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RIESZ TRANSFORMS AND LITTLEWOOD–PALEY SQUARE FUNCTION ASSOCIATED TO SCHRÖDINGER OPERATORS ON NEW WEIGHTED SPACES

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Abstract

Let $\mathcal{L} = -\Delta + \mathcal{V}$ be a Schrödinger operator on $\mathbb{R}^n, n \geq 3$, where $\mathcal{V}$ is a potential satisfying an appropriate reverse Hölder inequality. In this paper, we prove the boundedness of the Riesz transforms and the Littlewood–Paley square function associated with Schrödinger operators $\mathcal{L}$ in some new function spaces, such as new weighted Bounded Mean Oscillation (BMO) and weighted Lipschitz spaces, associated with $\mathcal{L}$. Our results extend certain well-known results.

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1. Introduction

In this paper we consider the Schrödinger operator defined by $\mathcal{L} = -\Delta + \mathcal{V}$, which is a Schrödinger operator acting on $L^2(\mathbb{R}^n)$ $(n \geq 3)$, where $\mathcal{V}$ is a nonnegative potential in the reverse Hölder class B_q for some $q \geq n/2$. Recall that given $0 \leq \mathcal{V} \in L^q_{\text{loc}}(\mathbb{R}^n)$ with $1 < q \leq \infty$, $\mathcal{V}$ is said to be in the reverse Hölder class B_q if there exists a constant $C = C(q, \mathcal{V}) > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B \mathcal{V}^q\,dx\right)^{1/q} \leq \frac{C}{|B|}\int_B \mathcal{V}\,dx$$

holds for every ball $B \subset \mathbb{R}^n$.

In recent years, the theory of singular integrals related to Schrödinger operators has attracted a great deal of attention of many mathematicians; see [2–7, 9, 10, 12–15, 18, 20, 23] and references therein. For the classical case of $\mathcal{V} = 0$, the Riesz transforms are bounded on $L^p(w)$ for all $1 < p < \infty$ and $w \in A_p$, where A_p is the Muckenhoupt class of weights, and bounded on the weighted BMO spaces $\text{BMO}^\beta(w)$

for a suitable index β and a suitable weight w; see [16, 17]. When $\mathcal{V}$ is a nonnegative polynomial, the L^p boundedness of Riesz transforms was obtained in [23]. Later, Zhen investigated the boundedness of the Riesz transforms for Schrödinger operators with potentials in the class B_q. The boundedness of the Riesz transforms on the weighted Bounded Mean Oscillation (BMO) spaces was obtained in [2]. The condition $\mathcal{V} \in B_q$ for some $1 < q < \infty$ is essential in the theory of singular integrals related to Schrödinger operators.

In particular, when $\mathcal{V} \in B_{n/2}$, the theory of new weights for singular integrals related to $\mathcal{L}$ was introduced in [3]. Let us recall its brief definition as follows (see [18]).

Let $\mathcal{V} \in B_{n/2}$. The critical radius function $\rho(\cdot)$ is defined by

$$\rho(x) = \sup\Big\{r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} \mathcal{V}(y)\, dy \le 1\Big\}, \quad x \in \mathbb{R}^n.$$

Following [3], for $\theta \ge 0$ and $1 < p < \infty$, the class of weights $A_p^{\mathcal{L},\theta}$ is defined as those weights w such that

$$\Big(\int_B w\, dz\Big)^{1/p}\Big(\int_B w^{-1/(p-1)}\, dz\Big)^{1/p'} \le C|B|\Psi_\theta(B)$$

for every ball $B = B(x, r)$, where $\Psi_\theta(B) = (1 + r/\rho(x))^\theta$.

When $p = 1$, the class $A_1^{\mathcal{L},\theta}$ is defined as those weights w such that

$$\frac{1}{|B|} \int_B w\, dz \le Cw(y)\Psi_\theta(B) \quad \text{a.e. } y \in B$$

for every ball $B = B(x, r)$.

For $p \ge 1$, they define the class

$$A_p^{\mathcal{L},\infty} = \bigcup_{\theta \ge 0} A_p^{\mathcal{L},\theta}.$$

Note that the classes $A_p^{\mathcal{L},\theta}$ are necessarily increasing with θ and, if $\theta = 0$, they coincide with the Muckenhoupt class A_p. In general, the class $A_p^{\mathcal{L},\theta}$ is strictly larger than the class A_p when $\theta > 0$; see [3].

It was proved in [3] that the Riesz transform $\mathcal{R} = \nabla\mathcal{L}^{-1/2}$ and the Littlewood–Paley square function defined by

$$\mathcal{G}_{\mathcal{L}}(f)(x) = \Big(\int_0^\infty |t^2\mathcal{L}e^{-t^2\mathcal{L}}f(x)|^2\, \frac{dt}{t}\Big)^{1/2} \tag{1.1}$$

are bounded on $L^p(w)$ for $w \in A_p^{\mathcal{L},\infty}$, $1 < p < \infty$, and of weak type (1,1) for $w \in A_1^{\mathcal{L},\infty}$.

The purpose of this paper is to prove the boundedness of the Riesz transform and the Littlewood–Paley square function related to $\mathcal{L}$ on new weighted BMO spaces and new weighted Lipschitz spaces with more general weights than $A_p^{\mathcal{L},\infty}$. To proceed, we first define the new weights in what follows.

Definition 1.1. Let $\theta \geq 0$ and $1 \leq p < \infty$. A function $w : \mathbb{R}^n \to \mathbb{R}$ is said to be in the class $D_p^{\mathcal{L},\theta}$ if there exists a constant $C = C(w) > 0$ such that, for every ball $B = B(x, r) \subset \mathbb{R}^n$ and $t > 1$,

$$w(tB) \leq C t^{np} w(B) \Psi_\theta(B),$$

where tB denotes the ball with the same center as B and t times its radius, and $w(E) = \int_E w(x)\, dx$ for $E \subset \mathbb{R}^n$.

For $1 \leq p < \infty$, we set

$$D_p^{\mathcal{L},\infty} = \bigcup_{\theta \geq 0} D_p^{\mathcal{L},\theta}.$$

Definition 1.2. For $0 \leq \beta < 1$, $\theta \geq 0$, and for a weight $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the weighted BMO spaces $\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta}(w)$ are defined for every function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying the inequalities

$$\int_B |f - f_B|\, dy \leq C w(B) |B|^{\beta/n} \quad \text{when } r < \rho(x) \tag{1.2}$$

and

$$\int_B |f|\, dy \leq C w(B) |B|^{\beta/n} \Psi_\theta(B) \quad \text{when } r \geq \rho(x) \tag{1.3}$$

for every ball $B = B(x, r)$ in $\mathbb{R}^n$, where f_B stands for the mean of f over B. The norm $\|\cdot\|_{\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta}(w)}$ can be defined as the maximum of the two infimums of the constants that satisfy (1.2) and (1.3).

Notice that, when $w \in A_p$ and $\theta = 0$, the space $\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta}(w)$ has been introduced in [1]. Moreover, if $w \in A_p^{\mathcal{L},\infty}$ for some $p > 1$, then w may not satisfy the doubling condition and thus our $\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta}(w)$ space is a significant extension of those in [1].

On the other hand, for $\beta > 0$ and $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, as in [1], the Lipschitz-type space $\Lambda_{\mathcal{L}}^{\beta}(w)$ can be defined as the set of all functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that there exist positive constants C_1 and C_2 satisfying the inequalities

$$|f(x) - f(y)| \leq C_1 [\mathcal{W}_\beta(x, |x - y|) + \mathcal{W}_\beta(y, |x - y|)],$$

provided that $|x - y| \leq \rho(x)$, and

$$\int_{B(x,\rho(x))} |f|\, dy \leq C_2 |B(x, \rho(x))|^{\beta/n} w(B(x, \rho(x)))$$

for all $x \in \mathbb{R}^n$, where

$$\mathcal{W}_\beta(x, r) = \int_{B(x,r)} \frac{w(y)}{|y - x|^{n-\beta}}\, dy.$$

This paper is devoted to proving the boundedness of the Riesz transform and the square function on the new weighted spaces $\mathrm{BMO}_{\mathcal{L}}^{\beta}(w)$ and $\Lambda_{\mathcal{L}}^{\beta}(w)$ (see Theorems 5.2 and 5.4). Note that when w satisfies the doubling condition, we have $\Lambda_{\mathcal{L}}^{\beta}(w) = \mathrm{BMO}_{\mathcal{L}}^{\beta}(w)$; see [1, Proposition 4]. Thus, Theorem 5.2 is an extension of [2, Theorem 1]. It should be emphasized that the method used in [2] is based on the $\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta}(w)$-boundedness of $\nabla(-\Delta)^{-1/2}$, $(-\Delta)^{-1/2}\nabla$ when $w \in A_p$ and $\theta = 0$. This, however, may not be applicable to our setting due to the new weights $w \in D_\sigma^{\mathcal{L},\infty}$.

