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COMPACTNESS OF SPACES OF CONVEX AND SIMPLE QUADRILATERALS

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Abstract

The space of shapes of quadrilaterals can be identified with $\mathbb{CP}^2$. We deal with the subset of $\mathbb{CP}^2$ corresponding to convex quadrilaterals and the subset which corresponds to simple (that is, without self-intersections) quadrilaterals. We provide a complete description of the topological closures in $\mathbb{CP}^2$ of both spaces. Although the interior of each space is homeomorphic to a disjoint union $\mathbb{R}^4 \sqcup \mathbb{R}^4$, their closures are topologically different. In particular, the boundary of the space corresponding to convex quadrilaterals is homeomorphic to a pair of three-dimensional spheres glued along a Möbius strip while the boundary of the space corresponding to simple quadrilaterals is more complicated.

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1. Introduction

Let $n \geq 3$ be an integer. We can view $\mathbb{C}^n$ as the space of n-gons (with marked vertices) contained in the plane $\mathbb{C}$ by identifying an n-tuple $(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ with the n-gon whose consecutive vertices are $z_1, z_2, \ldots, z_{n-1}$ and z_n. In fact we allow all possible degenerations of the polygons with the exception of the degeneration of the polygon to a point. Define an equivalence relation via $(z_1, z_2, \ldots, z_n) \sim (w_1, w_2, \ldots, w_n)$ if and only if there exist $a, b \in \mathbb{C}$ with $a \neq 0$ so that $w_i = az_i + b$ for all $i = 1, \ldots, n$. The *space of shapes of n-gons* is the quotient space

$$(\mathbb{C}^n \smallsetminus \{(z_1, \ldots, z_n) \mid z_1 = \cdots = z_n\})/\sim.$$

Obviously, this quotient space is biholomorphic to the complex projective space $\mathbb{P}_{\mathbb{C}}(V)$ of the hyperplane $V = \{(z_1, \ldots, z_n) \mid z_1 = 0\}$. Then the space of shapes of n-gons can be identified with $\mathbb{CP}^{n-2} := \mathbb{P}_{\mathbb{C}}(\mathbb{C}^{n-1})$. The space of shapes arises when we are interested only in the shape, that is, when we do not care about the size or placement of the polygon in the plane.

The topology (and a certain geometry) of some subsets of the space of shapes $\mathbb{CP}^{n-2}$ has been investigated by various authors. We mention some of the earliest and seminal

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references. The topology of subsets corresponding to n-gons with fixed side lengths is studied in [5] (achieving a complete description for the cases $n = 4$, 5 and 6). The structure of the subsets which corresponds to n-gons whose sides are parallel to fixed directions has been discussed in [1]. The paper [6] is devoted to the study of the subset of $\mathbb{CP}^3$ determined by certain (star-shaped) pentagons. Subsets corresponding to n-gons obtained by unfolding of polyhedra with fixed conic angles appeared in [9].

Our goal is to understand the subsets of $\mathbb{CP}^2$ corresponding to convex and simple (that is, without self-intersections) 4-gons (or quadrilaterals), as well as their topological closures. The paper is organised in three sections. After some preliminaries, we present a brief overview of our results, in Section 2. Section 3 deals with convex quadrilaterals and Section 4 with simple quadrilaterals.

This paper is based on the doctoral thesis [3] of the first author under the direction of J. C. Gómez-Larrañaga. In particular, the results concerning convex quadrilaterals are to be found in [3].

2. Preliminaries and overview

From now on, we shall use η to denote the quotient projection to the space of shapes $\eta : \mathbb{C}^n \setminus \{(z, z, \dots, z)\} \to \mathbb{CP}^{n-2}$ and $[z_1, z_2, \dots, z_n]$ to denote the equivalence class $\eta(z_1, z_2, \dots, z_n)$. The elements of the space of shapes $\mathbb{CP}^{n-2}$ will be called *shapes*. We also refer to n-tuples in $\mathbb{C}^n$ as *n-gons* if $n > 2$, or *quadrilaterals* if $n = 4$ and *triangles* if $n = 3$. Given $Z = (z_1, \dots, z_n) \in \mathbb{C}^n$, it is convenient to consider the set

$$\mathfrak{c}(Z) = \{\overline{z_1z_2} \cup \overline{z_2z_3} \cup \dots \cup \overline{z_{n-1}z_n} \cup \overline{z_nz_1}\} \subset \mathbb{C},$$

where $\overline{ab}$ denotes the segment between a and b.

Definition 2.1 (Simple polygons). An n-gon $Z \in \mathbb{C}^n$ is *simple* if its vertices are pairwise different and $\mathfrak{c}(Z)$ determines a Jordan curve. We denote the set of simple n-gons by $S(n) \subset \mathbb{C}^n$. If $Z \in S(n)$, the bounded component of $\mathbb{C} \setminus \mathfrak{c}(Z)$ will be denoted by $\mathfrak{int}(Z)$, and the set $\mathfrak{c}(Z) \cup \mathfrak{int}(Z)$ will be denoted by $\mathfrak{p}(Z)$. A simple n-gon is *positively-oriented* if the labelling of its vertices is counterclockwise, and it is *negatively-oriented* if the labelling is clockwise.

Definition 2.2 (Convex polygons). An n-gon $Z = (z_1, \dots, z_n)$ is *convex* if it is simple and $\overline{z_kz_l} \subset \mathfrak{p}(Z)$ for all $k, l \in \{1, \dots, n\}$. We denote the set of convex n-gons by $K(n) \subset \mathbb{C}^n$.

Clearly, $K(3) = S(3)$ and $K(n) \subsetneq S(n)$ for $n > 3$.

Remark 2.3 (Local chart for shapes of simple polygons). There is exactly one representative in the plane $\{(0, 1, z_3, z_4, \dots, z_n)\} \subset \mathbb{C}^n$ for every shape of simple polygons. Hence the chart $\{[z_1, z_2, z_3, \dots, z_n] \in \mathbb{CP}^{n-2} \mid z_1 \neq z_2\} \to \mathbb{C}^{n-2}$, given by

$$[z_1, z_2, z_3, \dots, z_n] \mapsto \left(\frac{z_3 - z_1}{z_2 - z_1}, \dots, \frac{z_n - z_1}{z_2 - z_1}\right),$$

endows $\eta(S(n))$ with a coordinate system. Obviously, we may endow $\eta(S(n))$ with other coordinate systems by setting different coordinates z_k and z_l as zero and one.

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REMARK 2.4 ($\mathbb{C}^n$ as a fibration over $\mathbb{CP}^{n-2}$). Note that

$$\eta^{-1}([z_1, z_2, \ldots, z_n]) = \{(az_1 + b, az_2 + b, \ldots, az_n + b) \in \mathbb{C}^n \mid a, b \in \mathbb{C}, a \neq 0\}.$$

Moreover, it is a fibre bundle

$$\mathbb{C}^* \times \mathbb{C} \longrightarrow \mathbb{C}^n \setminus \{(z, z, \ldots, z)\} \xrightarrow{\eta} \mathbb{CP}^{n-2},$$

where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The fibration is not a product because $\pi_1(\mathbb{CP}^{n-2} \times \mathbb{C}^* \times \mathbb{C}) = \mathbb{Z}$ but $\pi_1(\mathbb{C}^n \setminus \{(z, \ldots, z\}) = 0$.

