in the proof of Proposition 4.1.

Note that in the case $n = 1$ (where $U' = 1$) Z reduces to the generalized Tate integral for (a character of) GL_m considered by Godement and Jacquet [GJ72].

For any k, let

$$w_k = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & \cdot^{\cdot^{\cdot}} & \end{pmatrix} \in \mathrm{GL}_k\,.$$

Theorem 5.1. *The integral* $Z(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, \Phi, s)$ *has the following properties.*

(i) *The integral defining* $Z(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, \Phi, s)$ *is absolutely convergent for*

$$\mathrm{Re}\, s + \mathbf{e}(\pi) + \mathbf{e}(\pi') + 1 > m.$$

(ii) *The function*

$$\left(\prod_{i=0}^{m-1} L(s - i, \pi \times \pi') \right)^{-1} Z(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, \Phi, s)$$

is a Laurent polynomial in q^s, *hence entire.*

(iii) *If σ, σ' are unramified, $W_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}}(\sigma)$, $W'_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}^{-1}}(\sigma)$ are the unramified vectors such that $W_{\mathrm{Sh}}(e) = W_{\mathrm{Sh}}(e) = 1$, Φ is the characteristic function of $\mathrm{Mat}_{m,nm}(\mathcal{O})$ and ψ has conductor $\mathcal{O}$, then*

$$Z(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, \Phi, s) = c \prod_{i=0}^{m-1} L(s-i, \pi \times \pi')$$

where c is a measure-theoretic constant (depending only on F, m and n).

(iv) *We have a local functional equation*

$$Z(\widehat{W_{\mathrm{Sh}}}, \widehat{W'_{\mathrm{Sh}}}, \hat{\Phi}, m-s) = \omega_{\pi'}(-1)^{(n-1)m} \left(\prod_{i=0}^{m-1} \gamma(s-i, \pi \times \pi', \psi) \right) Z(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, \Phi, s) \quad (22)$$

where $\widehat{W_{\mathrm{Sh}}} \in \mathcal{W}^{\Psi_{U'}^{-1}}(\sigma^\vee)$ is given by $\widehat{W_{\mathrm{Sh}}}(g) = W_{\mathrm{Sh}}(w_{nm}^t g^{-1})$ and $\widehat{\Phi}$ is the Fourier transform

$$\widehat{\Phi}(X) = \int_{\mathrm{Mat}_{m,nm}(F)} \Phi(Y) \psi(\mathrm{tr}^t Y w_m X) \, dY.$$

We will prove the theorem below by relating $Z(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, \Phi, s)$ to the usual Rankin–Selberg integrals.

5.2 A result of Jacquet, Piatetski-Shapiro and Shalika

Recall the $\mathrm{GL}_n \times \mathrm{GL}_n$ local Rankin–Selberg integrals studied by Jacquet, Piatetski-Shapiro and Shalika [JPSS83]. They are given by

$$Z^{\mathrm{GL}_n}(W, W', \Phi, s) = \int_{N_n \backslash \mathrm{GL}_n} W(g) W'(g) \Phi(e_n g) |\det g|^s \, dg$$

where $W \in \mathcal{W}^{\Psi_{N_n}}(\pi)$, $W' \in \mathcal{W}^{\Psi_{N_n}^{-1}}(\pi')$, $\Phi \in \mathcal{S}(F^n)$ and $s \in \mathbb{C}$. The integral converges for $\mathrm{Re}\, s + \mathbf{e}(\pi) + \mathbf{e}(\pi') > 0$ and admits a meromorphic continuation in s to a rational function in q^s. The quotient

$$\frac{Z^{\mathrm{GL}_n}(W, W', \Phi, s)}{L(s, \pi \times \pi')}$$

is a Laurent polynomial in q^s which can be made non-zero at any given $s \in \mathbb{C}$ by an appropriate choice of W, W', Φ. Moreover, we have a functional equation

$$Z^{\mathrm{GL}_n}(\widehat{W}, \widehat{W'}, \hat{\Phi}, 1-s) = \omega_{\pi'}(-1)^{n-1} \gamma(s, \pi \times \pi', \psi) Z^{\mathrm{GL}_n}(W, W', \Phi, s)$$

where $\widehat{W} \in \mathcal{W}^{\Psi_{N_n}}(\pi^\vee)$, $\widehat{W'} \in \mathcal{W}^{\Psi_{N_n}^{-1}}(\pi'^\vee)$ are given by

$$\widehat{W}(g) = W(w_n {}^t g^{-1}), \quad \widehat{W'}(g) = W'(w_n {}^t g^{-1})$$

and $\hat{\Phi}$ is the Fourier transform of Φ given by

$$\hat{\Phi}(y) = \int_{F^n} \Phi(x) \psi(\langle x, y \rangle) \, dx$$

where $\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle = \sum_i x_i y_i$ denotes the standard pairing on F^n.

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Slightly more generally, for $W \in \mathcal{W}^{\Psi_{N_M}}(\pi^{\otimes m})$, $W' \in \mathcal{W}^{\Psi_{N_M}^{-1}}(\pi'^{\otimes m})$, $\tilde{\Phi} \in \mathcal{S}((F^n)^m)$ and $(s_1, \dots, s_m) \in \mathbb{C}^m$, we write

$$Z^M(W, W', \tilde{\Phi}, (s_1, \dots, s_m)) = \int_{N_M \backslash M} W(l) W'(l) \tilde{\Phi}(e_n g_1, \dots, e_n g_n) \prod_{i=1}^m |\det g_i|^{s_i} \, dl$$

where $l = \operatorname{diag}(g_1, \dots, g_m) \in M$. This is a linear combination of products

$$\prod_{i=1}^m Z^{\mathrm{GL}_n}(W_i, W_i', \Phi_i, s_i)$$

where $W_i \in \mathcal{W}^{\Psi_{N_n}}(\pi)$, $W_i' \in \mathcal{W}^{\Psi_{N_n}^{-1}}(\pi')$ and $\Phi_i \in \mathcal{S}(F^n)$. Thus,

$$\text{the integral defining } Z^M(W, W', \tilde{\Phi}, (s_1, \dots, s_m)) \text{ is absolutely convergent provided that } \operatorname{Re} s_i + \mathbf{e}(\pi) + \mathbf{e}(\pi') > 0 \text{ for all } i. \tag{23}$$

Moreover, we have a functional equation

$$\begin{aligned} &Z^M(\widehat{W}^M, \widehat{W'}^M, \widehat{\tilde{\Phi}}^M, (1 - s_1, \dots, 1 - s_m)) \\ &\quad = \omega_{\pi'}(-1)^{m(n-1)} \prod_{i=1}^m \gamma(s_i, \pi \times \pi', \psi) Z^M(W, W', \tilde{\Phi}, (s_1, \dots, s_m)) \end{aligned} \tag{24}$$

where $\widehat{W}^M(l) = W(\operatorname{diag}(\overbrace{w_n, \dots, w_n}^{m}) \, {}^t l^{-1})$ and

$$\widehat{\tilde{\Phi}}^M(X_1, \dots, X_m) = \int_{(F^n)^m} \Phi(Y_1, \dots, Y_m) \psi\left(\sum_i \langle X_i, Y_i \rangle\right) dY_1 \cdots dY_m.$$

5.3 Proof of the theorem

The fulcrum for Theorem 5.1 is the following proposition.

PROPOSITION 5.2. *For any* $W_{\mathrm{Ze}} \in \mathcal{W}^{\Psi_N}(\sigma)$, $W'_{\mathrm{Ze}} \in \mathcal{W}^{\Psi_N^{-1}}(\sigma')$ *and* $\Phi \in \mathcal{S}(\operatorname{Mat}_{m,nm}(F))$, *we have*

$$Z(\mathcal{T}^{\psi} W_{\mathrm{Ze}}, \mathcal{T}^{\psi^{-1}} W'_{\mathrm{Ze}}, \Phi, s) = \int_{P \backslash G} Z^M((W_{\mathrm{Ze}})_g, (W'_{\mathrm{Ze}})_g, \tilde{\Phi}_g, (s - m + 1, \dots, s)) |\det g|^s \, dg \tag{25}$$

where $(W_{\mathrm{Ze}})_g = \delta_P^{-1/2} \delta' W_{\mathrm{Ze}}(\cdot g) \in \mathcal{W}^{\Psi_{N_M}}(\pi^{\otimes m})$, $(W'_{\mathrm{Ze}})_g = \delta_P^{-1/2} \delta' W'_{\mathrm{Ze}}(\cdot g) \in \mathcal{W}^{\Psi_{N_M}^{-1}}(\pi'^{\otimes m})$ *and* $\tilde{\Phi}_g$ *is given by (20). The integral on the right-hand side is absolutely convergent for* $\operatorname{Re} s + \mathbf{e}(\pi) + \mathbf{e}(\pi') + 1 > m$.