The paper is organized as follows. Section 2 gives some facts about the critical functions and the new weights. In Section 3 we establish the John–Nirenberg inequality for $\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta}(w)$ and other important properties related to these new spaces. Section 4 gives some kernel estimates. The boundedness of Riesz transforms and square functions on $\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}(w)$ and $\Lambda_{\mathcal{L}}^{\beta}(w)$ is established in Section 5.

In this paper, we denote by E^c the set $\mathbb{R}^n \setminus E$ and by χ_E its characteristic function. All the positive constants are signified as C although they may be different on the same line. We write $A \lesssim B$ and $A \sim B$ if there exist some positive constants C, C' such that $A \leq CB$ and $C'A \leq B \leq CA$, respectively.

2. Preliminaries

Let us now recall some properties of the critical radius functions (see [18, Lemma 1.4]).

Proposition 2.1. *Let $\mathcal{V} \in B_{n/2}$. Then there exist $C > 0$ and $k_0 \geq 1$ such that*

$$C^{-1}\rho(x)\Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{-k_0} \leq \rho(y) \leq C\rho(x)\Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{k_0/(k_0+1)}$$

for all $x, y \in \mathbb{R}^n$.

Corollary 2.2. *Let $\mathcal{V} \in B_{n/2}$. Then there exist $C > 0$ and $k_0 \geq 1$ such that*

$$1 + \frac{R}{\rho(x)} \leq C\Big(1 + \frac{R}{\rho(x_0)}\Big)^{k_0+1}$$

for all $x_0 \in \mathbb{R}^n$ and $x \in B(x_0, R)$.

We also note that the class of weights $A_p^{\mathcal{L},\infty}$ satisfies the following properties (see [19]).

Proposition 2.3. *The following statements hold:*

(i) $A_p^{\mathcal{L},\theta} \subset A_q^{\mathcal{L},\theta}$ *for all* $1 \leq p \leq q < \infty$;

(ii) *if* $w \in A_p^{\mathcal{L},\infty}$ *with* $p > 1$, *then there exists* $\epsilon > 0$ *such that* $w \in A_{p-\epsilon}^{\mathcal{L},\infty}$;

(iii) $w \in A_p^{\mathcal{L},\theta}$ *if and only if* $w^{1-p'} \in A_{p'}^{\mathcal{L},\theta}$, *where* $\theta \geq 0$ *and* $1/p + 1/p' = 1$.

The relationship between the classes $A_p^{\mathcal{L},\infty}$ and $D_p^{\mathcal{L},\infty}$ is given in the lemma below.

Lemma 2.4. *Let $\theta \geq 0$ and $1 \leq p < \infty$. If $w \in A_p^{\mathcal{L},\theta}$, then $w \in D_{p(1+\theta/n)}^{\mathcal{L},\theta p}$.*

Proof. Let $t \geq 1$ and $B = B(x, r)$ for some $x \in \mathbb{R}^n$. We have

$$(\chi_B)_{tB} = \frac{1}{|tB|}\int_{tB} \chi_B\,dx = \frac{1}{|tB|}\int_{tB} \chi_B w^{1/p} w^{-1/p}\,dx.$$

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Applying Hölder's inequality with exponents p and p',

$$\begin{aligned}[(\chi_B)_{tB}]^p &\lesssim |tB|^{-p}\Big(\int_{tB}\chi_B^p w\,dx\Big)\Big(\int_{tB} w^{-p'/p}\,dx\Big)^{p/p'}\\ &\lesssim |tB|^{-p}\Big(\int_{tB}\chi_B^p w\,dx\Big)|tB|^p w(tB)^{-1}\Psi_{\theta p}(tB)\\ &\lesssim w(tB)^{-1}\Big(\int_{tB}\chi_B^p w\,dx\Big)\Psi_{\theta p}(tB)\\ &\lesssim w(tB)^{-1}w(B)t^{\theta p}\Psi_{\theta p}(B).\end{aligned}$$

Thus,

$$w(tB) \lesssim t^{np(1+\theta/n)}w(B)\Psi_{\theta p}(B),$$

which completes the proof of the lemma. □

3. $\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta}(w)$ space and John–Nirenberg inequality

Let $\beta \in [0, 1)$, $\theta \ge 0$, and $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We first define the space $\widehat{\mathrm{BMO}}_{\mathcal{L}}^{\beta,\theta}$ as the set of all functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying

$$\int_B |f - f_B|\,dy \le Cw(B)|B|^{\beta/n}\Psi_\theta(B) \tag{3.1}$$

for every ball $B = B(x, r) \subset \mathbb{R}^n$. The norm $\|\cdot\|_{\widehat{\mathrm{BMO}}_{\mathcal{L}}^{\beta,\theta}(w)}$ on $\widehat{\mathrm{BMO}}_{\mathcal{L}}^{\beta,\theta}(w)$ is defined to be the infimum of the constants that satisfy (3.1).

Using the same argument as in the proof of [17, Theorem 3], we come up with the following result.

Proposition 3.1. *Let $\theta, \theta_1, \theta_2 \ge 0$, $p, \sigma \ge 1$, and $w \in A_p^{\mathcal{L},\theta} \cap D_\sigma^{\mathcal{L},\theta_1}$.*

(1) *If $p = 1$, then there exist positive constants M and C such that*

$$w(x \in B : |f(x) - f_B|w^{-1}(x) > \alpha) \le M\exp\Big(\frac{-C\alpha}{|B|^{\beta/n}\|f\|_{\widehat{\mathrm{BMO}}_{\mathcal{L}}^{\beta,\theta_2}(w)}\Psi_\phi(B)}\Big)w(B)$$

for any $\alpha > 0$, $f \in \widehat{\mathrm{BMO}}_{\mathcal{L}}^{\beta,\theta_2}(w)$ with $\beta \in [0, 1)$, and for any ball B.

(2) *If $1 < p < \infty$, then there is a constant $K > 0$ such that*

$$w(x \in B : |f(x) - f_B|w^{-1}(x) > \alpha) \le K\Big(1 + \frac{\alpha}{|B|^{\beta/n}\|f\|_{\widehat{\mathrm{BMO}}_{\mathcal{L}}^{\beta,\theta_2}(w)}\Psi_\phi(B)}\Big)^{-p'} w(B)$$

for any $\alpha > 0$, $f \in \widehat{\mathrm{BMO}}_{\mathcal{L}}^{\beta,\theta_2}(w)$ with $\beta \in [0, 1)$, and for any ball B.

The next results follow from some basic facts (and hence we will skip the proof).

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THEOREM 3.2. *Let $\theta, \theta_1, \theta_2 \geq 0$, $p, \sigma \geq 1$, and $w \in A_p^{\mathcal{L},\theta} \cap D_\sigma^{\mathcal{L},\theta_1}$. Then, for every $\nu \in (1, p'] \setminus \{\infty\}$, there exists a constant $C_\nu > 0$ such that*

$$\Big(\int_B |f - f_B|^\nu w^{1-\nu}\, dy\Big)^{1/\nu} \leq C_\nu \|f\|_{\widehat{\mathrm{BMO}}_{\mathcal{L}}^{\beta,\theta}(w)} |B|^{\beta/n} w(B)^{1/\nu} \Psi_\phi(B)$$

for all $B = B(x, r) \subset \mathbb{R}^n$ and for all $f \in \widehat{\mathrm{BMO}}_{\mathcal{L}}^{\beta,\theta_2}(w)$ with $\beta \in [0, 1)$. Here we recall that

$$\phi = \theta_1(k_0 + 2) + \theta_2 + \theta(k_0 + 1), \quad p' = \frac{p}{p-1}.$$

The next proposition is very crucial and will be used frequently in this paper.

PROPOSITION 3.3. *Let $\sigma \geq 1$, $\beta \in [0, 1)$, $\theta_1, \theta_2 \geq 0$, $w \in D_\sigma^{\mathcal{L},\theta_1}$, and $f \in \mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}(w)$. Then there exists a constant $C > 0$ such that for every ball $B = B(x, r)$,*

$$\int_B |f|\, dy \leq C \|f\|_{\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}(w)} w(B) |B|^{\beta/n} \Psi_{\theta_2}(B) \max\Big\{1, \Big(\frac{\rho(x)}{r}\Big)^{n\sigma - n + \beta}\Big\},$$

provided that $\sigma > 1$ or $\beta > 0$, and

$$\int_B |f|\, dy \leq C \|f\|_{\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}(w)} w(B) |B|^{\beta/n} \Psi_{\theta_2}(B) \max\Big\{1, 1 + \log_2\Big(\frac{\rho(x)}{r}\Big)\Big\}$$

if $\sigma = 1$ and $\beta = 0$.