DEFINITION 2.5 (n-segments). The shape $[z_1, \ldots, z_n] \in \mathbb{CP}^{n-2}$ is an *n-segment* if $z_1, \ldots, z_n$ are collinear.

REMARK 2.6. The set of n-segments is a real projective space $\mathbb{RP}^{n-2}$ smoothly embedded in $\mathbb{CP}^{n-2}$ since every shape $[z_1, \ldots, z_n]$ with $z_1, \ldots, z_n$ collinear is represented by the n-gons $(0, \lambda x_2, \ldots, \lambda x_n)$ with $x_2, \ldots, x_n \in \mathbb{R}$ and $\lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$. The n-segments do not have a natural orientation, that is, it makes no sense to say whether an n-segment is positively-oriented or negatively-oriented.

EXAMPLE 2.7 (Shapes of triangles). The space of shapes of triangles is the complex projective line $\mathbb{CP}^1$. There are three options for $[0, 1, z] \in \mathbb{CP}^1$, depending on the imaginary part y of $z = x + iy$

$$(a) \quad y > 0, \quad (b) \quad y < 0, \quad (c) \quad y = 0.$$

The first option corresponds to positively-oriented triangles, the second to negatively-oriented triangles and the last to 3-segments. The 3-segment $[0, 0, 1]$ is the only shape that is not of the form $[0, 1, z]$, that is, it does not belong to the local chart of Remark 2.3. We conclude that the space of shapes of positively-oriented triangles is homeomorphic to an open disk and its closure is a closed disk. It follows, from Remark 2.4, that the space of positively-oriented triangles is homeomorphic to the product of an open disk with $\mathbb{C}^* \times \mathbb{C}$, since any fibre bundle over a disk is a product.

We aim to provide a description for the spaces of shapes of convex and simple quadrilaterals similar to that in Example 2.7 for triangles. This will be achieved in Section 3 for shapes of convex quadrilaterals and in Section 4 for shapes of simple quadrilaterals. Both sections are divided into two subsections. After some notation, we present an overview of some results contained in these subsections.

We use A°, $\overline{A}$ and ∂A to denote the familiar concepts of interior, closure and boundary of a set A, respectively. Let $S^+ \subset S(4)$ and $S^- \subset S(4)$, respectively, be the sets of positively- and negatively-oriented simple quadrilaterals. Obviously, $S^+ \cap S^- = \emptyset$ and the mapping $(z_1, z_2, z_3, z_4) \mapsto (z_4, z_3, z_2, z_1)$ defines a homeomorphism between S^+ and S^-. For this reason we work mainly with positively-oriented quadrilaterals. Set $\mathcal{S} = \eta(S^+)$ and $\mathcal{K} = \eta(K(4) \cap S^+)$. Denote the upper half-plane $\{x + iy \in \mathbb{C} \mid y > 0\}$ by $\mathbb{H}$. We use the standard notation $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ to mean the real and imaginary parts of a complex z, respectively.

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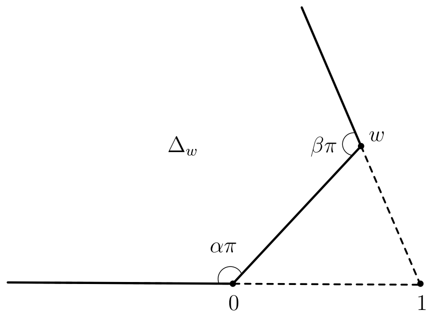

FIGURE 1. Infinite triangular region Δ_w determined by $w \in \mathbb{H}$.

We shall prove, in Sections 3.1 and 4.1, that both $\mathcal{K}^\circ \subset \mathbb{CP}^2$ and $\mathcal{S} \subset \mathbb{CP}^2$ are subspaces homeomorphic to $\mathbb{R}^4$. In fact, we construct a diffeomorphism from $\mathbb{H} \times \mathbb{H}$ to $\mathcal{K}^\circ$ and a homeomorphism from $\mathbb{C} \setminus \{r \in \mathbb{R} \mid r < 1\} \times \mathbb{H}$ to $\mathcal{S}$. A major technique for constructing both homeomorphisms involves using Schwarz–Christoffel integrals. Sections 3.2 and 4.2 contain results which are combinatorial and topological in nature. In Section 3.2, we show that $\overline{\mathcal{K}}$ is a topologically embedded four-dimensional closed ball in $\mathbb{CP}^2$ (Theorem 3.3), while, in Section 4.2, we see that this does not happen to $\overline{\mathcal{S}}$ (Theorem 4.2). We also prove that $\overline{\eta(K(4))} \subset \mathbb{CP}^2$ is homeomorphic to a pair of four-dimensional closed balls glued along a Möbius strip. In particular, the boundary of $\eta(K(4)) \subset \mathbb{CP}^2$ is characterised as being a pair of three-dimensional spheres glued along a certain Möbius strip (Theorem 3.8). The topology of the boundary of $\eta(S(4)) \subset \mathbb{CP}^2$ is also described (Section 4.2).

3. Convex quadrilaterals

3.1. $\mathcal{K}^\circ$ homeomorphic to $\mathbb{R}^4$. If $Z = (z_1, z_2, z_3, z_4)$, then Definition 2.2 implies that $\overline{z_1 z_3} \subset \operatorname{int}(Z)$ and $\overline{z_2 z_4} \subset \operatorname{int}(Z)$ whenever $\eta(Z) \in \mathcal{K}^\circ$.

THEOREM 3.1. *$\mathcal{K}^\circ$ is a subspace of $\mathbb{CP}^2$ homeomorphic to $\mathbb{R}^4$.*

PROOF. Consider local coordinates, defined as in Remark 2.3. Let $Z = (0, 1, w, z)$ with $w = a + ib$ and $z = x + iy$. If $\eta(Z) \in \mathcal{K}^\circ$, then $w \in \mathbb{H}$ and $z \in \Delta_w$, where the region $\Delta_w = \mathbb{H} \cap \{x + iy \in \mathbb{C} \mid bx - ay < 0\} \cap \{x + iy \in \mathbb{C} \mid bx + (1 - a)y < b\} \subset \mathbb{C}$ is determined by w (see Figure 1).

Note that $\mathcal{K}^\circ = \cup_{w \in \mathbb{H}} \{[0, 1, w, z] \mid z \in \Delta_w\}$ and that this union is disjoint. We construct a homeomorphism $F : \mathbb{H} \times \mathbb{H} \to \mathcal{K}^\circ$ of the form $F(w, z) = (w, f_w(z))$ (working with the coordinate system for $\mathcal{K}$ given in Remark 2.3) by using homeomorphisms $f_w : \mathbb{H} \to \Delta_w$, for all $w \in \mathbb{H}$. A natural candidate for f_w comes from the so called Schwarz–Christoffel mapping

$$f_w(z) = w \int_0^z \zeta^{\alpha-1}(\zeta - 1)^{\beta-1} d\zeta \Bigg/ \int_0^1 \zeta^{\alpha-1}(\zeta - 1)^{\beta-1} d\zeta, \tag{3.1}$$

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which defines a diffeomorphism from $\mathbb{H}$ to Δ_w, where $\alpha\pi$ and $\beta\pi$ are the interior angles of Δ_w (see Figure 1). We refer the reader to [8, pages 235–245] for a detailed discussion of the Schwarz–Christoffel mapping. Notice that both $\alpha = \alpha(w)$ and $\beta = \beta(w)$ are positive real analytic functions of w.