Proof. Write $Z(\mathcal{T}^{\psi} W_{\mathrm{Ze}}, \mathcal{T}^{\psi^{-1}} W'_{\mathrm{Ze}}, \Phi, s)$ as

$$\int_{D \backslash G} \int_{U' \backslash D} \mathcal{T}^{\psi} W_{\mathrm{Ze}}(lg) \mathcal{T}^{\psi^{-1}} W'_{\mathrm{Ze}}(lg) |\det l|^{s-m} \, dl \; \Phi(\eta g) |\det g|^s \, dg.$$

By Proposition 4.1 we get

$$\begin{aligned} &\int_{D \backslash G} \int_{N_D \backslash D} W_{\mathrm{Ze}}(lg) W'_{\mathrm{Ze}}(lg) |\det l|^{s-m} \, dl \; \Phi(\eta g) |\det g|^s \, dg \\ &\quad = \int_{N_D \backslash G} W_{\mathrm{Ze}}(g) W'_{\mathrm{Ze}}(g) \Phi(\eta g) \, dg |\det g|^s \, dg. \end{aligned}$$

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We write this as

$$\int_{P\backslash G}\int_{N_M\backslash M} W_{\mathrm{Ze}}(lg)W'_{\mathrm{Ze}}(lg)\int_{U_D\backslash U}\Phi(\eta ulg)\,du\;|\det l|^s\delta_P(l)^{-1}\,dl\;|\det g|^s\,dg.$$

The required identity now follows from (19). For convergence, as in the proof of Proposition 4.1, we may assume that $\Phi \geqslant 0$, $s \in \mathbb{R}$, $\pi' = \bar{\pi}$ and $W_2 = \overline{W_1}$, so that all the integrands considered above are non-negative. Therefore, the manipulations are justified for $s + 2\mathbf{e}(\pi) + 1 > m$ by (23). $\square$

Proposition 5.2 immediately implies the first part of Theorem 5.1 (absolute convergence). In view of Remark 3.6, Proposition 5.2 also reduces the second and third parts of Theorem 5.1 (analyticity and unramified computation) to the analogous statements for the usual Rankin–Selberg integrals.

Remark 5.3. If σ and σ' are unramified, then

$$\prod_{i=0}^{m-1} L(s-i,\pi\times\pi') = L\left(s-\frac{m-1}{2},\pi\times\sigma'\right) = L\left(s-\frac{m-1}{2},\sigma\times\pi'\right).$$

However, in general for $\pi,\pi' \in \mathrm{Irr}_{\mathrm{gen}}\,\mathrm{GL}_n$, the equality

$$L\left(s-\frac{m-1}{2},\pi\times\sigma'\right) = L\left(s-\frac{m-1}{2},\sigma\times\pi'\right)$$

does not always hold.

Finally, we prove the functional equation (last part of Theorem 5.1).

For any $W_{\mathrm{Ze}} \in \mathcal{W}^{\Psi_N}(\sigma)$, define $\widehat{W_{\mathrm{Ze}}} \in \mathcal{W}^{\Psi_N^{-1}}(\sigma^\vee)$ by $\widehat{W_{\mathrm{Ze}}}(g) = W_{\mathrm{Ze}}(w_{nm}\,{}^tg^{-1})$. Then $\widehat{\mathcal{T}W_{\mathrm{Ze}}} = \mathcal{T}(\widehat{W_{\mathrm{Ze}}})$. Note that $w_{mn} = \mathrm{diag}(\overbrace{w_n,\dots,w_n}^{m})w_{m,n}$ where $w_{m,n} = \iota(w_m)$; write $g' = w_{m,n}\,{}^tg^{-1}$, $g \in G$. Then, for any $g \in G$, we have

$$(\widehat{W_{\mathrm{Ze}}})_g(l) = \widehat{(W_{\mathrm{Ze}})_{g'}}^{M}(w_{m,n}lw_{m,n}^{-1}),\quad l \in M,$$

and by Fourier inversion

$$\widetilde{(\widehat{\Phi})}_g(v_1,\dots,v_m) = |\det g|^{-m}\widehat{\tilde{\Phi}_{g'}}^{M}(v_m,\dots,v_1),\quad v_1,\dots,v_m \in F^n.$$

The last part of Theorem 5.1 therefore follows from Proposition 5.2 and the functional equation (24) using the change of variable $g \mapsto g'$ in the integral on the right-hand side of (25).

This finishes the proof of Theorem 5.1.

6. More analytic results

In this section we prove some more analytic properties of the zeta integrals defined in the previous section, as well as the bilinear forms of § 4. Some of these properties are well known in the case $m = 1$. However, there are also some new phenomena.

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6.1 Relation between zeta integrals and $\mathcal{B}_{\mathrm{Sh}}$

Recall that $Q = D \rtimes M'^{\Psi}$, $\delta_Q|_D = \delta_D = |\det|^m$ and $\delta_Q|_{M'^{\Psi}} = 1$. Hence, we can write $Z(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, \Phi, s)$ as

$$\int_{Q\backslash G}\int_{U'\backslash D}\int_{M'^{\Psi}} W_{\mathrm{Sh}}(lpg)W'_{\mathrm{Sh}}(lpg)\Phi(\eta lg)|\det l|^s\,dl\ |\det p|^{s-m}\,dp\ \ |\det g|^s\,dg.$$

Using Lemma 3.9 and the identification $\iota : \mathrm{GL}_m \to M'^{\Psi}$, we get

$$Z(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, \Phi, s) = \int_{Q\backslash G} \mathcal{B}_{\mathrm{Sh}}(W_{\mathrm{Sh}}(\cdot g), W'_{\mathrm{Sh}}(\cdot g), s-m) f_{\Phi,\omega_\pi\omega_{\pi'},s}(g)\,dg \tag{26}$$

where, for any character ω of F^*,

$$f_{\Phi,\omega,s}(g) = \int_{\mathrm{GL}_m} \Phi'_g(l)\omega(\det l)|\det l|^{ns}\,dl\ |\det g|^s$$

and $\Phi'_g \in \mathcal{S}(\mathrm{Mat}_{m,m}(F))$ is given by $\Phi'_g(X) = \Phi(\mu(X)g)$ where $\mu(X) \in \mathrm{Mat}_{m\times nm}$ is the matrix whose ith row is $\sum_{j=1}^m X_{i,j}e_{nj}$. Note that $\Phi \mapsto f_{\Phi,\omega,s}$ is an intertwining map from $\mathcal{S}(\mathrm{Mat}_{m,nm}(F)) \otimes |\det|^s$ to $\mathrm{Ind}_Q^G \nu_s$ where ν_s is the character on Q such that $\nu_s|_D = |\det|^{s-m/2}$ and $\nu_s \circ \iota = \omega^{-1} \circ \det$.

LEMMA 6.1. *There exist $W_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}}(\sigma)$, $W'_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}^{-1}}(\sigma')$ and $\Phi \in \mathcal{S}(\mathrm{Mat}_{m,nm}(F))$ such that $Z(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, \Phi, s) \equiv 1$.*

Proof. This follows from Corollary 3.13 and (26) by taking W_{Sh} such that $W_{\mathrm{Sh}}|_D$ is supported in $U'\Omega$ for a small neighborhood Ω of e and Φ supported in a small neighborhood of η. □

Let $\mathrm{ord}_{\mathcal{B}}(s) = \mathrm{ord}_{\mathcal{B};\sigma,\sigma'}(s)$ be the maximal order of pole of $\mathcal{B}_i(W_i, W'_i, \cdot)$ at s for $i = 0, \dots, (m-1)n$ as we vary $W_i \in \mathcal{W}^{\Psi_{U_i}}(\sigma)$, $W'_i \in \mathcal{W}^{\Psi_{U_i}^{-1}}(\sigma')$. (Recall that this does not depend on i by (16).) By Corollary 3.13, we have $\mathrm{ord}_{\mathcal{B}}(s) \geqslant 0$ for all s.

Similarly, let $\mathrm{ord}_Z(s) = \mathrm{ord}_{Z;\sigma,\sigma'}(s) \geqslant 0$ be the maximal order of pole of $Z(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, \Phi, \cdot)$ at s as we vary $W \in \mathcal{W}^{\Psi_{U'}}(\sigma)$, $W' \in \mathcal{W}^{\Psi_{U'}^{-1}}(\sigma')$ and $\Phi \in \mathcal{S}(\mathrm{Mat}_{m,nm}(F))$. By Lemma 6.1, we have $\mathrm{ord}_Z(s) \geqslant 0$ for all s. We can sharpen this as follows.

PROPOSITION 6.2. *The bilinear form $\mathcal{B}_i(\cdot,\cdot,s)$ on $\mathcal{W}^{\Psi_{U_i}}(\sigma) \times \mathcal{W}^{\Psi_{U_i}^{-1}}(\sigma')$ admits meromorphic continuation in s to a rational function in q^s. Moreover, for every $s \in \mathbb{C}$, we have $\mathrm{ord}_{\mathcal{B}}(s-m) \leqslant \mathrm{ord}_Z(s)$ with equality unless $\omega_\pi\omega_{\pi'} = |\cdot|^{j-ns}$ for some $j \in \{0, \dots, m-1\}$, in which case $\mathrm{ord}_Z(s) \leqslant \mathrm{ord}_{\mathcal{B}}(s-m) + 1$. In particular, if $\pi' = \pi^\vee$, then $\mathcal{B}_0(\cdot,\cdot)$ is defined if and only if $Z(\cdot,\cdot,\cdot,s)$ is holomorphic at $s = m$ for all data.*

Proof. It is enough to prove the meromorphic continuation for $i = (m-1)n$, that is, for $\mathcal{B}_{\mathrm{Sh}}$. This case follows from equality (26). Indeed, taking $\omega = \omega_\pi\omega_{\pi'}$ and Φ to be the characteristic function of a small neighborhood of η, $f_{\Phi,\omega,s}$ is supported in $Q\Omega$ for a small neighborhood Ω of e and hence $Z(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, \Phi, s)$ is a non-zero constant multiple of $\mathcal{B}_{\mathrm{Sh}}(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, s-m)$. We also get that $\mathrm{ord}_{\mathcal{B}}(s-m) \leqslant \mathrm{ord}_Z(s)$ for all s.