PROOF. In the case of $r \geq \rho(x)$, the conclusion can be obtained from (1.3). Now we consider the case of $r < \rho(x)$. Let j_0 be such that

$$2^{j_0 - 1} < \frac{\rho(x)}{r} \leq 2^{j_0}.$$

Then

$$\begin{aligned}
\frac{1}{|B|} \int_B |f|\, dy &\lesssim \frac{1}{|2^{j_0} B|} \int_{2^{j_0} B} |f|\, dy + \sum_{j=0}^{j_0 - 1} \frac{1}{|2^j B|} \int_{2^j B} |f(y) - f_{2^j B}|\, dy \\
&\lesssim \|f\|_{\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}} \sum_{j=0}^{j_0} w(2^j B) |2^j B|^{\beta/n - 1} \Psi_{\theta_2}(2^j B) \\
&\lesssim \|f\|_{\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}} \sum_{j=0}^{j_0} w(2^j B) |2^j B|^{\beta/n - 1}.
\end{aligned}$$

By using $w \in D_\sigma^{\mathcal{L},\theta_1}$ and $\Psi_{\theta_1}(B) \lesssim 1$, we deduce that

$$\begin{aligned}
\int_B |f|\, dy &\lesssim \|f\|_{\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}} w(B) |B|^{\beta/n} \sum_{j=0}^{j_0} 2^{j(n\sigma - n + \beta)} \\
&\lesssim \|f\|_{\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}} w(B) |B|^{\beta/n} \Big(\frac{\rho(x)}{r}\Big)^{n\sigma - n + \beta}
\end{aligned}$$

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as long as $\sigma > 1$ or $\beta > 0$. Furthermore, if $\sigma = 1$ and $\beta = 0$, then

$$\sum_{j=0}^{j_0} 2^{j(n\sigma-n+\beta)} = j_0 + 1 \le 1 + \log_2\Big(\frac{\rho(x)}{r}\Big).$$

Combining these estimates, we obtain the desired estimates. □

Now set $\mathcal{B} = \{B(x, r) \subset \mathbb{R}^n : x \in \mathbb{R}^n, r < \rho(x)\}$. We have the following proposition.

Proposition 3.4. *Let $\beta \in [0, 1)$, $p, \sigma \ge 1$, $\theta, \theta_1, \theta_2 \ge 0$, and $w \in A_p^{\mathcal{L},\theta} \cap D_\sigma^{\mathcal{L},\theta_1}$. Assume that $f \in \mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}(w)$. Then, for each $\nu \in [1, p'] \setminus \{\infty\}$, there exists a constant $C_\nu > 0$ such that*

$$\Big(\frac{1}{w(B)}\int_B |f - f_B|^\nu w^{1-\nu}\,dy\Big)^{1/\nu} \le C_\nu \|f\|_{\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}(w)} |B|^{\beta/n}, \quad B \in \mathcal{B}$$

and

$$\Big(\frac{1}{w(B)}\int_B |f|^\nu w^{1-\nu}\,dy\Big)^{1/\nu} \le C_\nu \|f\|_{\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}(w)} \Psi_\phi(B)|B|^{\beta/n}, \quad B \notin \mathcal{B}. \tag{3.2}$$

Proof. By the continuous inclusion $\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}(w) \subset \widehat{\mathrm{BMO}}_{\mathcal{L}}^{\beta,\theta_2}(w)$, it suffices to prove that the left-hand side of (3.2) is dominated by $\|f\|_{\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}(w)}$.

Since $A_p^{\mathcal{L},\theta} \subset A_{\nu'}^{\mathcal{L},\theta}$ with $1/\nu + 1/\nu' = 1$, for every ball $B \notin \mathcal{B}$,

$$\begin{aligned}\Big(\frac{1}{w(B)}\int_B |f|^\nu w^{1-\nu}\,dy\Big)^{1/\nu} &\le \Big(\frac{1}{w(B)}\int_B |f - f_B|^\nu w^{1-\nu}\Big)^{1/\nu} dy + |f|_B\Big(\frac{w^{1-\nu}(B)}{w(B)}\Big)^{1/\nu} \\ &\le \|f\|_{\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}(w)} |B|^{\beta/n}[\Psi_\phi(B) \\ &\quad + w(B)^{1/\nu'}(w^{1-\nu}(B))^{1/\nu}|B|^{-1}\Psi_{\theta_2}(B)] \\ &\lesssim \|f\|_{\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}(w)} |B|^{\beta/n}\Psi_\phi(B),\end{aligned}$$

where the second inequality follows from Theorem 3.2 and thus we complete the proof of the proposition. □

Proposition 3.5. *Let $p, \sigma \ge 1$, $\theta, \theta_1, \theta_2 \ge 0$, and $w \in A_p^{\mathcal{L},\theta} \cap D_\sigma^{\mathcal{L},\theta_1}$. Then, for each $\nu \in [1, p'] \setminus \{\infty\}$, there exists $C_\nu > 0$ such that:*

(i) *if $\sigma > 1$ or $\beta > 0$, then*

$$\Big(\int_B |f|^\nu w^{1-\nu}\,dy\Big)^{1/\nu} \le C\|f\|_{\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}(w)} w(B)^{1/\nu}|B|^{\beta/n}\Psi_\phi(B)\max\Big\{1, \Big(\frac{\rho(x)}{r}\Big)^{n\sigma-n+\beta}\Big\}$$

for every ball $B = B(x, r)$ and $f \in \mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}(w)$;

(ii) *if $\sigma = 1$ and $\beta = 0$, then*

$$\Big(\int_B |f|^\nu w^{1-\nu}\,dy\Big)^{1/\nu} \le C\|f\|_{\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}(w)} w(B)^{1/\nu}|B|^{\beta/n}\Psi_\phi(B)\max\Big\{1, 1 + \log_2\Big(\frac{\rho(x)}{r}\Big)\Big\}$$

for every ball $B = B(x, r)$ and $f \in \mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}(w)$.

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Proof. Let $f \in \mathrm{BMO}^{\beta,\theta_2}_{\mathcal{L}}(w)$ and B be a ball in $\mathbb{R}^n$. In the case of $r \geq \rho(x)$, our conclusion can be deduced from (3.2). If $r < \rho(x)$, we can write

$$f = f - f_B + \sum_{j=0}^{j_0-1} (f_{2^j B} - f_{2^{j+1} B}) + f_{2^{j_0} B},$$

where $2^{j_0-1} < \rho(x)/r \leq 2^{j_0}$. This implies that

$$\Big(\int_B |f|^v w^{1-v}\, dy\Big)^{1/v} \leq \sum_{i=1}^{3} I_i,$$

where

$$I_1 = \Big(\int_B |f - f_B|^v w^{1-v}\, dy\Big)^{1/v},$$

$$I_2 = [w^{1-v}(B)]^{1/v} \sum_{j=0}^{j_0-1} |f_{2^j B} - f_{2^{j+1} B}|,$$

$$I_3 = [w^{1-v}(B)]^{1/v} |f_{2^{j_0} B}|.$$

The first term can be estimated easily by using Proposition 3.4. For the second term, since $w \in A^{\mathcal{L},\theta}_{v'}$ and $\Psi_\theta(B) \lesssim 1$,

$$[w^{1-v}(B)]^{1/v} \lesssim \frac{|B|}{w(B)^{1/v'}}.$$

This implies that

$$\begin{aligned} I_2 &\lesssim |B| w(B)^{-1/v'} \sum_{j=0}^{j_0-1} \frac{1}{|2^j B|} \int_{2^{j+1} B} |f - f_{2^{j+1} B}|\, dy \\ &\lesssim |B| w(B)^{-1/v'} \|f\|_{\mathrm{BMO}^{\beta,\theta_2}_{\mathcal{L}}(w)} \sum_{j=0}^{j_0-1} |2^j B|^{\beta/n-1} w(2^j B) \\ &\lesssim |B|^{\beta/n} w(B)^{1/v} \|f\|_{\mathrm{BMO}^{\beta,\theta_2}_{\mathcal{L}}(w)} \sum_{j=0}^{j_0-1} 2^{j(\beta+n\sigma-n)}. \end{aligned}$$

If $\beta > 0$ or $\sigma > 1$,

$$I_2 \lesssim |B|^{\beta/n} w(B)^{1/v} \|f\|_{\mathrm{BMO}^{\beta,\theta_2}_{\mathcal{L}}(w)} \Big(\frac{\rho(x)}{r}\Big)^{n\sigma+\beta-n}.$$