From here on, the proof that F is a homeomorphism could be completed by means of a purely topological argument, as in the proof of Theorem 4.1 below. However, a proof that F is a smooth diffeomorphism can be given as follows. We regard F as a function between subsets of $\mathbb{R}^4$. We claim that F has continuous partial derivatives. For the partial derivatives with respect to x and y, note that f_w is a holomorphic function of $z = x + iy$. The smoothness of $\int_0^1 \zeta^{\alpha-1}(\zeta-1)^{\beta-1}d\zeta$ with respect to the variables α and β is proved in [10, Section 17.3] via the smoothness of the gamma function. It only remains to prove that $\int_0^z \zeta^{\alpha-1}(\zeta-1)^{\beta-1}d\zeta$ has continuous partial derivatives with respect to α and β when $z \neq 1$. It is obvious that the partial derivative of $\int_0^z \zeta^{\alpha-1}(\zeta-1)^{\beta-1}d\zeta$ with respect to β exists and that it is continuous if the curve $\zeta(t)$ does not pass through one. Then it is sufficient to check the uniform convergence of the improper integral

$$\int_0^z \frac{\partial}{\partial\alpha}\zeta^{\alpha-1}(\zeta-1)^{\beta-1}d\zeta = \int_0^z \zeta^{\alpha-1}(\zeta-1)^{\beta-1}\log\zeta\, d\zeta$$

(see [10, Section 17.2] for a discussion on differentiation of improper integrals with respect to a parameter). Near zero, the absolute value of this improper integral can be bounded from above by an expression involving

$$-\int_0^1 \frac{\ln r}{r^{1-\alpha}}dr = -\lim_{\epsilon\to 0}\left[\frac{r^\alpha}{\alpha^2}(\alpha\ln r - 1)\right]_{r=\epsilon}^1 = \frac{1}{\alpha^2} \quad (\alpha > 0),$$

thereby proving the uniform convergence on compact sets of the parameter α, and the claim is established. The Jacobian matrix of F is

$$DF = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \dfrac{\partial}{\partial a}\mathrm{Re}(f_w) & \dfrac{\partial}{\partial b}\mathrm{Re}(f_w) & \mathrm{Re}\left(\dfrac{\partial f_w}{\partial z}\right) & -\mathrm{Im}\left(\dfrac{\partial f_w}{\partial z}\right) \\ \dfrac{\partial}{\partial a}\mathrm{Im}(f_w) & \dfrac{\partial}{\partial b}\mathrm{Im}(f_w) & \mathrm{Im}\left(\dfrac{\partial f_w}{\partial z}\right) & \mathrm{Re}\left(\dfrac{\partial f_w}{\partial z}\right) \end{pmatrix},$$

which has determinant $|\partial f_w/\partial z|^2 \neq 0$. We conclude that $F : \mathbb{H}\times\mathbb{H} \to \mathcal{K}^\circ$ is a diffeomorphism, by the inverse function theorem for functions of several real variables. Indeed, F is a C^∞-diffeomorphism (compare with the smoothness of Eulerian integrals discussed in [10, Section 17.3]). □

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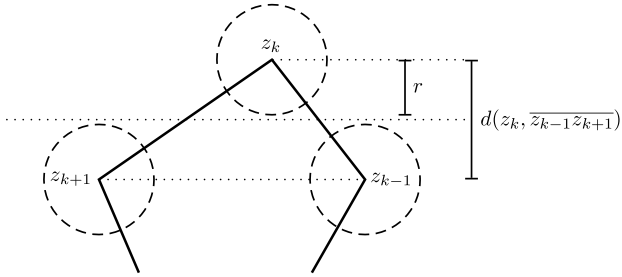

FIGURE 2. If $2r \leq d(z_k, \overline{z_{k-1}z_{k+1}})$, then w_k lies above $\overline{w_{k-1}w_{k+1}}$ for any triple of points $w_{k-1} \in B_r(z_{k-1})$, $w_k \in B_r(z_k)$ and $w_{k+1} \in B_r(z_{k+1})$.

3.2. Closure of $\mathcal{K}$.

LEMMA 3.2. *There are only three types of quadrilateral in* $\partial K(4) \subset \mathbb{C}^4$*, namely:*

(I) *a simple quadrilateral of the form* $Z = (z_1, z_2, z_3, z_4)$ *with* $\overline{z_{k-1}z_{k+1}} \subset \mathfrak{c}(Z)$ *for exactly one* $k \in \{1, 2, 3, 4\}$ *(subscripts* $k-1 = 0$ *and* $k+1 = 5$ *should be taken, respectively, as* 4 *and* 1*);*

(II) *a quadrilateral of the form* (z_1, z_2, z_3, z_4) *with* $z_k = z_{k+1}$ *for exactly one* $k \in \{1, 2, 3, 4\}$ *(subscript* $k+1 = 5$ *taken as* 1*); and*

(III) *a quadrilateral of the form* (z_1, z_2, z_3, z_4) *with collinear points* z_1*,* z_2*,* z_3*,* z_4*, which is the limit of convex quadrilaterals (recall that not all* 4*-segments are limits of convex quadrilaterals; for example,* $[0, 2/3, 1/3, 1]$*).*

PROOF. Let $Z = (z_1, z_2, z_3, z_4)$ be a quadrilateral in $\partial K(4)$.

First, suppose that Z is a simple quadrilateral. We analyse two cases.

Case 1. When $\overline{z_{k-1}z_{k+1}} \subset \mathfrak{int}(Z) \cup \{z_{k-1}, z_{k+1}\}$ for all $k = 1, 2, 3, 4$, we define $2r = \min_{k=1,2,3,4} d\,(z_k, \overline{z_{k-1}z_{k+1}})$, where $d\left(z, \overline{ab}\right)$ denotes the distance between the point $z \in \mathbb{C}$ and the segment $\overline{ab} \subset \mathbb{C}$. Clearly, $\prod_{k=1}^4 B_r(z_k)$ is a neighbourhood of $Z \subset \mathbb{C}^4$ contained in $K(4)$ (see Figure 2), where $B_r(z) \subset \mathbb{C}$ is the ball of radius $r > 0$ centred at z. Hence $Z \notin \partial K(4)$.

Case 2. When there exists a $k \in \{1, 2, 3, 4\}$ such that $\mathfrak{p}(Z) \cap \overline{z_{k-1}z_{k+1}} = \{z_{k-1}, z_{k+1}\}$, we again define r as in Case 1. Analogously, $\prod_{k=1}^4 B_r(z_k)$ is a neighbourhood of Z contained in $K(4)$, and therefore $Z \notin \partial K(4)$.