On the other hand, $f_{\Phi,\omega,s}(g)$ is a generalized Tate integral with respect to GL_m, and hence

$$L\left(ns - \frac{m-1}{2}, \omega \circ \det{}_{\mathrm{GL}_m}\right) f_{\Phi,\omega,s} = \left(\prod_{i=0}^{m-1} L(ns-i,\omega)\right) f_{\Phi,\omega,s}$$

is entire. We get from (26) that $\mathrm{ord}_Z(s) \leqslant \mathrm{ord}_{\mathcal{B}}(s-m)$ unless $\omega = |\cdot|^{j-ns}$ for some $j \in \{0, \dots, m-1\}$, in which case $\mathrm{ord}_Z(s) \leqslant \mathrm{ord}_{\mathcal{B}}(s-m) + 1$. The corollary follows. □

Remark 6.3. Note that if π and π' are tempered, then it follows from Theorem 5.1 part (ii) that $\mathrm{ord}_Z(s) = 0$ unless $\mathrm{Re}\, s \in \frac{1}{2}\mathbb{Z}$ and $\mathrm{Re}\, s \leqslant m$. Thus, in general, many poles of $f_{\Phi,\omega_\pi\omega_{\pi'},s}$ do not contribute a pole for $Z(\cdot,\cdot,\cdot,s)$.

Remark 6.4. In general, we do not know what precisely is the fractional ideal of $\mathbb{Z}[q^{\pm s}]$ generated by

$$Z(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, \Phi, s) \quad \text{where } W_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}}(\sigma),\ W'_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}}(\sigma'),\ \Phi \in \mathcal{S}(\mathrm{Mat}_{m,nm}(F)).$$

If both π and π' are unitarizable, then we expect that this ideal is generated by $\prod_{i=0}^{m-1} L(s-i, \pi \times \pi')$, that is, part (ii) of Theorem 5.1 is tight in this case.

Example 6.5. Consider $n = m = 2$ and $\pi = |\cdot| \times |\cdot|^{-1} \in \mathrm{Irr}_{\mathrm{gen}}\, \mathrm{GL}_2$. Then $\pi = \pi^\vee$ and $L(s, \pi \times \pi^\vee) = L(s, \mathbf{1}_{F^*})^2 L(s+2, \mathbf{1}_{F^*}) L(s-2, \mathbf{1}_{F^*})$. Therefore, $L(s, \pi \times \pi')L(s-1, \pi \times \pi')$ has a pole at $s = 2$. However, we do not know whether $Z(\cdot,\cdot,\cdot,s)$ is holomorphic at $s = 2$, or equivalently (by Proposition 6.2) whether $\mathcal{B}_0(\cdot,\cdot)$ is well defined. Recall that $\mathrm{Sp}(\pi, 2)$ is not unramified in this case (cf. Remark 2.6).

6.2 More results in the (AT) case

Proposition 6.6. *Suppose that π is (AT) and let $\pi' = \pi^\vee$. Then, for any $W_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}}(\sigma)$, $W'_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}^{-1}}(\sigma')$, we have*

$$Z(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, \Phi, m) = \mathcal{B}_{\mathrm{Sh}}(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}})\hat{\Phi}(0)$$

where both sides are well defined.

Proof. By the first part of Theorem 5.1, the integral defining $Z(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, \Phi, m)$ is absolutely convergent. Moreover, since the modulus function of D is $|\det|^m$, we can write

$$\begin{aligned} Z(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, \Phi, m) &= \int_{U'\backslash G} W_{\mathrm{Sh}}(g)W'_{\mathrm{Sh}}(g)\Phi(\eta g)|\det g|^m\, dg \\ &= \int_{D\backslash G}\int_{U'\backslash D} W_{\mathrm{Sh}}(pg)W'_{\mathrm{Sh}}(pg)\, dp\ \Phi(\eta g)|\det g|^m\, dg. \end{aligned}$$

For $\pi' = \pi^\vee$, by Theorem 4.3 we get

$$\mathcal{B}_{\mathrm{Sh}}(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}) \int_{D\backslash G} \Phi(\eta g)|\det g|^m dg = \hat{\Phi}(0)\mathcal{B}_{\mathrm{Sh}}(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}})$$

as required. ∎

From the functional equations (22) we deduce the following corollary.

Corollary 6.7. *Suppose that π is (AT) and let $\pi' = \pi^\vee$. Then $\mathrm{ord}_Z(0)$ is equal to the order of the zero of the product of γ-factors on the right-hand side of (22) at $s = 0$.*

Example 6.8. If $\pi \in \mathrm{Irr}_{\mathrm{sqr}}\, \mathrm{GL}_n$ corresponds to a segment of length k and $\pi' = \pi^\vee$, then $\mathrm{ord}_Z(0) = \min(m, k)$. Indeed, in this case $\left(\prod_{j=1}^{k} (1 - q^{f(s-j)})/(1 - q^{-f(s+j-1)})\right)\gamma(s, \pi \times \pi^\vee, \psi)$ is entire for a suitable integer $f > 0$ depending on π.

Under mild assumptions, we can give a lower bound for the real part of the first location of a pole.

Lemma 6.9. *Suppose that* ω_π *is unitary and let* $\pi' = \overline{\pi}$. *Then, for suitable* $W_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}}(\sigma)$, $W'_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}^{-1}}(\sigma^\vee)$ *and* $\Phi \in \mathcal{S}(\mathrm{Mat}_{m,nm}(F))$, $Z(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, \Phi, s)$ *has at least one pole for* $\mathrm{Re}\, s \geqslant m-1$.

Proof. Indeed, taking $W'_{\mathrm{Sh}} = \overline{W_{\mathrm{Sh}}}$ and $\Phi \geqslant 0$, the right-hand side of (25) is a power series in q^{-s} with non-negative coefficients a_k which vanish for $k \ll 0$. Assume to the contrary that $Z(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, \Phi, s)$ is holomorphic throughout $\mathrm{Re}\, s \geqslant m-1$. Then the power series would converge at $s = m-1$. However, the integral on the right-hand side of (25) diverges at $s = m-1$ since it contains $\int_{F^*} \tilde{\Phi}_g(\lambda e_n, e_n, \dots, e_n)|\lambda|^{s-m+1}\, d\lambda$ as an inner integral. We obtain a contradiction. □

Corollary 6.10. *Suppose* $n, m > 1$, $\pi' = \overline{\pi}$ *and* ω_π *is unitary. Then any* $\mathcal{B}_i$ *admits a pole in the right half plane* $\mathrm{Re}\, s \geqslant -1$. *Hence, there exists* $W_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}}(\pi)$ *such that the integral defining* $\mathcal{B}_{\mathrm{Sh}}(W_{\mathrm{Sh}}, \overline{W_{\mathrm{Sh}}}, s)$ *diverges for all* $s \leqslant -1$. *In particular,* $\int_{U'M'^\Psi\backslash G} |W_{\mathrm{Sh}}(g)|^2\, dg$ *diverges.*

Proof. Indeed, by Lemma 6.9 we have $\mathrm{ord}_Z(s) > 0$ for some s with $\mathrm{Re}\, s \geqslant m-1$. Hence, by Proposition 6.2, $\mathrm{ord}_{\mathcal{B}}(s-m) > 0$ for that s (since $n, m > 1$). Therefore, the integral defining $\mathcal{B}_{\mathrm{Sh}}(W_{\mathrm{Sh}}, \overline{W_{\mathrm{Sh}}}, s)$ diverges for all $s \leqslant -1$ (cf. (15)). In particular,

$$\int_{U'M'^\Psi\backslash G} |W_{\mathrm{Sh}}(g)|^2\, dg = \int_{Q\backslash G} \int_{U'\backslash D} |W_{\mathrm{Sh}}(g)|^2 |\det g|^{-m}\, dg$$

diverges. □

Our final result in this section is the following lemma.

Lemma 6.11. *Suppose that* π *is (AT) and let* $\pi' = \pi^\vee$. *Then* $\mathrm{ord}_Z(0) = \mathrm{ord}_{\mathcal{B}}(-m) + 1$. *In particular, if* π *is supercuspidal, then* $\mathcal{B}$ *is holomorphic at* $s = -m$ *(cf. Example 6.8). Moreover, let*

$$Z^*(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, \Phi) = \lim_{s\to 0}(q^s - 1)^{\mathrm{ord}_Z(0)} Z(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, \Phi, s)$$

and

$$\mathcal{B}^*_{\mathrm{Sh}}(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}) = \lim_{s\to -m}(q^{s+m} - 1)^{\mathrm{ord}_{\mathcal{B}}(-m)} \mathcal{B}_{\mathrm{Sh}}(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, s).$$

Then there exists a constant c *(depending only on* F, m *and* n*) such that*

$$Z^*(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, \Phi) = c\Phi(0) \int_{Q\backslash G} \mathcal{B}^*_{\mathrm{Sh}}(W_{\mathrm{Sh}}(\cdot g), W'_{\mathrm{Sh}}(\cdot g), -m)\, dg$$

for all $W_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}}(\sigma)$, $W'_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}^{-1}}(\sigma^\vee)$ *and* $\Phi \in \mathcal{S}(\mathrm{Mat}_{m,nm}(F))$.

Proof. By the local functional equation and Proposition 6.6, $Z^*(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, \Phi) = 0$ if $\Phi(0) = 0$. Therefore, the argument in the proof of Corollary 6.10 (taking Φ supported near η, which localizes $f_{\Phi,\mathbf{1}_{F^*},s}$ near Q) shows that $\mathrm{ord}_Z(0) = \mathrm{ord}_{\mathcal{B}}(-m) + 1$. Since $\mathrm{Res}_{s=0} f_{\Phi,\mathbf{1}_{F^*},s}$ is proportional to $\Phi(0)$, we get the required relation from (26). □

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We may view

$$\int_{Q\backslash G} \mathcal{B}^*_{\mathrm{Sh}}(W_{\mathrm{Sh}}(\cdot g), W'_{\mathrm{Sh}}(\cdot g), -m)\, dg$$

as a regularization of

$$\int_{M'^{\Psi}U'\backslash G} W_{\mathrm{Sh}}(g) W'_{\mathrm{Sh}}(g)\, dg.$$

(Recall that the latter diverges for $W'_{\mathrm{Sh}} = \overline{W_{\mathrm{Sh}}}$ if $m > 1$.)