If $\beta = 0$ and $\sigma > 1$,

$$I_2 \lesssim |B|^{\beta/n} w(B)^{1/v} \|f\|_{\mathrm{BMO}^{\beta,\theta_2}_{\mathcal{L}}(w)} \Big(1 + \log_2\Big(\frac{\rho(x)}{r}\Big)\Big).$$

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On the other hand, by $r2^{j_0} \geq \rho(x)$,

$$\begin{aligned} I_3 &\lesssim |B| w(B)^{-1/\nu'} w(2^{j_0}B) |2^{j_0}B|^{\beta/n-1} \Big(1 + \frac{2^{j_0} r}{\rho(x)}\Big)^{\theta_2} \\ &\lesssim |B|^{\beta/n} w(B)^{1/\nu} 2^{j_0(\beta+n\sigma-n)} \\ &\lesssim |B|^{\beta/n} w(B)^{1/\nu} \Big(\frac{\rho(x)}{r}\Big)^{\beta+n\sigma-n}. \end{aligned}$$

Here we has used $2^{j_0} r \sim \rho(x)$. Finally, by combining these estimates, we obtain the desired result. □

4. Some kernel estimates

In what follows, we denote by $k_t(x, y)$ the kernel of $e^{-t\mathcal{L}}$ with $t > 0$. Some estimates of k_t are presented below. The following lemma is essentially taken from [8].

Lemma 4.1. *Suppose that $V \in B_q$ for some $q > n$. For every $N > 0$, there exist constants $C > 0$ and $c > 0$ such that for all $x, y \in \mathbb{R}^n$ and $t > 0$,*

$$|\nabla k_t(x, y)| \leq C t^{-(n+1)/2} e^{-|x-y|^2/ct} \Big(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\Big)^{-N}.$$

By [11], we have the following lemma.

Lemma 4.2. *Suppose that $V \in B_q$ for some $q > n/2$. For every $N > 0$, there exist constants $C > 0$ and $c > 0$ such that for all $x, y \in \mathbb{R}^n$ and $t > 0$,*

$$|k_t(x, y)| + t|\partial_t k_t(x, y)| \leq C t^{-n/2} e^{-|x-y|^2/ct} \Big(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\Big)^{-N}.$$

The following proposition plays an important role in proving the BMO boundedness of the Riesz transform.

Proposition 4.3. *Suppose that $V \in B_q$ for some $q > n$. For every $N, M > 0$, there exist constants $C > 0$ and $\alpha > 0$ such that for $|h| \leq M\sqrt{t}, 0 < \delta < 1 - n/q$,*

$$|\nabla k_t(x + h, y) - \nabla k_t(x, y)| \leq C \Big(\frac{|h|}{\sqrt{t}}\Big)^{\delta} t^{-(n+1)/2} e^{-|x-y|^2/\alpha t} \Big(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\Big)^{-N}.$$

Proof. Fix $t > 0$ and $x, y \in \mathbb{R}^n$. We now consider the equation

$$\partial_t u + \mathcal{L} u = 0. \tag{4.1}$$

By [8, page 18], there exist $C > 0, m_0 > 1$ such that for any solution u to (4.1),

$$\Big(\int_{B(x,R)} |\nabla^2 u|^q\Big)^{1/q} \leq C R^{n/q-2} \Big\{\Big(\frac{R}{\rho(x)}\Big)^{m_0} + 1\Big\} \sup_{B(x,2R)} |u| + C R^{n/q} \sup_{B(x,2R)} |\partial_t u|.$$

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Taking $u(x,t) = k_t(x,y)$ and then using the imbedding theorem of Morrey, we deduce that

$$\begin{aligned}|\nabla k_t(x+\xi,y) - \nabla k_t(x,y)| &\lesssim |\xi|^{1-n/q}\Big(\int_{B(x,R)} |\nabla^2 k_t(z,y)|^q\,dz\Big)^{1/q}\\ &\lesssim \Big(\frac{|\xi|}{R}\Big)^{1-n/q}\frac{1}{R}\Big[\Big\{\Big(\frac{R}{\rho(x)}\Big)^{m_0}+1\Big\}\sup_{z\in B(x,2R)}|k_t(z,y)|\\ &\qquad + \frac{R^2}{t}\sup_{z\in B(x,2R)}|t\partial_t k_t(z,y)|\Big]\end{aligned}$$

for all $x,y\in\mathbb{R}^n$, $|\xi|<2R$.

We now consider three cases.

Case 1: $|x-y|\geq\sqrt{t}$. Let $R=\sqrt{t}/8$. If $z\in B(x,2R)$, then $|z-y|\geq|x-y|-|x-z|\geq |x-y|-\sqrt{t}/4$. Since $|x-y|/\sqrt{t}\geq 1$,

$$\frac{|z-y|^2}{t}\geq\frac{|x-y|^2}{t}-\frac{|x-y|}{2\sqrt{t}}+\frac{1}{16}\geq\frac{|x-y|^2}{2t}+\frac{1}{16}.$$

Using Lemma 4.2,

$$\begin{aligned}|\nabla k_t(x+h,y)-\nabla k_t(x,y)| &\lesssim \Big(\frac{|h|}{\sqrt{t}}\Big)^{1-n/q} t^{-(n+1)/2}e^{-|x-y|^2/\alpha t}\Big(1+\frac{\sqrt{t}}{\rho(y)}\Big)^{-\gamma}\Big[\Big(\frac{\sqrt{t}}{\rho(x)}\Big)^{m_0}+1\Big]\\ &\lesssim \Big(\frac{|h|}{\sqrt{t}}\Big)^{\delta} t^{-(n+1)/2}e^{-|x-y|^2/\alpha t}\Big(1+\frac{\sqrt{t}}{\rho(y)}\Big)^{-\gamma}\Big(1+\frac{\sqrt{t}}{\rho(x)}\Big)^{m_0}. \qquad (4.2)\end{aligned}$$

Note that $1+\sqrt{t}/\rho(z)+\sqrt{t}/\rho(y)\geq 1+\sqrt{t}/\rho(z)\geq C(1+\sqrt{t}/\rho(x))^{1/(k_0+1)}$ for all $z\in B(x,2R)$. This along with Lemma 4.2 implies that

$$|\nabla k_t(x+h,y)-\nabla k_t(x,y)|\lesssim\Big(\frac{|h|}{\sqrt{t}}\Big)^{\delta} t^{-(n+1)/2}e^{-|x-y|^2/\alpha t}\Big(1+\frac{\sqrt{t}}{\rho(x)}\Big)^{-(\gamma/(k_0+1)+m_0)}. \qquad (4.3)$$

Setting $\lambda(\gamma)=\frac{1}{2}\min(\gamma,\gamma/(k_0+1)+m_0)$, we note that $(1+\sqrt{t}/\rho(x))(1+\sqrt{t}/\rho(y))\geq 1+\sqrt{t}/\rho(x)+\sqrt{t}/\rho(y)$. We then multiply side by side of (4.2) and (4.3) to find that

$$|\nabla k_t(x+h,y)-\nabla k_t(x,y)|\lesssim\Big(\frac{|h|}{\sqrt{t}}\Big)^{\delta} t^{-(n+1)/2}e^{-|x-y|^2/\alpha t}\Big(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\Big)^{-\lambda(\gamma)+m_0/2}.$$

Choosing γ such that $\lambda(\gamma)-m_0/2>N$,

$$|\nabla k_t(x+h,y)-\nabla k_t(x,y)|\lesssim\Big(\frac{|h|}{\sqrt{t}}\Big)^{\delta} t^{-(n+1)/2}e^{-|x-y|^2/\alpha t}\Big(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\Big)^{-N}.$$

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Case 2: $|x-y| \le \sqrt{t}$ and $|h| \le |x-y|/4$.

Let $R = |x-y|/8$. If $z \in B(x_0, 2R)$, then $|z-y| \sim |x-y|$. Using Lemma 4.2,

$$\begin{aligned}|\nabla k_t(x+h,y) - \nabla k_t(x,y)| &\lesssim \Big(\frac{|h|}{|x-y|}\Big)^{1-n/q} \frac{\sqrt{t}}{|x-y|} t^{-(n+1)/2} e^{-2|x-y|^2/\alpha t}\Big(1+\frac{\sqrt{t}}{\rho(y)}\Big)^{-2\gamma} \\ &\quad \times \Big[\Big(\frac{|x-y|}{\rho(x)}\Big)^{m_0} + 1 + \frac{|x-y|^2}{t}\Big] \\ &\lesssim \Big(\frac{|h|}{|x-y|}\Big)^{\delta} \frac{\sqrt{t}}{|x-y|} t^{-(n+1)/2} e^{-|x-y|^2/\alpha t} e^{-|x-y|^2/\alpha t} \\ &\quad \times \Big(1+\frac{\sqrt{t}}{\rho(y)}\Big)^{-\gamma}\Big(1+\frac{\sqrt{t}}{\rho(y)}\Big)^{-\gamma}\Big(1+\frac{\sqrt{t}}{\rho(x)}\Big)^{m_0},\end{aligned}$$

where we used $|h|/|x-y| \le 1, |x-y|/\sqrt{t} \le 1$ in the second inequality.