From the above two cases, a simple quadrilateral $Z \in \partial K(4)$ is of the type I.

Finally, suppose that Z is not a simple quadrilateral. If $\mathfrak{c}(Z)$ is a Jordan curve, then Z is of the type II. If $\mathfrak{c}(Z)$ is not a Jordan curve, then Z is of the type III. □

A quadrilateral is said to have *angle equal to* π *at* z_k if it is of the type I of Lemma 3.2. The closed ball $\{p \in \mathbb{R}^4 \mid |p| \leq 1\}$ is denoted by $\mathbb{B}^4$.

THEOREM 3.3. $\overline{\mathcal{K}}$ *is a subspace of* $\mathbb{CP}^2$ *homeomorphic to* $\mathbb{B}^4$.

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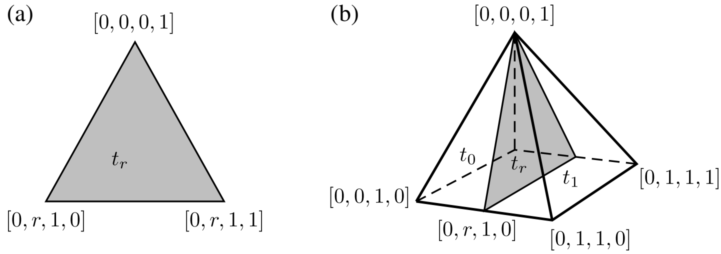

FIGURE 3. (a) Triangle t_r corresponding to shapes of the form $[0, r, 1, z_4]$, for a fixed $0 < r < 1$. The left edge of ∂t_r corresponds to shapes with $z_4 < 0$, its right edge to shapes with $z_4 > 1$, its lower edge to shapes with $0 < z_4 < 1$ and the upper vertex to $z_4 = \infty$. (b) Pyramid T_2 with a shaded slice t_r. Two faces of T_2 are the triangles t_0 and t_1.

PROOF. First we show that $\partial\mathcal{K} \subset \mathbb{CP}^2$ is an embedded three-dimensional sphere $\mathbb{S}^3$.

We begin by studying shapes of quadrilaterals having angle equal to π at z_2; other cases are similar. Shapes having angle equal to π at z_2 are of the form $[0, r, 1, z_4]$, where $0 < r < 1$ and $z_4 \in \mathbb{H}$. By fixing r and varying $z_4 \in \mathbb{H}$, we get a set that is homeomorphic to an open triangle t_r° whose boundary ∂t_r corresponds to shapes of the form $[0, r, 1, z_4]$, where $z_4 \in \mathbb{R} \cup \infty$ (see Figure 3). In fact, ∂t_r corresponds to shapes of type III. When $r = 0$ and z varies in $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$, we get shapes of type II. Analogously, we get shapes of type II when $r = 1$ and z varies in $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$. Hence the closure of shapes having angle equal to π at z_2 forms a pyramid $T_2 = \cup_{0 \le r \le 1} t_r$ (see Figure 3).

When the angle equal to π lies at z_1, z_3 and z_4, a similar procedure provides pyramids $T_1 = \{[z_1, 1, z_3, 0] \mid 0 \le z_1 \le 1, z_3 \in \overline{\mathbb{H}}\}$, $T_3 = \{[z_1, 0, z_3, 1] \mid 0 \le z_3 \le 1, z_1 \in \overline{\mathbb{H}}\}$ and $T_4 = \{[1, z_2, 0, z_4] \mid 0 \le z_4 \le 1, z_2 \in \overline{\mathbb{H}}\}$, respectively (see Figure 4). By Lemma 3.2, it is clear that all the possible shapes in $\partial\mathcal{K}$ belong at least to one of these pyramids. In order to reconstruct $\partial\mathcal{K}$, it is necessary to carry out the corresponding identifications between ∂T_1, ∂T_2, ∂T_3 and ∂T_4. Pyramids T_1 and T_3 are only identified by their bases, and, similarly, so are T_2 and T_4, giving two octahedra O_{13} and O_{24} such that ∂O_{13} is to be identified with ∂O_{24} by a homeomorphism. The identifications between lateral faces are listed below (identifications are by means of changes of coordinates between charts similar to that in Remark 2.3).

$$\begin{array}{rcl}
\{z_2 = 0, z_4 \in \overline{\mathbb{H}}\} \subset T_2 & \longleftrightarrow & \{z_1 = 1, z_3 \in \overline{\mathbb{H}}\} \subset T_1 \\
\{z_2 = 1, z_4 \in \overline{\mathbb{H}}\} \subset T_2 & \longleftrightarrow & \{z_3 = 0, z_1 \in \overline{\mathbb{H}}\} \subset T_3 \\
\{0 \le z_2 \le 1, z_4 \le 0\} \subset T_2 & \longleftrightarrow & \{0 \le z_1 \le 1, z_3 \ge 1\} \subset T_1 \\
\{0 \le z_2 \le 1, z_4 \ge 1\} \subset T_2 & \longleftrightarrow & \{0 \le z_3 \le 1, z_1 \le 0\} \subset T_3 \\
\{z_4 = 0, z_2 \in \overline{\mathbb{H}}\} \subset T_4 & \longleftrightarrow & \{z_3 = 1, z_1 \in \overline{\mathbb{H}}\} \subset T_3 \\
\{z_4 = 1, z_2 \in \overline{\mathbb{H}}\} \subset T_4 & \longleftrightarrow & \{z_1 = 0, z_3 \in \overline{\mathbb{H}}\} \subset T_1 \\
\{0 \le z_4 \le 1, z_2 \le 0\} \subset T_4 & \longleftrightarrow & \{0 \le z_3 \le 1, z_1 \ge 1\} \subset T_3 \\
\{0 \le z_4 \le 1, z_2 \ge 1\} \subset T_4 & \longleftrightarrow & \{0 \le z_1 \le 1, z_3 \le 0\} \subset T_1.
\end{array}$$

We conclude that $\partial\mathcal{K}$ is homeomorphic to $\mathbb{S}^3$. Moreover, the construction shows that $\partial\mathcal{K}$ is embedded in $\mathbb{CP}^2$.

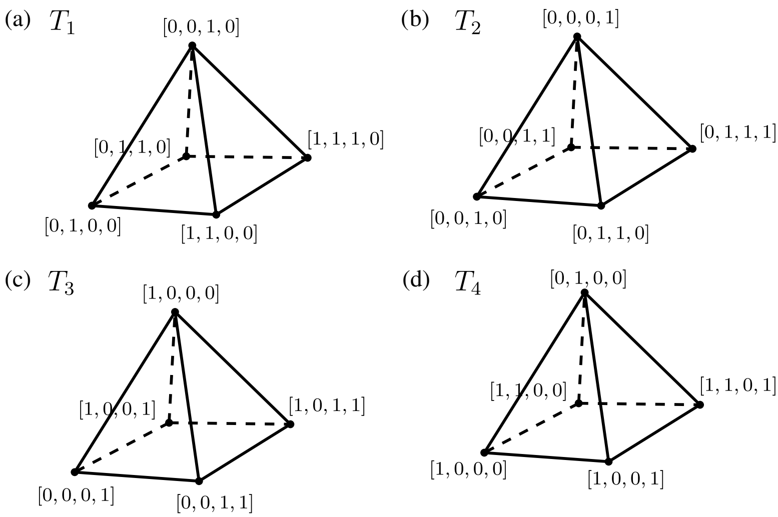

FIGURE 4. $\partial\mathcal{K}$ is obtained by identifying boundaries of these pyramids.