7. The case $n = m = 2$

Given $\sigma = \mathrm{Sp}(\pi, m)$, it is natural to ask what is the asymptotic behavior of a function in $\mathcal{W}^{\Psi_{U'}}(\sigma)$ or (what is essentially the same thing) in $\mathcal{K}^{\psi}(\sigma)$. In the case $n = 2$ or if $n = 3$ and $m = 2$, P'^{Ψ} is a spherical subgroup of G and the problem can in principle be analyzed by the methods of [SV17]. We will only treat here the case where $m = n = 2$ and π is supercuspidal, in a self-contained way, without appealing to the general results of [SV17]. For $n > 2$ and $m > 1$ (excluding the case $n = 3$ and $m = 2$), P'^{Ψ} is no longer a spherical subgroup and the problem seems to be more difficult than the analogous problem for $\mathcal{W}^{\Psi_N}(\sigma)$. We have little to say about it.

We note that in the case where $n = 2$ and σ is unramified, an explicit formula for the unramified W_{Sh} was given by Sato [Sat05]. This is a special case of a formula of Sakellaridis [Sak06]. In general, it would be an interesting problem to obtain such an explicit formula in the unramified case for any m, n. Once again, this goes beyond the scope of [Sak13].

For the rest of this section we consider the very special case where $n = m = 2$. Fix an infinite-dimensional $\pi \in \mathrm{Irr}\,\mathrm{GL}_2$ and $\sigma = \mathrm{Sp}(\pi, 2)$. The transition map $\mathcal{T} : \mathcal{W}^{\Psi_N}(\sigma) \to \mathcal{W}^{\Psi_{U'}}(\sigma)$ is

$$W_{\mathrm{Ze}} \mapsto W_{\mathrm{Sh}} = \int_F W_{\mathrm{Ze}}(\bar{u}(x)\cdot)\, dx \quad \text{where } \bar{u}(x) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & x & 1 & \\ & & & 1 \end{pmatrix}.$$

Recall that $\pi^\vee \simeq \pi\omega_\pi^{-1}$. Let $\pi_s = \pi|\cdot|^s$. Fix a pairing $\Delta : \pi_{1/2} \otimes \pi_{-1/2} \to \mathbb{C}$ such that $\Delta(\pi_{1/2}(g)v_1 \otimes \pi_{-1/2}(g)v_2) = \omega_\pi(\det g)\Delta(v_1 \otimes v_2)$ for all $g \in \mathrm{GL}_2$. For any $v \in \pi_{1/2} \otimes \pi_{-1/2}$, let $\mathrm{MC}_v : \mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathbb{C}$ be the twisted matrix coefficient $\mathrm{MC}_v(g_1, g_2) = \Delta((\pi_{1/2}(g_1) \otimes \pi_{-1/2}(g_2))(v))$. Thus, $v \mapsto \mathrm{MC}_v$ defines an equivariant map from $\pi_{1/2} \otimes \pi_{-1/2}$ to $\mathrm{Ind}_{(Z\times Z)\,\mathrm{GL}_2^{\mathrm{diag}}}^{\mathrm{GL}_2 \times \mathrm{GL}_2} \chi$ where Z is the center of GL_2, $\mathrm{GL}_2^{\mathrm{diag}}$ is GL_2 diagonally embedded in $\mathrm{GL}_2 \times \mathrm{GL}_2$ and $\chi((\lambda_1 I_2, \lambda_2 I_2)(g, g)) = |\lambda_1/\lambda_2|\omega_\pi(\lambda_1\lambda_2 \det g)$. If π is supercuspidal, then the image is contained in $\mathrm{ind}_{(Z\times Z)\,\mathrm{GL}_2^{\mathrm{diag}}}^{\mathrm{GL}_2 \times \mathrm{GL}_2} \chi$. If π is (AT), then upon identifying $\pi_{1/2} \otimes \pi_{-1/2}$ with $\mathcal{W}^{\Psi_{N_M}}(\pi_{1/2} \otimes \pi_{-1/2})$ we may realize Δ as the convergent integral

$$\Delta(W) = \int_{F^*} W(\mathrm{diag}(1, -t, 1, t))\omega_\pi(t)^{-1}\, d^*t, \quad W \in \mathcal{W}^{\Psi_{N_M}}(\pi_{1/2} \otimes \pi_{-1/2}). \tag{27}$$

It follows from the Schur orthogonality relations that if $\pi, \pi' \in \mathrm{Irr}_{\mathrm{cusp}}\,\mathrm{GL}_2$ with $\omega_\pi\omega_{\pi'} = 1$, then π' is equivalent to $\pi^\vee$ if and only if

$$\int_{Z\backslash \mathrm{GL}_2} \mathrm{MC}_v(g, 1)\, \mathrm{MC}_{v'}(g, 1) \frac{dg}{|\det g|} \neq 0 \tag{28}$$

for some $v \in \pi_{1/2} \otimes \pi_{-1/2}$, $v' \in \pi'_{1/2} \otimes \pi'_{-1/2}$.

Recall that in the case at hand, $Q = P' = P^w$ where $w = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}$, and that ω_π^Ψ is the character of P'^Ψ whose restriction to U' is $\Psi_{U'}$ and whose composition with ι is $\omega_\pi \circ \det$. Also, $\|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\| = \max(|a|, |b|, |c|, |d|)$.

Proposition 7.1. *Suppose that $\pi \in \mathrm{Irr}_{\mathrm{cusp}} \mathrm{GL}_2$. Then we have a short exact sequence of Q-modules*

$$0 \to \mathrm{ind}_{P'^\Psi}^{Q} \omega_\pi^\Psi \to \sigma|_Q \xrightarrow{A} \pi_{1/2} \otimes \pi_{-1/2} \to 0$$

where Q acts on $\pi_{1/2} \otimes \pi_{-1/2}$ through M' (identified with $\mathrm{GL}_2 \times \mathrm{GL}_2$ via κ). Upon identifying $\sigma|_Q$ with $\mathcal{K}^\psi(\sigma)$, the map A is characterized by the property that, for any $L \in \mathcal{K}^\psi(\sigma)$, there exists $c > 0$ such that

$$L(\kappa(g_1, g_2)) = \mathrm{MC}_\varphi(g_1, g_2) \quad \text{for all } g_1, g_2 \in \mathrm{GL}_2(F) \text{ such that } \|g_2^{-1} g_1\| \leqslant c \tag{29}$$

where $\varphi = A(L)$. Moreover,

$$L(\kappa(\cdot, 1)) \text{ is compactly supported in } \{g \in \mathrm{GL}_2(F) : \|g\| \geqslant c\}. \tag{30}$$

Proof. First note that property (29) determines φ uniquely (if it exists). It then also follows that if (29) is satisfied, then A necessarily intertwines the Q-action. Moreover, if (30) is satisfied, then $\varphi = 0$ if and only if L is compactly supported modulo P'^Ψ. Also note that in relation (29) it is enough to consider $g_2 = 1$ since both sides are $(\mathrm{GL}_2^{\mathrm{diag}}, \omega_\pi \circ \det)$-equivariant. (For simplicity write $g = g_1$.)

Recall that, by Lemma 3.2, there exists a constant $C_1 > 1$ such that $L(\kappa(g, 1)) = 0$ unless $\|g\| \leqslant C_1$.

Suppose that $L = \mathcal{T}(W_{\mathrm{Ze}})|_Q$. Write $g = u'(y) \operatorname{diag}(t_1, t_2) k$ where $u'(y) = \left(\begin{smallmatrix} 1 & y \\ & 1 \end{smallmatrix}\right)$ with $k \in \mathrm{GL}_2(\mathcal{O})$. We claim that there exists C_3 such that

$$W_{\mathrm{Ze}}(\bar{u}(x)\kappa(g, 1)) = |x|^{-1} \omega_\pi(x) W_{\mathrm{Ze}}(\operatorname{diag}(1, -x^{-1}, 1, x^{-1}) \operatorname{diag}(g, I_2) w) \tag{31}$$

for all $x \in F$ such that $|x| > C_3 |t_2|$.

Indeed, write

$$\bar{u}(x) = u(x^{-1}) \operatorname{diag}(1, 1, x, x) \operatorname{diag}(1, -x^{-1}, 1, x^{-1}) w u(x^{-1}) \quad \text{where } u(y) = \begin{pmatrix} 1 & & & \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Then

$$\bar{u}(x)\kappa(g, 1) = u(x^{-1}) \operatorname{diag}(1, 1, x, x) \operatorname{diag}(1, -x^{-1}, 1, x^{-1}) w \kappa(g, 1) u(t_2 x^{-1})^{\kappa(k, 1)}$$

where the superscript denotes conjugation. Our claim follows since $w\kappa(g, 1) w^{-1} = \operatorname{diag}(g, I_2)$.

Next, we show that there exists a compact set C of $\mathrm{GL}_2(F)$ such that if $\|g\| \leqslant C_1$ and $g \notin C$, then both sides of (31) vanish if $|x| \leqslant C_3 |t_2|$.

First, note that the condition $\|g\| \leqslant C_1$ means that $|t_1|, |t_2|, |t_2 y| \leqslant C_1$. Now

$$\operatorname{diag}(1, -x^{-1}, 1, x^{-1}) \operatorname{diag}(g, I_2) \in N \operatorname{diag}(t_1, -x^{-1} t_2, 1, x^{-1}) \mathrm{GL}_4(\mathcal{O}).$$

Therefore, if the right-hand side of (31) is non-zero, then by the supercuspidality of π, x and $t_1 t_2^{-1}$ are confined to a compact subset of F^*. Since $|x| \leqslant C_3 |t_2|$, we infer that t_2 belongs to a compact set of F^*, and hence also t_1. Finally, $|y|$ is bounded since $|t_2 y| \leqslant C_1$. Hence, g belongs to a compact set.