In view of Corollary 2.2,

$$\begin{aligned}|\nabla k_t(x+h,y) - \nabla k_t(x,y)| &\lesssim \Big(\frac{|h|}{|x-y|}\Big)^{\delta} \frac{\sqrt{t}}{|x-y|} t^{-(n+1)/2} e^{-|x-y|^2/\alpha t} e^{-|x-y|^2/\alpha t} \\ &\quad \times \Big(1+\frac{\sqrt{t}}{\rho(y)}\Big)^{-\gamma}\Big(1+\frac{\sqrt{t}}{\rho(x)}\Big)^{-\gamma/(k_0+1)+m_0} \\ &\lesssim \Big(\frac{|h|}{\sqrt{t}}\Big)^{\delta}\Big(\frac{\sqrt{t}}{|x-y|}\Big)^{\delta+1} t^{-(n+1)/2} e^{-|x-y|^2/\alpha t} e^{-|x-y|^2/\alpha t} \\ &\quad \times \Big(1+\frac{\sqrt{t}}{\rho(y)}\Big)^{-\gamma}\Big(1+\frac{\sqrt{t}}{\rho(x)}\Big)^{-\gamma/(k_0+1)+m_0}.\end{aligned}$$

Put $\lambda(\gamma) = \min(\gamma, \gamma/(k_0+1) - m_0)$. Using the inequality $e^{-x^2} \le C_\eta x^\eta$ for all $x \ge 1$, $\eta > 0$,

$$\begin{aligned}|\nabla k_t(x+h,y) - \nabla k_t(x,y)| &\lesssim \Big(\frac{|h|}{\sqrt{t}}\Big)^{\delta} t^{-(n+1)/2} e^{-|x-y|^2/\alpha t}\Big(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\Big)^{-\lambda(\gamma)} \\ &\lesssim \Big(\frac{|h|}{\sqrt{t}}\Big)^{\delta} t^{-(n+1)/2} e^{-|x-y|^2/\alpha t}\Big(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\Big)^{-N} \qquad (4.4)\end{aligned}$$

as long as $\lambda(\gamma) > N$.

Case 3: $|x-y|/4 < |h| \le \sqrt{t}$. By the semigroup property,

$$k_t(x+h,y) - k_t(x,y) = \int_{\mathbb{R}^n} k_{t/2}(y,z)[k_{t/2}(x+h,z) - k_{t/2}(x,z)]\,dz.$$

Thus,

$$\begin{aligned}|\nabla k_t(x+h,y) - \nabla k_t(x,y)| &\le \int_{\mathbb{R}^n} k_{t/2}(y,z)|\nabla k_{t/2}(x+h,z) - \nabla k_{t/2}(x,z)|\,dz \\ &= \int_{|z-x|\le 4|h|} \ldots + \int_{|z-x|>4|h|} \ldots = S_1 + S_2.\end{aligned}$$

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By Lemmas 4.1 and 4.2,

$$S_1 \lesssim t^{-(n+1)/2}\Big(\frac{|h|}{\sqrt{t}}\Big)^n\Big(1+\frac{\sqrt{t}}{\rho(y)}\Big)^{-2\gamma}.$$

Hence, from the assumption $|x-y|/4<|h|<\sqrt{t}$,

$$\begin{aligned}S_1 &\lesssim t^{-(n+1)/2}\Big(\frac{|h|}{\sqrt{t}}\Big)^n\Big(1+\frac{\sqrt{t}}{\rho(y)}\Big)^{-2\gamma}\\ &\lesssim t^{-(n+1)/2}\Big(\frac{|h|}{\sqrt{t}}\Big)^\delta\Big(1+\frac{\sqrt{t}}{\rho(y)}\Big)^{-2\gamma}e^{-|x-y|^2/\alpha t}e^{|x-y|^2/\alpha t}\\ &\lesssim t^{-(n+1)/2}\Big(\frac{|h|}{\sqrt{t}}\Big)^\delta\Big(1+\frac{\sqrt{t}}{\rho(y)}\Big)^{-2\gamma}e^{-|x-y|^2/\alpha t}.\end{aligned}$$

We now take care of S_2. Due to $|z-x|>4|h|>|x-y|$ and $|h|\le\sqrt{2}\sqrt{t/2}:=M\sqrt{t/2}$, applying (4.4),

$$\begin{aligned}S_2 &\lesssim \int_{|z-x|>4|h|}k_{t/2}(y,z)\Big(\frac{|h|}{\sqrt{t}}\Big)^\delta t^{-(n+1)/2}e^{-|x-y|^2/\alpha t}\,dz\\ &\lesssim \Big(1+\frac{\sqrt{t}}{\rho(y)}\Big)^{-2\gamma}\Big(\frac{|h|}{\sqrt{t}}\Big)^\delta t^{-(n+1)/2}e^{-|x-y|^2/\alpha t}\int_{|z-x|>4|h|}t^{-n/2}e^{-|y-z|^2/\alpha t}\,dz\\ &\lesssim \Big(1+\frac{\sqrt{t}}{\rho(y)}\Big)^{-2\gamma}\Big(\frac{|h|}{\sqrt{t}}\Big)^\delta t^{-(n+1)/2}e^{-|x-y|^2/\alpha t}.\end{aligned}$$

Hence,

$$S_1+S_2\lesssim\Big(1+\frac{\sqrt{t}}{\rho(y)}\Big)^{-2\gamma}\Big(\frac{|h|}{\sqrt{t}}\Big)^\delta t^{-(n+1)/2}e^{-|x-y|^2/\alpha t}.$$

Using Lemma 4.2,

$$\begin{aligned}S_1+S_2 &\lesssim \Big(1+\frac{\sqrt{t}}{\rho(y)}\Big)^{-\gamma}\Big(1+\frac{\sqrt{t}}{\rho(x)}\Big)^{-\gamma/(k_0+1)}\Big(\frac{|h|}{\sqrt{t}}\Big)^\delta t^{-(n+1)/2}e^{-|x-y|^2/\alpha t}\\ &\lesssim \Big(\frac{|h|}{\sqrt{t}}\Big)^\delta t^{-(n+1)/2}e^{-|x-y|^2/\alpha t}\Big(1+\frac{\sqrt{t}}{\rho(y)}+\frac{\sqrt{t}}{\rho(x)}\Big)^{-N},\end{aligned}$$

provided that $\min(\gamma,\gamma/(k_0+1))>N$. This completes our proof. □

Setting $q_t(x,y)=k_t(x,y)-p_t(x,y)$, by the Kato–Trotter formula (see [11]),

$$q_t(x,y)=\int_0^t\int_{\mathbb{R}^n}p_s(y,z)V(z)k_{t-s}(x,z)\,dz\,ds.$$

Therefore,

$$\nabla q_t(x,y)=\int_0^t\int_{\mathbb{R}^n}p_s(y,z)V(z)\nabla k_{t-s}(x,z)\,dz\,ds.$$

From Proposition 4.1, using the arguments in [11, Propositions 2.16 and 2.17], we have the following results.

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Proposition 4.4. *There exist a rapidly decaying function $\psi \geq 0$ and $C > 0$ such that*

$$\nabla q_t(x,y) \leq C\Big(\frac{\sqrt{t}}{\rho(x)}\Big)^{2-n/q} \frac{1}{\sqrt{t}}\psi_t(x-y)$$

for all x, y in $\mathbb{R}^n$ and $t > 0$.

Proposition 4.5. *For every $0 < \delta < 1 - n/q$ and $M > 0$, there exist a rapidly decaying function ψ and $C > 0$ such that*

$$|\nabla q_t(x+h,y) - \nabla q_t(x,y)| \leq C\Big(\frac{|h|}{\rho(y)}\Big)^{\delta} \frac{1}{\sqrt{t}}\psi_t(x-y)$$

for all $x, y \in \mathbb{R}^n$ and $t > 0$, with $|h| < M\rho(x)$ and $|h| < |x-y|/4$.

5. Boundedness of Riesz transforms and square functions

5.1. Rieszs transforms. Let $\mathcal{K}(x,y)$ denote the kernel of $\mathcal{R}$. The following lemma is essentially taken from [2].