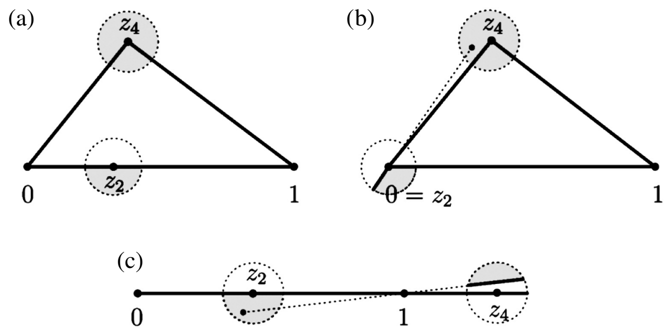

FIGURE 5. Neighbourhoods in $\mathbb{C}^2$ of shapes $[0, z_2, 1, z_4] \in \partial\mathcal{K}$ (using a coordinate system for $\mathcal{K}$ that is similar to that in Remark 2.3). The shaded region describes the intersection of a neighbourhood with $\overline{\mathcal{K}}$. In (b), the region of possible values for z_2 varies according to z_4 while, in (c), the possible values for z_4 vary according to z_2.

Shapes of type I, II and III have neighbourhoods that are homeomorphic to the closed upper half-space $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_4 \geq 0\}$ (see Figure 5). Then $\overline{\mathcal{K}}$ is a manifold with a boundary that is homeomorphic to $\mathbb{S}^3$, the interior of which is diffeomorphic to $\mathbb{R}^4$. Therefore $\overline{\mathcal{K}}$ is homeomorphic to $\mathbb{B}^4$ (see [2, page 371]). □

REMARK 3.4. Figure 5 shows that the neighbourhoods of shapes of types II and III are homeomorphic to the upper half-space $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_4 \geq 0\}$, but are not diffeomorphic to it. Hence $\mathbb{B}^4$ does not admit a smooth inclusion $i : \mathbb{B}^4 \hookrightarrow \mathbb{CP}^2$ such that $i(\mathbb{B}^4) = \overline{\mathcal{K}}$.

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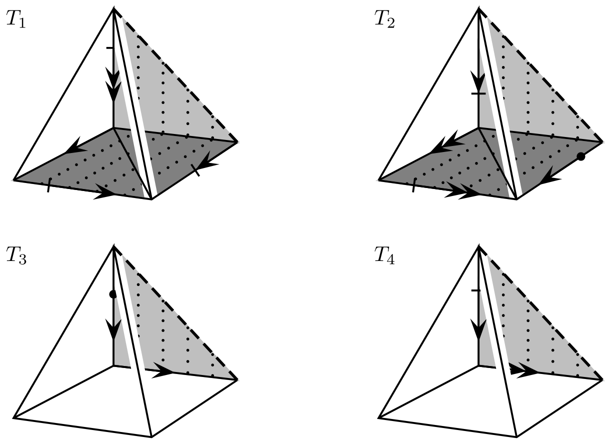

FIGURE 6. Q is obtained by carrying out the identifications indicated by the arrows. Each back face of a pyramid is identified with a front face of another pyramid. The dashed edges of back faces form the boundary of Q. The diagonals that lie in the bases of T_1 and T_2 can be thought of as the components of the middle circle of the Möbius strip Q. Leaves of a foliation that is transversal to the middle circle are shown by dotted lines. Since the leaves vary from 0 to $\pi/2$ in one octahedron, and from $\pi/2$ to π in the other, it follows that Q is a standard Möbius strip.

COROLLARY 3.5. *$\eta^{-1}(\overline{\mathcal{K}}) \subset \mathbb{C}^4$ is homeomorphic to $\mathbb{B}^4 \times \mathbb{C}^* \times \mathbb{C}$.*

PROOF. It follows, from Theorem 3.3 and Remark 2.4, since any fibration over $\mathbb{B}^4$ is the trivial one. □

Contrast Corollary 3.5 with the nontrivial global bundle structure of Remark 2.4.

DEFINITION 3.6 (Convex 4-segment). A *convex 4-segment* is a shape of type III. We denote the set of convex 4-segments by $Q \subset \mathbb{CP}^2$.

There are an infinite number of topologically different embeddings of the Möbius strip into $\mathbb{R}^3$. One embedding is obtained by rotating a small segment in such a way that its centre is moving in time $0 \le t \le 2\pi$ along $\gamma(t) = (\cos t, \sin t, 0) \in \mathbb{R}^3$ while it remains orthogonal to $\gamma'(t)$ and forms angle $t/2$ with the plane xy. If an embedding of the Möbius strip is isotopic to this one, then it will be called a *standard* Möbius strip. Embeddings of Möbius strips into $\mathbb{R}^3$ have been classified up to ambient isotopy [7].

COROLLARY 3.7. *Q is a subspace of $\partial\mathcal{K}$ homeomorphic to a closed Möbius strip. Moreover, Q is the standard Möbius strip in $\mathbb{S}^3 = \partial\mathcal{K}$.*

PROOF. Identify the corresponding faces of the pyramids constructed in Theorem 3.3. Figure 6 should convince the reader that Q is in fact a standard Möbius strip. □

THEOREM 3.8. *The subspace $\overline{\eta(K(4))} \subset \mathbb{CP}^2$ is characterised as the quotient of the disjoint union $\mathbb{B}^4 \sqcup \mathbb{B}^4$ by identifying each point in a standard Möbius strip $Q \subset \mathbb{S}^3 = \partial\mathbb{B}^4$ with its corresponding point in the other copy of $\partial\mathbb{B}^4$.*

PROOF. Recall that the mapping $(z_1, z_2, z_3, z_4) \mapsto (z_4, z_3, z_2, z_1)$ defines a homeomorphism between the sets of positively- and negatively-oriented simple quadrilaterals, so Theorem 3.3 and Corollary 3.7 are valid for shapes of negatively-oriented quadrilaterals. To complete the proof, use the fact that the 4-segments of type III are the only shapes in $\partial\mathcal{K}$ that are limits of positively- and negatively-oriented convex quadrilaterals. $\square$

4. Simple quadrilaterals

4.1. $\mathcal{S}$ homeomorphic to $\mathbb{R}^4$.

THEOREM 4.1. *$\mathcal{S}$ is a subspace of $\mathbb{CP}^2$ homeomorphic to $\mathbb{R}^4$.*

PROOF. Consider local coordinates, defined as in Remark 2.3. Let $Z = (0, 1, w, z)$ with $w = a + ib$ and $z = x + iy$. If $Z \in S^+$, then $w \in \mathbb{C} \setminus \{r \in \mathbb{R} \mid r < 1\}$ and $z \in \Delta_w$, where $\Delta_w \subset \mathbb{C}$ is determined by w according to three possibilities.