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On the other hand,

$$\bar{u}(x)\kappa(g,1) \in N \operatorname{diag}(t_1,1,t_2,1)\bar{u}(t_2^{-1}x)K$$

and since $|t_2^{-1}x| \leqslant C_3$ we infer from the supercuspidality of π that if the left-hand side of (31) is non-zero, then t_1, t_2 belong to a compact subset of F^*. As before, g belongs to a compact set. Our claim follows.

In conclusion, (31) holds for all $x \in F$ provided that $\|g\| \leqslant C_1$ and $g \notin C$. Integrating (31) over $x \in F$, we conclude that if $\|g\| \leqslant C_1$ and $g \notin C$, then

$$L(\kappa(g,1)) = \int_{F^*} \omega_\pi(t^{-1}) W_{\mathrm{Ze}}(\operatorname{diag}(1,-t,1,t)\operatorname{diag}(g,I_2)w)\, d^*t.$$

By (27), this is equal to $\mathrm{MC}_\varphi(g,1)$ where $\varphi \in \mathcal{W}^{\Psi_{N_M}}(\pi_{1/2} \otimes \pi_{-1/2})$ is the restriction of $W_{\mathrm{Ze}}(\cdot w)$ to M. Thus, (29) and (30) hold. In view of Corollary 3.13, this proves the proposition. (Note that $W_{\mathrm{Ze}} \mapsto \varphi$ is Q-equivariant since w conjugates P to Q.) □

Remark 7.2. It follows from (the proof of) Proposition 7.1 that there exists a non-zero $W_{\mathrm{Ze}} \in \mathcal{W}^{\Psi_N}(\sigma)$ that vanishes on M (in which case $\mathcal{T}W_{\mathrm{Ze}}(\cdot w)|_Q \in \mathcal{K}^\psi(\sigma)$ is compactly supported modulo P'^Ψ). This can be also shown directly by realizing σ as the image of the intertwining operator

$$\pi_{1/2} \times \pi_{-1/2} \to \pi_{-1/2} \times \pi_{1/2}$$

and taking the image of a suitable vector in $\pi_{1/2} \times \pi_{-1/2}$ that is supported in the big cell.

Corollary 7.3. *Suppose that $\pi \in \operatorname{Irr}_{\mathrm{cusp}} \mathrm{GL}_2$ and let $\pi' = \pi^\vee$. Then the poles of the bilinear form $\mathcal{B}_i(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, s)$, as we vary $W_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}}(\sigma)$ and $W'_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}^{-1}}(\sigma')$, coincide with those of $L(s+1, \pi \times \pi^\vee)$.*

Proof. We may assume without loss of generality that π is unitary. Then

$$\mathcal{B}(W_{\mathrm{Sh}}, \overline{W_{\mathrm{Sh}}}, s-1) = \int_{\mathrm{GL}_2} |\det g|^{s-1} |W_{\mathrm{Sh}}(\kappa(g,1))|^2\, dg.$$

By Proposition 7.1, the analytic properties are governed by those of

$$\int_{\mathrm{GL}_2 : \|g\| \leqslant 1} |\det g|^{s-1} |\mathrm{MC}_\varphi(g,1)|^2\, dg,$$

which can be written as

$$\int_{Z\backslash \mathrm{GL}_2} \left(\int_{F^*:|\lambda| \leqslant \|g\|^{-1}} |\lambda|^{2s}\, d^*\lambda \right) |\mathrm{MC}_\varphi(g,1)|^2 |\det g|^{s-1}\, dg$$
$$= \frac{1-q^{-1}}{1-q^{-2s}} \int_{Z\backslash \mathrm{GL}_2} \|g\|^{-2s} |\mathrm{MC}_\varphi(g,1)|^2 |\det g|^{s-1}\, dg.$$

Thus, the poles are simple and are confined to $q^{2s}=1$. If $q^s = 1$, then the residue is clearly non-zero. If $q^s = -1$, then the residue is a constant multiple of

$$\int_{Z\backslash \mathrm{GL}_2} |\mathrm{MC}_\varphi(g,1)|^2 \omega(\det g) |\det g|^{-1}\, dg$$

where ω is the non-trivial quadratic unramified character of F^*. Thus, by (28), the residue is non-zero if and only if $\pi \simeq \pi\omega$. This matches exactly with the poles of $L(s, \pi \times \pi^\vee)$ [JPSS83, Proposition 8.1]. □

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8. Global heuristics

Let F be a number field with ring of adeles $\mathbb{A}$. We consider $G = \mathrm{GL}_{nm}$ as a group over F and write $G(\mathbb{A})^1 = \{g \in G(\mathbb{A}) : |\det g| = 1\}$. As before, let Q be the stabilizer of $\mathrm{span}\{e_{ni} : i = 1, \dots, m\}$ in G – a maximal non-standard parabolic subgroup of G of type $((n-1)m, m)$. For any $\Phi \in \mathcal{S}(\mathrm{Mat}_{m,nm}(\mathbb{A}))$ and a Hecke character ω of $F^*\backslash\mathbb{A}^*$, consider the degenerate normalized Eisenstein series that is given by

$$\begin{aligned}\mathcal{E}_{\Phi,\omega}(g,s) &= \int_{\mathrm{GL}_m(F)\backslash\mathrm{GL}_m(\mathbb{A})} \sum_{\gamma\in M_{m,mn}(F):\mathrm{rk}\,\gamma=m} \Phi(z^{-1}\gamma g)|\det z|^{-ns}\omega(\det z)^{-1}|\det g|^s\,dz\\ &= \sum_{\gamma\in Q(F)\backslash G(F)} f_{\Phi,\omega,s}(\gamma g)\end{aligned}$$

for $\mathrm{Re}\, s \gg 0$ (more precisely, $\mathrm{Re}\, s > m$ if ω is unitary) where, as in § 6.1,

$$f_{\Phi,\omega,s}(g) = \int_{\mathrm{GL}_m(\mathbb{A})} \Phi'_g(l)\omega(\det l)|\det l|^{ns}\,dl\ |\det g|^s$$

and $\Phi'_g \in \mathcal{S}(\mathrm{Mat}_{m,m}(\mathbb{A}))$ is given by $\Phi'_g(X) = \Phi(\mu(X)g)$, where $\mu(X) \in \mathrm{Mat}_{m,nm}(\mathbb{A})$ is the matrix whose ith row is $\sum_{j=1}^m X_{i,j}e_{nj}$. By the method of Tate's thesis (which goes back to Riemann), $\mathcal{E}_{\Phi,\omega}$ admits a meromorphic continuation with finitely many (simple) poles and a functional equation

$$\mathcal{E}_{\Phi,\omega}(g,s) = \mathcal{E}_{\hat{\Phi},\omega^{-1}}({}^t g^{-1}, m-s).$$

As before, let $P = M \ltimes U$ be the standard maximal parabolic subgroup of G of type $(\overbrace{n,\dots,n}^{m})$ and let $|\cdot|_M : M(\mathbb{A}) \to \mathbb{R}^m_{>0}$ be the homomorphism

$$|\mathrm{diag}(l_1,\dots,l_m)| = (|\det l_1|,\dots,|\det l_m|).$$

We extend $|\cdot|_M$ to a left-$U(\mathbb{A})$- and right-K-invariant function $|\cdot|_P$ on $G(\mathbb{A})$ where K is the standard maximal compact subgroup of $G(\mathbb{A})$. For any $x = (x_1,\dots,x_m) \in \mathbb{R}^m_{>0}$ and $\lambda = (\lambda_1,\dots,\lambda_m) \in \mathbb{C}^m$, we write $x^\lambda = \prod_i x_i^{\lambda_i}$.

Let $\pi = \otimes\pi_v$ be an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$. Let $\phi : G(\mathbb{A}) \to \mathbb{C}$ be a smooth function such that, for all $g \in G(\mathbb{A})$, the function $l \in M(\mathbb{A}) \mapsto \delta_P(l)^{-1/2}\phi(lg)$ belongs to the space of $\overbrace{\pi\otimes\cdots\otimes\pi}^{m}$. The Eisenstein series

$$E(\phi,\lambda,g) = \sum_{\gamma\in P(F)\backslash G(F)} \phi(\gamma g)|\gamma g|_P^\lambda$$

converges if $\mathrm{Re}(\lambda_i - \lambda_{i+1}) > n$ for all $i = 1,\dots,m-1$ and admits a meromorphic continuation to $\mathbb{C}^m$. The limit

$$\varphi(g) = \lim_{\lambda\to((m-1)/2,\dots,(1-m)/2)} (\lambda_1-\lambda_2-1)\cdots(\lambda_{m-1}-\lambda_m-1)E(\phi,\lambda,g) \tag{32}$$

exists and is a square-integrable automorphic form on $G(F)\backslash G(\mathbb{A})^1$ which is non-zero for a suitable ϕ. As we vary ϕ, we obtain an irreducible automorphic representation of $G(\mathbb{A})$ whose local components are $\mathrm{Sp}(\pi_v, m)$. (It is well known that as we vary over π and $m \geqslant 1$, these representations furnish the entire automorphic discrete spectrum of the general linear group

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[MW89].) Similarly, let π' be another irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ and let ϕ' and φ' be analogous functions with respect to π'.

Formally, we would have liked to consider the integral

$$\int_{G(F)\backslash G(\mathbb{A})^1} \varphi(g)\varphi'(g)\mathcal{E}_{\Phi,\omega}(g,s)\,dg \tag{33}$$

where $\omega = \omega_\pi\omega_{\pi'}$. For $m = 1$, this is of course the classical Rankin–Selberg integral. Unfortunately, for $m > 1$ this integral does not converge as none of the functions that appear in the integrand is rapidly decreasing. A suitable regularization (in the spirit of [Zag81] or later accounts) is therefore needed in order to make sense of (33). We will not pursue this matter here. Instead, we will be content with a *purely heuristic* argument, anticipating what a possible regularization of (33) would yield.