Lemma 5.1. *If $V \in B_q$ with $q > n$, then for every k there exists a constant C such that*

$$|\mathcal{K}(x,y)| \leq \frac{C}{\Big(1 + \dfrac{|x-z|}{\rho(x)}\Big)^k} \frac{1}{|x-z|^n}.$$

Now we are ready to prove the first main result.

Theorem 5.2. *Let $\mathcal{V} \in B_n$ and $\beta \in (0,1)$, $\theta \geq 0$ such that $0 \leq \beta + \theta < 1 - n/q_0$, where $q_0 = \sup\{q : \mathcal{V} \in B_q\}$. Assume that the weight function $w \in D_\sigma^{\mathcal{L},\infty} \cap A_p^{\mathcal{L},\infty}$ for some $p \in (1,\infty)$ and*

$$1 \leq \sigma < 1 + \frac{1 - n/q_0 - \beta - \theta}{n}.$$

Then the Riesz transform $\mathcal{R}$ is bounded from $\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta}(w)$ into $\Lambda_{\mathcal{L}}^{\beta}(w)$.

Proof. We use the idea in [1]. Note that

$$\mathcal{R}f(x) = \nabla\mathcal{L}^{-1/2}f(x) = \int_0^\infty \nabla e^{-t\mathcal{L}}f(x)\,\frac{dt}{\sqrt{t}} = \int_{\mathbb{R}^n}\int_0^\infty \nabla k_t(x,y)\,\frac{dt}{\sqrt{t}}f(y)\,dy.$$

Let $f \in \mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}(w)$. We will see that for x and y in $\mathbb{R}^n$,

$$|\mathcal{R}f(x) - \mathcal{R}f(y)| \lesssim \|f\|_{\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta}(w)}[\mathcal{W}_\beta(x,|x-y|) + \mathcal{W}_\beta(y,|x-y|)], \tag{5.1}$$

provided that $|x-y| \leq \rho(x)$, and

$$\int_{B(x,\rho(x))} |\mathcal{R}f(u)|\,du \leq \|f\|_{\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}}\rho(x)^\beta w(B(x,\rho(x))). \tag{5.2}$$

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Since $w \in D_\sigma^{\mathcal{L},\infty} \cap A_\infty^{\mathcal{L},\infty}$, there exist $\theta_1, \theta_3 \geq 0$ and $1 < p < \infty$ such that $w \in D_\sigma^{\mathcal{L},\theta_1} \cap A_p^{\mathcal{L},\theta_3}$. Thus, $w^{1-p'} \in A_{p'}^{\mathcal{L},\theta_3}$.

Note that we may choose $q > n$ and $\beta + \theta_2 < \delta_0 < 1 - n/q$ such that

$$\sigma < 1 + \frac{\delta_0 - \beta - \theta_2}{n}.$$

Suppose that $\|f\|_{\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}} = 1$; let us start with (5.2). For $B = B(x, \rho(x))$, we write $f = f_1 + f_2$, with $f = f\chi_{2B}$.

By [3, Theorem 3], $\mathcal{R}$ is bounded on $L^{p'}(w^{1-p'})$. Thus,

$$\begin{aligned}\int_B |\mathcal{R}f_1| &\lesssim w(B)^{1/p}\Big(\int_B |\mathcal{R}f_1|^{p'} w^{1-p'}\Big)^{1/p'}\\ &\lesssim w(B)^{1/p}\Big(\int_{2B} |f|^{p'} w^{1-p'}\Big)^{1/p'}\\ &\lesssim w(B)\rho(x)^\beta,\end{aligned}$$

where in the last inequality we used Proposition 3.5 and the fact that $w \in D_\sigma^{\mathcal{L},\theta_1}$.

Applying Lemma 5.1,

$$\begin{aligned}\int_B |\mathcal{R}f_2| &\lesssim \int_B \int_{(2B)^c} |\mathcal{K}(x,z) f(z)|\,dz\,dx\\ &\lesssim \int_B \int_{(2B)^c} \Big(\frac{\rho(x)}{|x-z|}\Big)^k \frac{1}{|x-z|^n} |f(z)|\,dz\,dx\\ &\lesssim \rho(x_0)^{k+n} \int_{(2B)^c} \frac{f(z)}{|x_0 - z|^{k+n}}\,dz,\end{aligned}$$

where we used $\rho(x) \sim \rho(x_0)$ and $|x - z| \sim |x_0 - z|$.

Setting $B_j = 2^j B$, due to $w \in D_\sigma^{\mathcal{L},\theta_1}$,

$$\begin{aligned}\int_{(2B)^c} \frac{|f(z)|}{|x-z|^{n+k}}\,dz &\leq \sum_{j=1}^\infty \int_{B_{j+1}\setminus B_j} \frac{|f(z)|}{|x-z|^{n+k}}\,dz\\ &\leq \rho(x)^{-n-k} \sum_{j=1}^\infty 2^{-j(n+k)} \int_{B_{j+1}} |f(z)|\,dz\\ &\lesssim \rho(x)^{-n+\beta-k} w(B) \sum_{j=1}^\infty 2^{-j(n-\beta+k-n\sigma-\theta_2)}.\end{aligned}$$

Taking k to be sufficiently large, the last sum is finite and hence (5.2) holds true. To see (5.1), for $|x - y| \leq \rho(x)$,

$$\begin{aligned}|\mathcal{R}f(x) - \mathcal{R}f(y)| &\leq \Big|\int_0^{\rho(x)^2} \int_{\mathbb{R}^n} [\nabla k_t(x,z) - \nabla k_t(y,z)] f(z)\,dz\,\frac{dt}{\sqrt{t}}\Big|\\ &\quad + \Big|\int_{\rho(x)^2}^\infty \int_{\mathbb{R}^n} [\nabla k_t(x,z) - \nabla k_t(y,z)] f(z)\,dz\,\frac{dt}{\sqrt{t}}\Big|\\ &:= A_1 + A_2. \end{aligned} \tag{5.3}$$

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If $t > \rho(x)^2$, then, from $|x-y| \le \rho(x)$, we have $|x-y| < \sqrt{t}$. By Proposition 4.3,

$$A_2 \lesssim \int_{\rho(x)^2}^{\infty} \int_{\mathbb{R}^n} |k_t(x,z) - k_t(y,z)|\,|f(z)|\,dz\,\frac{dt}{t}$$
$$\lesssim |x-y|^{\delta} \int_{\rho(x)^2}^{\infty} \int_{\mathbb{R}^n} e^{-|x-z|^2/ct}|f(z)|\,dz t^{(-n-\delta)/2}\,\frac{dt}{t}$$

for all $0 < \delta < \delta_0$.

Moreover, for $B = B(x, \sqrt{t})$,

$$\int_{\mathbb{R}^n} e^{-|x-z|^2/ct}|f(z)|\,dz \lesssim \int_B |f| + t^{M/2} \sum_{k=0}^{\infty} \int_{2^{k+1}B\setminus 2^k B} \frac{|f(z)|}{|x-z|^M}\,dz$$

for some $M > 1$ which is fixed later.

Since $f \in \mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}(w)$ and $t > \rho(x)^2$,

$$\int_B |f| \lesssim w(B) t^{\beta/2} \Psi_{\theta_2}(B).$$

To deal with the sum in k, we use $w \in D_{\sigma}^{\mathcal{L},\theta_1}$ to obtain

$$t^{M/2} \sum_{k=0}^{\infty} \int_{2^{k+1}B\setminus 2^k B} \frac{|f(z)|}{|x-z|^M}\,dz \lesssim \sum_{k=0}^{\infty} 2^{-kM} \int_{2^{k+1}B} |f|$$
$$\lesssim t^{\beta/2}\Psi_{\theta_2}(B) w(B) \sum_{k=0}^{\infty} 2^{-k(M-\beta-n\sigma-\theta_2)},$$

and the sum being finite for sufficiently large M.