(1) If $b < 0$, then $\Delta_w = \mathbb{H} \cap \{x + iy \in \mathbb{C} \mid bx + (1 - a)y > b\}$.
(2) If $b = 0$, then $\Delta_w = \mathbb{H}$ (in this case, $a > 1$).
(3) If $b > 0$, then $\Delta_w = \{x + iy \in \mathbb{C} \mid bx - ay < 0\} \cup \operatorname{int}(0, 1, w) \cup \overline{0w} \setminus \{0, w\}$ (see Figure 7).

We construct a homeomorphism $F : (\mathbb{C} \setminus \{r \in \mathbb{R} \mid r < 1\}) \times \mathbb{H} \to \mathcal{S}$ of the form $F(w, z) = (w, f_w(z))$ by using the diffeomorphisms $f_w : \mathbb{H} \to \Delta_w$ defined for all $w \in \mathbb{C} \setminus \{r \in \mathbb{R} \mid r < 1\}$. The definition of f_w depends on the three possibilities for w.

(1) If $b < 0$, then $f_w(z) = (z - 1)^\theta + 1$, where $\theta\pi$ is the angle (with counterclockwise orientation and such that $0 < \theta\pi < \pi$) between the half-line $\{r \in \mathbb{R} \mid 1 \leq r\}$ and the line $bx + (1 - a)y = b$, and the branch of $(z - 1)^\theta = |z - 1|^\theta e^{i\theta \arg(z-1)}$ is chosen so that $-\pi < \arg(z - 1) < \pi$.
(2) If $b = 0$, then f_w is the identity on $\mathbb{H}$.
(3) If $b > 0$, then f_w comes from the Schwarz–Christoffel mapping: namely, there exists a real number $\tau > 1$ such that

$$f_w(z) = \int_0^z \zeta^{\alpha-1}(\zeta - 1)^{\beta-1}(\zeta - \tau)^{\gamma-1}d\zeta \Big/ \int_0^1 \zeta^{\alpha-1}(\zeta - 1)^{\beta-1}(\zeta - \tau)^{\gamma-1}d\zeta \quad (4.1)$$

defines a diffeomorphism between $\mathbb{H}$ and Δ_w, where $\alpha\pi, \beta\pi$ and $\gamma\pi$ are the interior angles of Δ_w, as in Figure 7.

Since F is a continuous bijection, the restriction of F to any compact set K determines a homeomorphism $F|_K : K \to F(K)$. Let $U = (\mathbb{C} \setminus \{r \in \mathbb{R} \mid r < 1\}) \times \mathbb{H}$. Now we shall prove that F^{-1} is a continuous map. Suppose, on the contrary, that there exist a point $p = (w_0, z_0) \in U$, an open ball $B_\rho(p) \subset U$ of radius $\rho > 0$ centred at p and a sequence $\{p_k\} \subset U \setminus \overline{B_\rho(p)}$ so that $F(p_k)$ converges to $F(p)$. Since the 3-sphere $\partial B_\rho(p)$ is a compact set, $F(\partial B_\rho(p))$ is a 3-sphere, which separates $\mathbb{C}^2$ into two components by the Jordan separation theorem (see [4, Section 2.B]). Due to the hypotheses on

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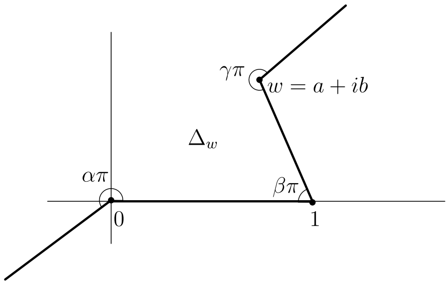

FIGURE 7. Region Δ_w whenever $w \in \mathbb{H}$. Note that $\alpha + \beta + \gamma = 3$ for every such w.

$\{p_k\}$, the connected sets $F(B_\rho(p))$ and $F(U \setminus \overline{B_\rho(p)})$ lie in the same component of $\mathbb{C}^2 \setminus F(\partial B_\rho(p))$. In particular, the image under F of the slice $\{w_0\} \times \mathbb{H}$ lies in just one of the two components determined in $\{w_0\} \times \mathbb{C}$ by the circle $F(\partial B_\rho(p) \cap (\{w_0\} \times \mathbb{H}))$. This is absurd because $F|_{\{w_0\}\times\mathbb{H}} : \{w_0\} \times \mathbb{H} \to \Delta_{w_0} \subset \{w_0\} \times \mathbb{C}$ is a bijection. □

4.2. Boundary of $\mathcal{S}$. We begin the study of the topology of $\partial\mathcal{S}$. Any subscript $j \notin \{1, 2, 3, 4\}$ should be taken modulo four from now on, so $j \in \{1, 2, 3, 4\}$. A shape $[z_1, z_2, z_3, z_4]$ belongs to $\partial\mathcal{S}(4)$ if and only if there is a vertex z_k which lies in the segments $\overline{z_{k+1}z_{k+2}} \cup \overline{z_{k+2}z_{k+3}}$. This situation is encoded by two pyramids (see Figure 3)

$$\begin{aligned} T_{k,k+1} &= \{z_{k+2} = 0,\ z_{k+3} = 1,\ 0 \le z_k \le 1,\ z_{k+1} \in \overline{\mathbb{H}}\},\\ T_{k,k-1} &= \{z_{k+1} = 0,\ z_{k+2} = 1,\ 0 \le z_k \le 1,\ z_{k-1} \in \overline{\mathbb{H}}\}. \end{aligned}$$

In fact, $\partial\mathcal{S}$ is the quotient of $T_{1,2} \cup T_{1,4} \cup T_{2,3} \cup T_{2,1} \cup T_{3,4} \cup T_{3,2} \cup T_{4,1} \cup T_{4,3}$ by identifying points corresponding to the same shape. Clearly, the pyramids are to be identified by their boundaries. It can be verified that the pyramids $T_{k,k+1}$ and $T_{k,k-1}$ are identified along three triangles adjacent to the vertex $[1, 0, 1, 0]$, for $k = 1, 2, 3, 4$. We explain these identifications for the pyramids $T_{2,1} = \{[z, r, 0, 1] \mid 0 \le r \le 1,\ z \in \overline{\mathbb{H}}\}$ and $T_{2,3} = \{[1, r', z', 0] \mid 0 \le r' \le 1, z' \in \overline{\mathbb{H}}\}$ (other cases are similar).

$$[z, r, 0, 1] = \left[1, \frac{1-r}{1-z}, \frac{1}{1-z}, 0\right] \quad \text{and} \quad \left[1, \frac{1-r}{1-z}, \frac{1}{1-z}, 0\right] \in T_{2,3}$$

if and only if both $0 \le (1 - r)/(1 - z) \le 1$ and $1/(1 - z) \in \overline{\mathbb{H}}$ hold. In this case, three triangular regions of $T_{2,1}$, which share the vertex $[0, 1, 0, 1]$, are identified with three triangular regions of $T_{2,3}$: namely,

$$\begin{aligned} \{r = 1\} &\longleftrightarrow \{r' = 0\},\\ \{z \le 0\} &\longleftrightarrow \{r' \le z' \le 1\},\\ \{0 \le z \le r\} &\longleftrightarrow \{1 \le z'\}. \end{aligned}$$

With these identifications between $T_{k,k+1}$ and $T_{k,k-1}$, a polyhedron P_k is obtained for each $k = 1, 2, 3, 4$ (see Figure 8).