As in the case $m = 1$, we unfold (formally) expression (33). For any $i = 1,\dots,m$, let $Q_i = L_i \ltimes V_i$ be the stabilizer of the flag

$$(\mathrm{span}\{e_{nj-k} : j = 1,\dots,m,\ k = 0,\dots,r-1\})_{r=1,\dots,i}$$

in G. Thus, $Q_1 = Q \supset Q_2 \supset \cdots \supset Q_{n-1} = Q_n = P'$ and $L_i \simeq \mathrm{GL}_{m(n-i)} \times L'_i$ with $L'_i \simeq \overbrace{\mathrm{GL}_m \times \cdots \times \mathrm{GL}_m}^{i}$. Let $p_i : Q_i \to L'_i$ be the resulting projection and let Q'_i be the inverse image of GL_m diagonally embedded in L'_i. In particular, $Q'_1 = Q_1 = Q$ and $Q'_n = M'^{\Psi} \ltimes U'$. Note that, for all $i = 1,\dots,n-1$, Q'_{i+1} is the stabilizer in Q_i of the character Ψ_{V_i} and V_i/V_{i-1} is abelian (and can be identified with $\mathrm{Mat}_{m,(n-i)m}$), where for consistency we let $V_0 = 0$.

In the first step we unfold (33) to write it as

$$\int_{Q(F)\backslash G(\mathbb{A})^1} \varphi(g)\varphi'(g)f_{\Phi,\omega,s}(g)\,dg = \int_{Q_1(F)\backslash G(\mathbb{A})^1}\int_{V_1(F)\backslash V_1(\mathbb{A})} \varphi(vg)\varphi'(vg)\,dv\ f_{\Phi,\omega,s}(g)\,dg$$

and expand

$$\int_{V_1(F)\backslash V_1(\mathbb{A})} \varphi(vg)\varphi'(vg)\,dv = \sum_{\chi\in PD(V_1(F)\backslash V_1(\mathbb{A}))} \varphi^{V_1,\chi}(g)\varphi'^{V_1,\chi^{-1}}(g)$$

where

$$\varphi^{V_1,\chi}(g) = \int_{V_1(F)\backslash V_1(\mathbb{A})} \varphi(vg)\chi(v)^{-1}\,dv.$$

The Pontryagin dual of the compact abelian group $V_1(F)\backslash V_1(\mathbb{A})$ is isomorphic to $\mathrm{Mat}_{m,(n-1)m}(F)$. We consider only the contribution from the non-degenerate χ, that is, those corresponding to matrices of rank m (anticipating that the degenerate ones will not contribute, either by the cuspidality of π or by the regularization procedure itself). The non-degenerate characters form a single orbit under $Q = Q_1$, namely the orbit of Ψ_{V_1}, and the stabilizer of Ψ_{V_1} is Q'_2. We thus get

$$\int_{Q'_2(F)\backslash G(\mathbb{A})^1} \varphi^{V_1,\Psi_{V_1}}(g)\varphi'^{V_1,\Psi_{V_1}^{-1}}(g)f_{\Phi,\omega,s}(g)\,dg$$

which we write as

$$\int_{Q'_2(F)\backslash G(\mathbb{A})^1}\int_{V_2(F)\backslash V_2(\mathbb{A})} \varphi^{V_1,\Psi_{V_1}}(ug)\varphi'^{V_1,\Psi_{V_1}^{-1}}(ug)\,du\ f_{\Phi,\omega,s}(g)\,dg.$$

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Once again, we expand the inner integral according to characters of the compact abelian group $V_2(\mathbb{A})/V_1(\mathbb{A})V_2(F)$ and consider only the non-degenerate characters. Continuing in this way, we get, for $k = 1, \dots, n$,

$$\int_{Q'_k(F)\backslash G(\mathbb{A})^1} \varphi^{V_{k-1},\Psi_{V_{k-1}}}(g)\varphi'^{V_{k-1},\Psi_{V_{k-1}}^{-1}}(g) f_{\Phi,\omega,s}(g)\, dg.$$

For $k = n$, we obtain

$$\int_{M'^{\Psi}(F)U'(\mathbb{A})\backslash G(\mathbb{A})^1} \varphi^{U',\Psi_{U'}}(g)\varphi'^{U',\Psi_{U'}^{-1}}(g) f_{\Phi,\omega,s}(g)\, dg.$$

Now $\varphi^{U',\Psi_{U'}}$ is $(M'^{\Psi}(\mathbb{A}), \omega_\pi \circ \det)$-equivariant (taking into account the identification $\iota : \mathrm{GL}_m \to M'^{\Psi}$). Therefore, up to a volume factor, we get

$$\int_{M'^{\Psi}(\mathbb{A})U'(\mathbb{A})\backslash G(\mathbb{A})} \varphi^{U',\Psi_{U'}}(g)\varphi'^{U',\Psi_{U'}^{-1}}(g) f_{\Phi,\omega,s}(g)\, dg. \tag{34}$$

This integral (which actually converges for $\mathrm{Re}\, s > m$ if ω is unitary) is Eulerian. Let S be a finite set of places of F containing all the archimedean ones such that, for all $v \notin S$, φ and φ' are $G(\mathcal{O}_v)$-invariant (and, in particular, π_v and π'_v are unramified), ψ_v has conductor $\mathcal{O}_v$, Φ is invariant under translation by $\mathrm{Mat}_{m,mn}(\mathcal{O}_v)$ and $\Phi(X) = 0$ unless $X_v \in \mathrm{Mat}_{m,mn}(\mathcal{O}_v)$. Using (26) and Theorem 5.1 part (iii), up to a measure-theoretic constant, the integral (34) is equal to

$$\left(\prod_{i=0}^{m-1} L^S(s-i, \pi\times\pi')\right) Z_S(\varphi^{U',\Psi_{U'}}|_{G(F_S)}, \varphi'^{U',\Psi_{U'}^{-1}}|_{G(F_S)}, \Phi|_{\mathrm{Mat}_{m,mn}(F_S)}, s)$$

where $L^S(s, \pi\times\pi')$ is the partial Rankin–Selberg L-function and, for any $W_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}}(\mathrm{Sp}(\pi_S, m))$ and $W'_{\mathrm{Sh}} \in \mathcal{W}^{\Psi_{U'}^{-1}}(\mathrm{Sp}(\pi'_S, m))$,

$$Z_S(W_{\mathrm{Sh}}, W'_{\mathrm{Sh}}, \Psi, s) = \int_{U'(F_S)\backslash G(F_S)} W_{\mathrm{Sh}}(g) W'_{\mathrm{Sh}}(g)\Phi(\eta g)|\det g|^s\, dg,$$

which is essentially the product over $v \in S$ of the integrals considered in § 5. (We tacitly assume that much of the analysis of §§ 3–5 carries over to the archimedean case.)

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Appendix A. Relation to intertwining operators

For this appendix assume that $\pi \in \mathrm{Irr}_{(AT)}\,\mathrm{GL}_n$. Let

$$\Pi = \pi|\cdot|^{(m-1)/2} \times \pi|\cdot|^{(m-3)/2} \times \cdots \times \pi|\cdot|^{(1-m)/2}$$

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be the standard module which admits $\sigma = \mathrm{Sp}(\pi, m)$ as the Langlands quotient. We realize Π in the subspace $\mathcal{W}^{\Psi_N}(\Pi)$ of $\mathrm{Ind}_N^G \Psi_N$ consisting of functions W such that $l \in M \mapsto \delta_P^{-1/2}(l)\delta'^{-1}(l)W(lg) \in \mathcal{W}^{\Psi_{N_M}}(\pi^{\otimes m})$ for all $g \in G$. Define an intertwining operator on $\mathcal{W}^{\Psi_N}(\Pi)$ by

$$W \mapsto MW(\cdot) = \int_U W(\tilde{w}_{m,n}^{-1} u \cdot)\, du \tag{A.1}$$

where $\tilde{w}_{m,n}$ is as in Remark 3.10. The integral defining MW is absolutely convergent and its image is $\mathcal{W}^{\Psi_N}(\sigma)$. Similarly, define $\mathcal{W}^{\Psi_N^{-1}}(\Pi^\vee) \simeq \Pi^\vee$ to be the subspace of $\mathrm{Ind}_N^G \Psi_N^{-1}$ consisting of functions $W^\vee$ such that $l \in M \mapsto \delta_P^{-1/2}(l)\delta'(l)W^\vee(lg) \in \mathcal{W}^{\Psi_{N_M}^{-1}}((\pi^\vee)^{\otimes m})$ for all $g \in G$. Then the bilinear form

$$\langle W, W^\vee \rangle = \int_{P\backslash G} \int_{N_M \backslash D_M} \delta_P(l)^{-1} W(lg) W^\vee(lg)\, dl\, dg, \quad W \in \mathcal{W}^{\Psi_N}(\Pi),\ W^\vee \in \mathcal{W}^{\Psi_N^{-1}}(\Pi^\vee),$$

converges absolutely and defines a G-invariant pairing on $\mathcal{W}^{\Psi_N}(\Pi) \times \mathcal{W}^{\Psi_N^{-1}}(\Pi^\vee)$ where $D_M = D \cap M$ is the product of m copies of the mirabolic subgroup of GL_n. Since $\mathcal{W}^{\Psi_N^{-1}}(\sigma^\vee)$ is the socle of $\mathcal{W}^{\Psi_N^{-1}}(\Pi^\vee)$, for any $W_{\mathrm{Ze}}^\vee \in \mathcal{W}^{\Psi_N^{-1}}(\sigma^\vee)$ the linear form $W \mapsto \langle W, W_{\mathrm{Ze}}^\vee \rangle$ factors through MW and it is a scalar multiple (independent of W) of $\mathcal{B}_0(MW, W_{\mathrm{Ze}}^\vee)$. In the rest of the appendix we prove the following identity.