Since $|x-y| \le \rho(x) < \sqrt{t}$ and $-n+\beta+n\sigma-\delta+\theta_2 < 0$, then, by choosing δ close to δ_0,

$$A_2 \lesssim |x-y|^{\delta} \int_{\rho(x)^2}^{\infty} w(B(x,\sqrt{t}))\Big(\frac{\sqrt{t}}{|x-y|}\Big)^{\theta_2} t^{(-n+\beta-\delta)/2}\,\frac{dt}{t}$$
$$\lesssim |x-y|^{\delta-n\sigma-\theta_2} w(B(x,|x-y|)) \int_{|x-y|^2}^{\infty} t^{(-n+\beta-\delta+n\sigma+\theta_2)/2}\,\frac{dt}{t}$$
$$\lesssim w(B(x,|x-y|))|x-y|^{-n+\beta}$$
$$\lesssim \mathcal{W}_{\beta}(x,|x-y|).$$

In order to deal with the second term of (5.3), we write

$$\Big|\int_0^{\rho(x)^2} \int_{\mathbb{R}^n} [\nabla k_t(x,z) - \nabla k_t(y,z)] f(z)\,dz\,\frac{dt}{\sqrt{t}}\Big| \le I + J,$$

where

$$I = \Big|\int_0^{\rho(x)^2} \int_{\mathbb{R}^n} [\nabla q_t(x,z) - \nabla q_t(y,z)] f(z)\,dz\,\frac{dt}{\sqrt{t}}\Big|$$

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and

$$J = \left| \int_0^{\rho(x)^2} \int_{\mathbb{R}^n} [\nabla p_t(x,z) - \nabla p_t(y,z)] f(z)\, dz \frac{dt}{\sqrt{t}} \right|.$$

To estimate I, for $B = B(x, 4|x-y|)$, we write

$$I \le I_1 + I_2 + I_3,$$

where

$$I_1 = \int_0^{\rho(x)^2} \int_{B^c} [\nabla q_t(x,z) - \nabla q_t(y,z)] f(z)\, dz \frac{dt}{\sqrt{t}},$$

$$I_2 = \int_0^{\rho(x)^2} \int_B |\nabla q_t(x,z)||f(z)|\, dz \frac{dt}{\sqrt{t}},$$

and

$$I_3 = \int_0^{\rho(x)^2} \int_B |\nabla q_t(y,z)||f(z)|\, dz \frac{dt}{\sqrt{t}}.$$

We denote $B_j = B(x, 2^{j+3}|x-y|)$ and j_0 as the integer part of $\log_2(\rho(x)/|x-y|)$. If $z \in B^c$, then, by applying Proposition 4.5, for $0 < \delta < \delta_0$, there exists a rapidly decaying function ψ such that

$$\begin{aligned} I_1 &\lesssim |x-y|^\delta \int_0^{\rho(x)^2} \int_{B^c} \frac{\psi_t(z-x)}{\rho(z)^\delta} |f(z)|\, dz \frac{dt}{t} \\ &\lesssim \left(\frac{|x-y|}{\rho(x)}\right)^\delta \int_0^{\rho(x)^2} \int_{B^c} \left(1 + \frac{|x-z|}{\rho(x)}\right)^{\delta k_0} \psi_t(z-x) |f(z)|\, dz \frac{dt}{t}, \end{aligned}$$

where the last inequality follows from Proposition 2.1.

Next,

$$\int_{B^c} \left(1 + \frac{|x-z|}{\rho(x)}\right)^{\delta k_0} \psi_t(z-x)|f(z)|\, dz = \sum_{j=0}^{\infty} \int_{B_j \setminus B_{j-1}} \left(1 + \frac{|x-z|}{\rho(x)}\right)^{\delta k_0} \psi_t(z-x)|f(z)|\, dz,$$

where $B_j = B(x, 2^{j+3}|x-y|)$. Thus, $I_1 \le I_{11} + I_{12}$, where

$$I_{11} = \left(\frac{|x-y|}{\rho(x)}\right)^\delta \int_0^{\rho(x)^2} \sum_{j=0}^{j_0} \int_{B_j \setminus B_{j-1}} \left(1 + \frac{|x-z|}{\rho(x)}\right)^{\delta k_0} \psi_t(z-x)|f(z)|\, dz \frac{dt}{t}$$

and j_0 is the integer part of $\log_2(\rho(x)/|x-y|)$.

If $j \le j_0$ and $z \in B_j \setminus B_{j-1}$, then $(1 + |x-z|/\rho(x))^{\delta k_0} \lesssim 1$. Moreover, since $\psi_t(z-x) \lesssim t^{\varepsilon/2}/|x-z|^{n+\varepsilon}$ for some $\varepsilon > 0$,

$$\begin{aligned} I_{11} &\lesssim \left(\frac{|x-y|}{\rho(x)}\right)^\delta \int_0^{\rho(x)^2} t^{\varepsilon/2} \frac{dt}{t} \sum_{j=0}^{j_0} \int_{B_j \setminus B_{j-1}} \frac{|f(z)|}{|x-z|^{n+\varepsilon}}\, dz \\ &\lesssim \frac{|x-y|^{\delta-n-\varepsilon}}{\rho(x)^{\delta-\varepsilon}} \sum_{j=0}^{j_0} 2^{-j(n+\varepsilon)} \int_{B_j} |f(z)|\, dz \end{aligned}$$

for sufficiently small ε.

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It follows from Proposition 3.5 and the fact that $w \in D_\sigma^{\mathcal{L},\theta_1}$, in the case $\sigma > 1$ or $\beta > 0$, that

$$\sum_{j=0}^{j_0} 2^{-j(n+\varepsilon)} \int_{B_j} |f(z)|\,dz \lesssim \sum_{j=0}^{j_0} 2^{-j(n+\varepsilon)} w(B_j)|B_j|^{\beta/n}\Big(\frac{\rho(x)}{2^{j+3}|x-y|}\Big)^{n\sigma-n+\beta}$$
$$\lesssim \frac{\rho(x)^{n\sigma-n+\beta}}{|x-y|^{n\sigma-n}} w(B) \sum_{j=0}^{j_0} 2^{-j\varepsilon}$$
$$\lesssim \frac{\rho(x)^{n\sigma-n+\beta}}{|x-y|^{n\sigma-n}} w(B),$$

where we used $\Psi_{\theta_2}(2^j B) \lesssim 1$ for all $j = \overline{0, j_0}$ in the first inequality.

From $1 \le \sigma < (\delta_0 - \beta - \theta_2)/n + 1$ and $|x-y| < \rho(x)$, by choosing ε small enough and δ close to δ_0, we have $\delta - \beta - n\sigma + n - \varepsilon > 0$. Therefore,

$$I_{11} \lesssim \Big(\frac{|x-y|}{\rho(x)}\Big)^{\delta-\beta-n\sigma+n-\varepsilon} \frac{w(B)}{|x-y|^{n-\beta}} \lesssim \frac{w(B)}{|x-y|^{n-\beta}}.$$

Similarly to the case of $\beta = 0$ and $\sigma = 1$, using Proposition 3.5 and the inequality

$$1 + \log_2(t) \lesssim t^{\varepsilon/2}, \quad t > 1/8, \tag{5.4}$$

we obtain the same estimate of I_{11} by an argument as above.

Next we estimate I_{12}. For $M > \delta k_0 + n\sigma + \beta$,

$$\psi_t(z-x) \lesssim t^{(M-n)/2}|z-x|^{-M}.$$

Also, if $z \in B_j \setminus B_{j-1}$ for $j > j_0$, then $|x-z| > \rho(x)$. Therefore,

$$I_{12} \le C\Big(\frac{|x-y|}{\rho(x)}\Big)^{\delta+\delta k_0} \int_0^{\rho(x)^2} t^{(M-n)/2} \frac{dt}{t} \sum_{j=j_0+1}^{\infty} 2^{j\delta k_0} \int_{B_j\setminus B_{j-1}} \frac{|f(z)|}{|z-x|^M}$$
$$\le C\frac{|x-y|^{\delta+\delta k_0-M}}{\rho(x)^{\delta+\delta k_0-M+n}} \sum_{j=j_0+1}^{\infty} 2^{-j(M-\delta k_0)} \int_{B_j} |f(z)|\,dz.$$

For $j > j_0$, the radius of B_j is $2^{j+3}|x-y| > \rho(x)$ and $\Psi_{\theta_2}(B) \lesssim 1$. Hence,

$$\int_{B_j} |f(z)|\,dz \lesssim w(B_j)|B_j|^{\beta/n}\Psi_{\theta_2}(B_j)$$
$$\lesssim 2^{j(n\sigma+\beta+\theta_2)}|x-y|^\beta w(B),$$

which in turn implies that

$$I_{12} \lesssim \Big(\frac{|x-y|}{\rho(x)}\Big)^{-M+\delta k_0+\delta+n} \frac{w(B)}{|x-y|^{n-\beta}} \sum_{j=j_0+1}^{\infty} 2^{-j(M-\delta k_0-n\sigma-\beta-\theta)}$$
$$\lesssim \Big(\frac{|x-y|}{\rho(x)}\Big)^{n-n\sigma+\delta-\beta-\theta} \frac{w(B)}{|x-y|^{n-\beta}}$$
$$\lesssim \frac{w(B)}{|x-y|^{n-\beta}},$$

with an appropriate choice of δ.