Then we identify the boundaries of P_2 and P_4 along the triangles with vertices $[1, 0, 0, 1]$, $[1, 0, 0, 0]$, $[1, 1, 0, 0]$ and $[1, 1, 0, 1]$, forming an octahedron, and also P_2

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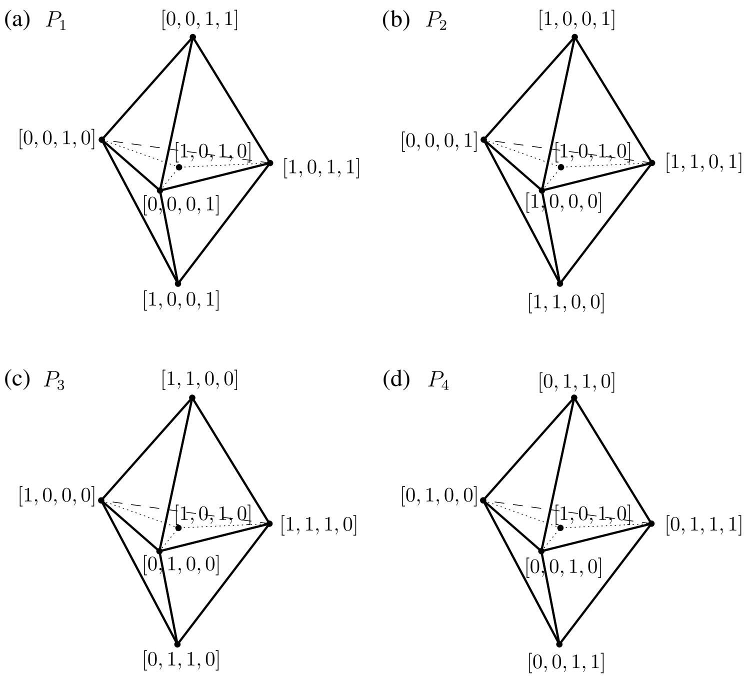

FIGURE 8. Polyhedra obtained via the identification of $T_{k,k+1}$ and $T_{k,k-1}$ along three triangles. Note that all of them have the same vertex at the centre.

and P_4 are to be identified along one closed curve. By way of illustration, we explain these identifications for the pyramid $T_{2,1} = \{[z, r, 0, 1]\}$.

Identifications between $T_{2,1}$ and $T_{4,3}$.

$$[z, r, 0, 1] = \left[0, 1, \frac{z}{z-r}, \frac{z-1}{z-r}\right] \quad \text{and} \quad \left[0, 1, \frac{z}{z-r}, \frac{z-1}{z-r}\right] \in T_{4,3}$$

if and only if both $z/(z-r) \in \overline{\mathbb{H}}$ and $0 \le (z-1)/(z-r) \le 1$ hold. In this case, one lateral face and one closed curve of $T_{2,1}$ are, respectively, identified with one lateral face and one closed curve of $T_{4,3}$: namely,

$$\{1 \le z\} \longleftrightarrow \{1 \le z'\},$$
$$\{r = 1, z \in \mathbb{R}\} \longleftrightarrow \{r' = 1, z' \in \mathbb{R}\},$$

where $T_{4,3} = \{[0, 1, z', r'] \mid 0 \le r' \le 1, z' \in \overline{\mathbb{H}}\}$.

Identifications between $T_{2,1}$ and $T_{4,1}$.

$$[z, r, 0, 1] = \left[\frac{r-z}{r}, 0, 1, \frac{r-1}{r}\right] \quad \text{and} \quad \left[\frac{r-z}{r}, 0, 1, \frac{r-1}{r}\right] \in T_{4,1}$$

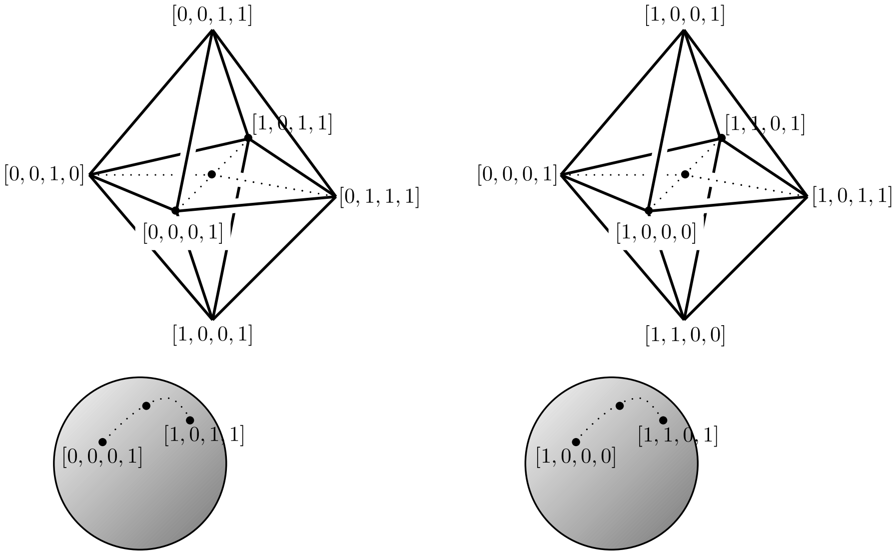

FIGURE 9. Octahedra and spheres obtained via the identification of P_k and P_{k+2}. Each octahedron has an equatorial square whose centre represents the shape $[0, 1, 0, 1]$. One diagonal of that equatorial square is to be identified with a simple curve in a two-dimensional sphere which corresponds to shapes of quadrilaterals such that $z_k = z_{k+2}$.

if and only if both $(r - z)/r \in \overline{\mathbb{H}}$ and $0 \leq (r - 1)/r \leq 1$ hold. In this case, the closed curve $\{r = 1, z \in \mathbb{R}\}$ of $T_{2,1}$ is identified with the closed curve $\{r' = 0, z' \in \mathbb{R}\}$ of $T_{4,1} = \{[z', 0, 1, r'] \mid 0 \leq r' \leq 1, z' \in \overline{\mathbb{H}}\}$.