Proposition A.1. *For any $W \in \mathcal{W}^{\Psi_N}(\Pi)$ and $W_{\mathrm{Ze}}^\vee \in \mathcal{W}^{\Psi_N^{-1}}(\sigma^\vee)$, we have*

$$\langle W, W_{\mathrm{Ze}}^\vee \rangle = \mathcal{B}_0(MW, W_{\mathrm{Ze}}^\vee). \tag{A.2}$$

The identity will follow from a series of identities proved below.

For $i = 1, \dots, m-1$, let U^i be the unipotent radical of the standard parabolic subgroup P^i of G of type $(in, n, n, \dots, n)$. Let $\bar{U}^i = {}^t U^i$ be its opposite.

Lemma A.2. *Let $\tau \in \mathrm{Irr}_{\mathrm{gen}} \mathrm{GL}_n$. Then, for any $W_{\mathrm{Ze}} \in \mathcal{W}^{\Psi_N}(\mathrm{Sp}(\tau, m))$, we have*

$$\int_{D \cap \bar{U}^i} W_{\mathrm{Ze}}(\bar{u})\, d\bar{u} = W_{\mathrm{Ze}}(\iota(\hat{w}_i)) \tag{A.3}$$

where the integrand on the left-hand side is compactly supported. Here

$$\hat{w}_i = \begin{pmatrix} & & & (-1)^{m-1} \\ & & \iddots & \\ & (-1)^i & & \\ I_i & & & \end{pmatrix} = \begin{pmatrix} I_{m-i} & \\ & \tilde{w}_i^{-1} \end{pmatrix} \tilde{w}_m \in \mathrm{SL}_m$$

where the signs on the upper right $(m-i) \times (m-i)$-corner are alternating. Thus,

$$\int_{D \cap \bar{U}^i} W_{\mathrm{Ze}}(\bar{u}\bar{v})\, d\bar{u} = \int_{D \cap \bar{U}^i} W_{\mathrm{Ze}}(\bar{u})\, d\bar{u}$$

for all $\bar{v} \in \bar{U}^i$. In particular, for $i = 1$,

$$\int_{\bar{U}_D} W_{\mathrm{Ze}}(\bar{u})\, d\bar{u} = W_{\mathrm{Ze}}(\tilde{w}_{m,n})$$

where $\bar{U}_D = \bar{U} \cap D$.

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Proof. Let $W^i_{\mathrm{Sh}} = \mathcal{T}_{(i-1)n} W_{\mathrm{Ze}} \in \mathcal{W}^{\Psi_{U_{(i-1)n}}}(\mathrm{Sp}(\tau, m))$. Recall that $U_{(i-1)n}$ is the subgroup of P^i consisting of matrices whose $n \times n$ blocks $A_{j,k}$ satisfy:

- $A_{j,j}$ is upper unitriangular for all $j = 1, \dots, m$;
- $A_{j,k}$ is strictly upper triangular if $j \neq k$ and $j, k \leqslant i$;
- $A_{j,k} = 0$ if $j > k$ and $j > i$.

(There are no conditions on $A_{j,k}$ if $k > j$ and $k > i$.)

The inverse transform in Proposition 3.5 gives

$$W_{\mathrm{Ze}}(g) = \int_{N \cap U_{(i-1)n} \backslash N_D} W^i_{\mathrm{Sh}}(ug)\, du.$$

We may replace the domain of integration by $(N \cap U_{(i-1)n} \cap \mathrm{GL}_{in}) \backslash (N_D \cap \mathrm{GL}_{in})$ where GL_{in} is embedded in G by $h \mapsto \left(\begin{smallmatrix} h & \\ & I_{(m-i)n} \end{smallmatrix}\right)$. Let $U'_i = U_{(i-1)n} \cap \mathrm{GL}_{in} = U' \cap \mathrm{GL}_{in}$ and $D_i = D \cap \mathrm{GL}_{in}$. Thus, the above integral can be taken over $N \cap U'_i \backslash N \cap D_i$, and by Lemma 3.11 the integrand is compactly supported.

The expression on the left-hand side of (A.3) is

$$\int_{D \cap \bar{U}^i} \int_{N \cap U'_i \backslash N \cap D_i} W^i_{\mathrm{Sh}}(u\bar{u})\, du\, d\bar{u}.$$

The same argument as in Lemma 3.8 shows that the function $W^i_{\mathrm{Sh}}(u\bar{u})$ is compactly supported in $\bar{u}$ uniformly in u. Thus, the above double integral is absolutely convergent. Changing the order of integration and making a change of variable in $\bar{u}$, we get

$$\int_{N \cap U'_i \backslash N \cap D_i} \int_{D \cap \bar{U}^i} W^i_{\mathrm{Sh}}(\bar{u}u)\, d\bar{u}\, du.$$

Notice that the partial integration over $U' \cap \bar{U}^i \subset D \cap \bar{U}^i$ is the composition of the transforms $\mathcal{T}_j$ defined in Proposition 3.5 for $j = (i-1)n, \dots, (m-1)n - 1$. Thus, the above expression is

$$\int_{N \cap U'_i \backslash N \cap D_i} \int_{U' \cap \bar{U}^i \backslash D \cap \bar{U}^i} W_{\mathrm{Sh}}(\bar{u}u)\, d\bar{u}\, du$$

where $W_{\mathrm{Sh}} = \mathcal{T} W_{\mathrm{Ze}}$. By Lemma 3.9, $W_{\mathrm{Sh}}(\iota(\hat{w}_i)g) = W_{\mathrm{Sh}}(g)$. The above expression becomes

$$\int_{N \cap U'_i \backslash N \cap D_i} \int_{U' \cap \bar{U}^i \backslash D \cap \bar{U}^i} W_{\mathrm{Sh}}(\iota(\hat{w}_i)\bar{u}u)\, d\bar{u}\, du = \int_{N \cap U' \backslash N_D} W_{\mathrm{Sh}}(u\iota(\hat{w}_i))\, du.$$

Now Lemma 3.11 gives (A.3). For the second part, we only need to note that, for all $\bar{v} \in \bar{U}^i$, we have $W_{\mathrm{Ze}}(\iota(\hat{w}_i)\bar{v}) = W_{\mathrm{Ze}}(\iota(\hat{w}_i))$. □

Write $\bar{U}$ as a (semidirect) product of abelian groups $\bar{U}_2 \bar{U}_3 \cdots \bar{U}_m$, where $\bar{U}_i$ consists of the elements $\bar{u}$ in $\bar{U}$ such that $\bar{u}_{j,k} = \delta_{j,k}$ if $j \leqslant n(i-1)$ or $j > ni$. For brevity, for any $i = 1, \dots, m$, we denote the iterated integral

$$\int_{\bar{U}_2 \cap D} \left(\int_{\bar{U}_3 \cap D} \cdots \left(\int_{\bar{U}_i \cap D} f(\bar{u}_i \cdots \bar{u}_3 \bar{u}_2)\, d\bar{u}_i \right) \cdots d\bar{u}_3 \right) d\bar{u}_2$$

(assuming convergence) by

$$\int^{\mathrm{it}}_{\bar{U} \cap D_i} f(\bar{u})\, du.$$

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Lemma A.3. *Let W_{Ze} be as before and let $\phi \in \mathcal{S}(\bar{U})$. Then*

$$\int_{\bar{U}_D}^{\mathrm{it}} \int_{\bar{U}} \phi(\bar{u}) W_{\mathrm{Ze}}(\bar{u}\bar{v})\, d\bar{u}\, d\bar{v} = W_{\mathrm{Ze}}(\tilde{w}_{m,n}) \int_{\bar{U}} \phi(\bar{u})\, du$$

where each integrand in the iterated integral on the left-hand side is compactly supported.

Proof. We show by descending induction on $i = 1, \dots, m$ that the left-hand side is equal to

$$\int_{\bar{U}\cap D_i}^{\mathrm{it}} \int_{\bar{U}^i \cap D} \int_{\bar{U}} \phi(\bar{u}) W_{\mathrm{Ze}}(\bar{v}\bar{u}\bar{v}')\, d\bar{u}\, d\bar{v}\, d\bar{v}'. \tag{A.4}$$

Note that the integrand is compactly supported in $\bar{u}$ and $\bar{v}$. For $i = m$ this is clear, while for $i = 1$ we obtain the statement of the lemma by Lemma A.2.