Under the identification, the circle $\{[z, 1, 0, 1] \in T_{2,1} \mid z \in \overline{\mathbb{R}}\}$ is the equator of a two-dimensional sphere whose hemispheres are described by $\{[z, 1, 0, 1] \in T_{2,1} \mid z \in \mathbb{H}\}$ and $\{[0, 1, z, 1] \in T_{4,3} \mid z \in \mathbb{H}\} = \{[z, 0, 1, 0] \in T_{4,1} \mid z \in \mathbb{H}\}$. Note that the part $\{[z, 1, 0, 1] \in T_{2,1} \mid 0 \leq z \leq 1\}$ of that equatorial circle is to be identified with the diagonal between $[1, 1, 0, 1]$ and $[1, 0, 0, 0]$ of the octahedron (see Figure 9). Similarly, P_1 and P_3 are to be identified along the two triangles with vertices $[0, 0, 1, 1]$, $[0, 0, 0, 1]$, $[1, 0, 0, 1]$ and $[1, 0, 1, 1]$ and also along one closed curve. With these identifications between P_k and P_{k+2}, an octahedron O_k is obtained, for $k = 1, 2$.

Finally, O_1 and O_2 have to be identified. We begin by explaining the remaining identifications for the pyramid $T_{2,1} = \{[z, r, 0, 1]\}$. We shall see that, with two exceptions (the triangle $\{r = 0\}$ and one triangle in the base of $T_{2,1}$), the faces of $T_{2,1}$ involved in the remaining identifications are triangles that have one vertex at $[0, 1, 0, 1]$, which, in turn, are identified with other triangles that also have a vertex at $[0, 1, 0, 1]$, and they will be identifications between the equatorial squares of the octahedra.

Identifications between $T_{2,1}$ and $T_{3,4}$.

$$[z, r, 0, 1] = \left[0, 1, \frac{z}{z - r}, \frac{z - 1}{z - r}\right] \quad \text{and} \quad [0, 1, r', z'] := \left[0, 1, \frac{z}{z - r}, \frac{z - 1}{z - r}\right] \in T_{3,4}$$

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if and only if both $0 \leq z/(z-r) \leq 1$ and $(z-1)/(z-r) \in \overline{\mathbb{H}}$ hold. In this case, two lateral faces of $T_{2,1}$ are identified with two lateral faces of $T_{3,4}$: namely,

$$\{r = 0\} \longleftrightarrow \{r' = 1\},$$
$$\{z \leq 0\} \longleftrightarrow \{1 \leq z'\}.$$

Identifications between $T_{2,1}$ and $T_{1,2}$. The shape $[z, r, 0, 1] \in T_{1,2}$ if and only if both $0 \leq z \leq 1$ and $r \in \overline{\mathbb{H}}$ hold. In this case, the two triangles in the base of $T_{2,1}$ are identified with the two triangles in the base of $T_{1,2}$.

Identifications between $T_{2,1}$ and $T_{3,2}$.

$$[z, r, 0, 1] = \left[1, \frac{1-r}{1-z}, \frac{1}{1-z}, 0\right] \quad \text{and} \quad [1, z', r', 0] := \left[1, \frac{1-r}{1-z}, \frac{1}{1-z}, 0\right] \in T_{3,2}$$

if and only if both $(1-r)/(1-z) \in \overline{\mathbb{H}}$ and $0 \leq 1/(1-z) \leq 1$ hold. In this case, the lateral face $\{z \leq 0\}$ of $T_{2,1}$ is identified with the triangular region $\{0 \leq z' \leq r'\}$ of the base of $T_{3,2}$.

Identifications between $T_{2,1}$ and $T_{1,4}$.

$$[z, r, 0, 1] = \left[\frac{r-z}{r}, 0, 1, \frac{r-1}{r}\right] \quad \text{and} \quad [r', 0, 1, z'] := \left[\frac{r-z}{r}, 0, 1, \frac{r-1}{r}\right] \in T_{1,4}$$

if and only if both $0 \leq (r-z)/r \leq 1$ and $(r-1)/r \in \overline{\mathbb{H}}$ hold. In this case, the triangular region $\{0 \leq z \leq r\}$ of the base of $T_{2,1}$ is identified with the lateral face $\{z' \leq 0\}$ of $T_{1,4}$.

One can conclude that each face of O_1 is identified with the face of O_2 that has the same vertices, and each triangle in the equatorial square of O_1 is identified with the triangle in the equatorial square of O_2 that has the same vertices. Thus we have proved the following result.

Theorem 4.2. *Let O_1 and O_2 be copies of an octahedron (by octahedron we mean the boundary and the interior), and let $\mathbb{S}^2_1$ and $\mathbb{S}^2_2$ be copies of a two-dimensional sphere. Consider the equatorial square in both copies of the octahedron. Let d_1 and d_2 denote the two diagonals of this equatorial square. Consider also a simple (that is, without self-intersections) path in both copies of the sphere (as in Figure 9). The subspace $\partial\mathcal{S} \subset \mathbb{CP}^2$ is homeomorphic to the quotient of the disjoint union $O_1 \sqcup O_2 \sqcup \mathbb{S}^2_1 \sqcup \mathbb{S}^2_2$ by identifying the boundary and the equatorial square of O_1 with the boundary and the equatorial square of O_2, respectively, via the identity mapping, and by identifying d_k with the simple path of $\mathbb{S}^2_k$, for $k = 1, 2$.*

Let $\mathcal{S}^- = \eta(S^-)$ denote the set of shapes corresponding to negatively-oriented simple quadrilaterals. Recall that $(z_1, z_2, z_3, z_4) \mapsto (z_4, z_3, z_2, z_1)$ defines a homeomorphism between the sets of positively- and negatively-oriented simple quadrilaterals.

Corollary 4.3. *Consider a simple path in two copies $\mathbb{S}^2_1$ and $\mathbb{S}^2_2$ of a two-dimensional sphere.Consider also two simple paths d_1 and d_2 in a projective plane $\mathbb{RP}^2$ that*

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intersect at one point. The subspace $\partial\mathcal{S} \cap \partial\mathcal{S}^- \subset \mathbb{CP}^2$ *is homeomorphic to the quotient of the disjoint union* $\mathbb{RP}^2 \sqcup \mathbb{S}^2_1 \sqcup \mathbb{S}^2_2$ *by identifying* d_k *with the simple path of* $\mathbb{S}^2_k$*, for* $k = 1, 2$.

Proof. The area of a quadrilateral Z such that $\eta(Z) \in \partial\mathcal{S} \cap \partial\mathcal{S}^-$ is necessarily equal to zero. If Z has area equal to zero and $\eta(Z) \in \partial\mathcal{S}$, then either Z is a 4-segment or Z is a quadrilateral $Z = [z_1, z_2, z_3, z_4]$ such that $z_1 = z_3$ or $z_2 = z_4$. These sets, respectively, correspond to $\mathbb{RP}^2$ (see Remark 2.6) and to the spheres $\mathbb{S}^2_1$ and $\mathbb{S}^2_2$ that were constructed in the proof of Theorem 4.2. Following the proof of Theorem 4.2, we may conclude that the identifications are along curves with the required properties. □

Remark 4.4 (Work in progress). We make a final comment about shapes of n-gons with $n > 4$. Theorem 3.1 can be generalised to $n > 4$ by the same method we have used here (the generalisation can be found in [3]). We have not succeeded in proving 'by hand' generalisations of other results shown herein. Currently, we are working on the generalisation of Theorem 2.3 by other methods.

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