For the induction step, we assume $i > 1$ and use Lemma A.2 to rewrite (A.4) as

$$\int_{\bar{U}\cap D_i}^{\mathrm{it}} \int_{\bar{U}^i \cap D} \int_{\bar{U}} \phi(\bar{u}) W_{\mathrm{Ze}}(\bar{v} p_i(\bar{u}) \bar{v}')\, d\bar{u}\, d\bar{v}\, d\bar{v}'$$

where $p_i : \mathrm{GL}_{in} \ltimes \bar{U}^i \to \mathrm{GL}_{in}$ is the projection. Now write $\bar{v}' = \bar{v}_1 \bar{v}_2$ where $\bar{v}_1 \in \bar{U}_i \cap D$ and $\bar{v}_2 \in \bar{U} \cap D_{i-1}$ and note that $p_i(\bar{u}) \in \mathrm{GL}_{in} \cap \bar{U}$ normalizes $\bar{U}_i \cap D$. Therefore, (A.4) is equal to

$$\begin{aligned}
&\int_{\bar{U}\cap D_{i-1}}^{\mathrm{it}} \int_{\bar{U}_i \cap D} \int_{\bar{U}^i \cap D} \int_{\bar{U}} \phi(\bar{u}) W_{\mathrm{Ze}}(\bar{v}\bar{v}_1 p_i(\bar{u}) \bar{v}_2)\, d\bar{u}\, d\bar{v}\, d\bar{v}_1\, d\bar{v}_2 \\
&\quad = \int_{\bar{U}\cap D_{i-1}}^{\mathrm{it}} \int_{\bar{U}^{i-1} \cap D} \int_{\bar{U}} \phi(\bar{u}) W_{\mathrm{Ze}}(\bar{v} p_i(\bar{u}) \bar{v}')\, d\bar{u}\, d\bar{v}\, d\bar{v}' \\
&\quad = \int_{\bar{U}\cap D_{i-1}}^{\mathrm{it}} \int_{\bar{U}^{i-1} \cap D} \int_{\bar{U}} \phi(\bar{u}) W_{\mathrm{Ze}}(\bar{v}\bar{u}\bar{v}')\, d\bar{u}\, d\bar{v}\, d\bar{v}'
\end{aligned}$$

as required. □

Denote by $\delta_{\bar{U}_D}(l)$ the character of D_M given by $d(l\bar{u}l^{-1}) = \delta_{\bar{U}_D}(l)\, d\bar{u}$ where $d\bar{u}$ is a Haar measure on $\bar{U}_D$.

Let $\mathcal{W}^{\Psi_N}(\Pi)_\sharp$ be the linear subspace of $\mathcal{W}^{\Psi_N}(\Pi)$ generated by the functions W of the form

$$W(g) = \begin{cases} \delta_P(l)^{1/2} \delta'(l) W'(l) \phi(\bar{u}) & \text{if } g = ul\bar{u},\ u \in U,\ l \in M,\ \bar{u} \in \bar{U}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\phi \in \mathcal{S}(\bar{U})$, $W' \in \mathcal{W}^{\Psi_{N_M}}(\pi^{\otimes m})$ and $W'|_{D_M}$ is compactly supported modulo N_M.

Lemma A.4. *For any $W \in \mathcal{W}^{\Psi_N}(\Pi)_\sharp$ and $W_{\mathrm{Ze}}^\vee \in \mathcal{W}^{\Psi_N^{-1}}(\sigma^\vee)$, we have*

$$\int_{\bar{U}_D}^{\mathrm{it}} \left\langle W, W_{\mathrm{Ze}}^\vee(\cdot\bar{v}) \right\rangle d\bar{v} = \int_{N_M \backslash D_M} MW(\tilde{w}_{m,n} l) W_{\mathrm{Ze}}^\vee(\tilde{w}_{m,n} l) \delta_{\bar{U}_D}^{-1}(l)\, dl. \tag{A.5}$$

Proof. The left-hand side is

$$\int_{\bar{U}_D}^{\mathrm{it}} \int_{\bar{U}} \int_{N_M \backslash D_M} \delta_P(l)^{-1} W(l\bar{u}) W_{\mathrm{Ze}}^\vee(l\bar{u}\bar{v})\, dl\, d\bar{u}\, d\bar{v} = \int_{\bar{U}_D}^{\mathrm{it}} \int_{N_M \backslash D_M} \int_{\bar{U}} W(\bar{u}l) W_{\mathrm{Ze}}^\vee(\bar{u}l\bar{v})\, d\bar{u}\, dl\, d\bar{v}.$$

As M normalizes $\bar{U}_i$ for any i, this equals

$$\int_{N_M\backslash D_M}\int_{\bar{U}_D}^{\mathrm{it}}\int_{\bar{U}} W(\bar{u}l)W^{\vee}_{\mathrm{Ze}}(\bar{u}\bar{v}l)\delta^{-1}_{\bar{U}_D}(l)\,d\bar{u}\,d\bar{v}\,dl.$$

Here we can interchange the order of integration as l is integrated over a fixed compact set, by the choice of W. The claim now follows from Lemma A.3 and (A.1). □

Let $\mathcal{W}^{\Psi_N^{-1}}(\sigma^{\vee})_{\flat}$ be the subspace of $\mathcal{W}^{\Psi_N^{-1}}(\sigma^{\vee})$ consisting of the functions $W^{\vee}_{\mathrm{Ze}}$ such that $W^{\vee}_{\mathrm{Ze}}|_D$ is contained in $\mathrm{ind}_{N_D}^{D}\Psi_N$ and $W^{\vee}_{\mathrm{Ze}}|_D$ is supported in $P\bar{U}\cap D$.

Lemma A.5. *For any $W_{\mathrm{Ze}}\in\mathcal{W}^{\Psi_N}(\sigma)$ and $W^{\vee}_{\mathrm{Ze}}\in\mathcal{W}^{\Psi_N^{-1}}(\sigma^{\vee})_{\flat}$, we have*

$$\int_{\bar{U}_D}\mathcal{B}_0(W_{\mathrm{Ze}},W^{\vee}_{\mathrm{Ze}}(\cdot\bar{v}))\,d\bar{v}=\int_{N_M\backslash D_M}W_{\mathrm{Ze}}(\tilde{w}_{m,n}l)W^{\vee}_{\mathrm{Ze}}(\tilde{w}_{m,n}l)\delta^{-1}_{\bar{U}_D}(l)\,dl.$$

Proof. Note that $P\bar{U}\cap D=U_DD_M\bar{U}_D$. Thus, the left-hand side is

$$\begin{aligned}\int_{\bar{U}_D}\int_{N_D\backslash D}W_{\mathrm{Ze}}(p)W^{\vee}_{\mathrm{Ze}}(p\bar{v})\,dp\,d\bar{v}&=\int_{\bar{U}_D}\int_{N_M\backslash D_M}\int_{\bar{U}_D}W_{\mathrm{Ze}}(\bar{u}l)W^{\vee}_{\mathrm{Ze}}(\bar{u}l\bar{v})\,d\bar{u}\,dl\,d\bar{v}\\&=\int_{N_M\backslash D_M}\int_{\bar{U}_D}\int_{\bar{U}_D}W_{\mathrm{Ze}}(\bar{u}l)W^{\vee}_{\mathrm{Ze}}(\bar{u}l\bar{v})\,d\bar{v}\,d\bar{u}\,dl\\&=\int_{N_M\backslash D_M}\int_{\bar{U}_D}\int_{\bar{U}_D}W_{\mathrm{Ze}}(\bar{u}l)W^{\vee}_{\mathrm{Ze}}(\bar{v}l)\delta^{-1}_{\bar{U}_D}(l)\,d\bar{v}\,d\bar{u}\,dl\end{aligned}$$

where we made a change of variable $\bar{v}\mapsto l^{-1}\bar{u}^{-1}\bar{v}l$. By the condition on $W^{\vee}_{\mathrm{Ze}}|_D$ and Lemma 3.8, the integrand is compactly supported, which justifies the previous steps. Applying Lemma A.2 for both integrals over $\bar{U}_D$, we get the required statement. □

Since (A.2) holds up to a scalar, in order to conclude Proposition A.1, it suffices, in view of Lemmas A.4 and A.5, to show the existence of $W\in\mathcal{W}^{\Psi_N}(\Pi)_{\sharp}$ and $W^{\vee}_{\mathrm{Ze}}\in\mathcal{W}^{\Psi_N^{-1}}(\sigma^{\vee})_{\flat}$ such that the right-hand side of (A.5) is non-zero. By Corollary 3.15, given $\phi\in\mathcal{S}(\bar{U}_D)$ and $W'\in\mathrm{ind}_{N_M}^{D_M}\Psi^{-1}_{N_M}$, there exists (a unique) $W^{\vee}_{\mathrm{Ze}}\in\mathcal{W}^{\Psi_N^{-1}}(\sigma^{\vee})_{\flat}$ such that

$$W^{\vee}_{\mathrm{Ze}}(ul\bar{v})=\phi(\bar{v})W'(l)\quad\forall u\in U_D,\ l\in D_M,\ \bar{v}\in\bar{U}_D.$$

Thus,

$$l\mapsto W^{\vee}_{\mathrm{Ze}}(\tilde{w}_{m,n}l)=\int_{\bar{U}_D}W^{\vee}_{\mathrm{Ze}}(\bar{v}l)\,d\bar{v}=\delta_{\bar{U}_D}(l)\int_{\bar{U}_D}W^{\vee}_{\mathrm{Ze}}(l\bar{v})\,d\bar{v}$$

can be taken to be an arbitrary function in $\mathrm{ind}_{N_M}^{D_M}\Psi^{-1}_{N_M}$ Thus, we only need to show that $MW(\tilde{w}_{m,n})$ is non-zero for some $W\in\mathcal{W}^{\Psi_N}(\Pi)_{\sharp}$. However, this is clear since $MW(\tilde{w}_{m,n})=\int_{\bar{U}}W(\bar{u})\,d\bar{u}$.

This finishes the proof of Proposition A.1.

Remark A.6. Let us return to the setup of § 8. It is well known that the Petersson inner product of cusp forms in π factorizes as the product over v of the Bernstein inner product on the Whittaker model of π_v. Now let φ be as in (32). The Ψ_Nth Fourier coefficient of φ is the Ψ_{N_M}th Whittaker coefficient of the constant term of φ, which is given by the iterated residue M_{-1} of the global intertwining operator. Proposition A.1 (assumed to work in the archimedean case as well) gives a factorization of the square of the Petersson norm of φ in terms of the local inner product (14) on the Zelevinsky model of $\mathrm{Sp}(\pi_v,m)$. Indeed, by the Maass–Selberg relations, the Petersson inner product is given by M_{-1} and Proposition A.1 will reduce the statement to the classical case.

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