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We next estimate I_2. Let $M > n$. From Proposition 4.4, for $t < \rho(x)^2$,

$$\nabla q_t(x,y) \lesssim \Big(\frac{\sqrt{t}}{\rho(x)}\Big)^{\delta} t^{-(n+1)/2}\Big(1+\frac{|x-z|}{\sqrt{t}}\Big)^{-M}. \tag{5.5}$$

Then we may write $I_2 = I_{21} + I_{22}$, where

$$I_{21} = \int_0^{|x-y|^2}\int_B |\nabla q_t(x,z)|\,|f(z)|\,dz\,\frac{dt}{t}$$

and

$$I_{22} = \int_{|x-y|^2}^{\rho(x)^2}\int_B |\nabla q_t(x,z)|\,|f(z)|\,dz\,\frac{dt}{t}.$$

We consider the case $\sigma > 1$ or $\beta > 0$. To take care of I_{21}, let $B_t = B(x,\sqrt{t})$ with $t \le \rho(x)^2$ and N be the integer part of $\log_2(4|x-y|/\sqrt{t})$. Using estimate (5.5),

$$\begin{aligned}\int_B |q_t(x,z)||f(z)|\,dz &\le \frac{t^{(\delta_0-n)/2}}{\rho(x)^{\delta_0}}\Big(\int_{B_t}|f| + t^{M/2}\int_{B\setminus B_t}\frac{|f(z)|}{|x-z|^M}\,dz\Big)\\ &\le \frac{t^{(\delta_0-n)/2}}{\rho(x)^{\delta_0}}\Big(\int_{B_t}|f| + t^{M/2}\sum_{j=0}^{N}\int_{2^{j+1}B_t\setminus 2^jB_t}\frac{|f(z)|}{|x-z|^M}\,dz\Big)\\ &\le \frac{t^{(\delta_0-n)/2}}{\rho(x)^{\delta_0}}\Big(\sum_{j=0}^{N+1}2^{-jM}\int_{2^jB_t}|f|\Big).\end{aligned}$$

Since $w \in D_\sigma^{\mathcal{L},\delta_1}$ and $\Psi_{\theta_2}(2^jB_t) \lesssim 1$ for all $t < \rho(x)^2$, $j \le N+1$,

$$\begin{aligned}\int_B |q_t(x,z)||f(z)|\,dz &\lesssim \frac{t^{(\delta_0-n\sigma)/2}}{\rho(x)^{\delta_0-n\sigma+n-\beta}}w(B_t)\sum_{j=0}^{N+1}2^{-j(M-n)}\\ &\lesssim \frac{t^{(\delta_0-n\sigma)/2}}{\rho(x)^{\delta_0-n\sigma+n-\beta}}w(B_t)\end{aligned}$$

as long as $M > n$.

Thus,

$$\begin{aligned}I_{21} &\lesssim \rho(x)^{-\delta_0+n\sigma-n+\beta}\int_0^{|x-y|^2} t^{(\delta_0-n\sigma)/2}w(B_t)\,\frac{dt}{t}\\ &\lesssim \rho(x)^{-\delta_0+n\sigma-n+\beta}\int_0^{|x-y|^2} t^{(\delta_0-\beta-n\sigma+n)/2}\mathcal{W}_\beta(x,\sqrt{t})\,\frac{dt}{t}\\ &\lesssim \rho(x)^{-\delta_0+n\sigma-n+\beta}\int_0^{|x-y|^2} t^{(\delta_0-\beta-n\sigma+n)/2}\,\frac{dt}{t}\mathcal{W}_\beta(x,|x-y|)\\ &\lesssim \Big(\frac{|x-y|}{\rho(x)}\Big)^{\delta_0-\beta-n\sigma+n}\mathcal{W}_\beta(x,|x-y|).\end{aligned}$$

Notice that $\delta_0 - \beta - n\sigma + n > 0$ and $|x - y| < \rho(x)$. We then have

$$I_{21} \lesssim \mathcal{W}_\beta(x, |x - y|).$$

To deal with I_{22}, we use (5.5) and Proposition 3.5 to obtain

$$\begin{aligned}
I_{22} &= \int_{|x-y|^2}^{\infty} \int_B |\nabla q_t(x,z)|\,|f(z)|\,dz\,\frac{dt}{t} \\
&\lesssim \rho(x)^{-\delta_0} \int_{|x-y|^2}^{\infty} t^{(\delta_0-n)/2}\,\frac{dt}{t} \int_B |f| \\
&\lesssim \frac{w(B)}{|x-y|^{n-\beta}} \Big(\frac{|x-y|}{\rho(x)}\Big)^{n-n\sigma+\delta_0-\beta} \\
&\lesssim \frac{w(B)}{|x-y|^{n-\beta}} \leq C\mathcal{W}_\beta(x, |x-y|),
\end{aligned}$$

where $n - n\sigma + \delta_0 - \beta > 0, |x - y| < \rho(x)$, and $\Psi_{\theta_2}(B) \lesssim 1$.

Thus,

$$I_2 \lesssim \mathcal{W}_\beta(x, |x - y|).$$

Similarly to the case of $\beta = 0$ and $\sigma = 1$, using Proposition 3.5 and inequality (5.4), we also arrive at the same estimate of I_2 by an argument as above.

Using the same argument as above,

$$I_3 \lesssim \mathcal{W}_\beta(x, |x - y|).$$

Taking the estimates of I_1, I_2, and I_3 into account, we find that

$$I \lesssim \mathcal{W}_\beta(x, |x - y|).$$

We now take care of J. To proceed, observe that

$$\int_{\mathbb{R}^n} [\nabla p_t(x,z) - \nabla p_t(y,z)] f_B\,dz = 0.$$

Hence,

$$J = \Big| \int_0^{\rho(x)^2} \int_{\mathbb{R}^n} [\nabla p_t(x,z) - \nabla p_t(y,z)](f(z) - f_B)\,dz\,\frac{dt}{\sqrt{t}} \Big| \leq J_1 + J_2,$$

where

$$J_1 = \Big| \int_0^{\rho(x)^2} \int_{2B} [\nabla p_t(x,z) - \nabla p_t(y,z)]|f(z) - f_B|\,dz\,\frac{dt}{\sqrt{t}} \Big|$$

and

$$J_2 = \Big| \int_0^{\rho(x)^2} \int_{(2B)^c} [\nabla p_t(x,z) - \nabla p_t(y,z)]|f(z) - f_B|\,dz\,\frac{dt}{\sqrt{t}} \Big|,$$

with $B = B(x, |x - y|)$.

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For $z \in (2B)^c$, we have $|y-z| \sim |x-z|$. Applying the mean value theorem,

$$|\nabla p_t(x,z) - \nabla p_t(y,z)| \lesssim \Big(\frac{e^{-u^2/(Ct)}}{t^{n/2+1}} + \frac{u^2 e^{-u^2/(Ct)}}{t^{n/2+2}}\Big)|x-y|,$$

with $\min(|x-z|, |y-z|) \le u \le \max(|x-z|, |y-z|)$.

Thus,

$$|\nabla p_t(x,z) - \nabla p_t(y,z)| \lesssim \Big(\frac{e^{-|x-z|^2/(Ct)}}{t^{n/2+1}} + \frac{|x-z|^2 e^{-|x-z|^2/(Ct)}}{t^{n/2+2}}\Big)|x-y|.$$

Using the change of variables $s = |x-z|^2/Ct$,

$$\begin{aligned}
J_2 &\le |x-y| \int_{(2B)^c} |f(z) - f_B| \int_0^{\rho(x)^2} \frac{e^{-|x-z|^2/(Ct)}}{t^{n/2+1}} \frac{dt}{\sqrt{t}}\, dz \\
&\quad + |x-y| \int_{(2B)^c} |f(z) - f_B|\, |x-z|^2 \int_0^{\rho(x)^2} \frac{e^{-|x-z|^2/(Ct)}}{t^{n/2+2}} \frac{dt}{\sqrt{t}}\, dz \\
&\le |x-y| \int_{(2B)^c} \frac{|f(z) - f_B|}{|x-z|^{n+1}}\, dz \int_0^\infty e^{-s} s^{((n+1)/2)-1}\, ds \\
&\quad + |x-y| \int_{(2B)^c} \frac{|f(z) - f_B|}{|x-z|^{n+1}}\, dz \int_0^\infty e^{-s} s^{((n+3)/2)-1}\, ds.
\end{aligned}$$

It is easy to see that the last expression is bounded by

$$C|x-y| \int_{(2B)^c} \frac{|f(z) - f_B|}{|x-z|^{n+1}}\, dz.$$

Since $w \in D_\sigma^{\mathcal{L},\theta_1}$, the above expression is bounded by

$$\begin{aligned}
C|x-y| \int_{(2B)^c} \frac{|f(z) - f_B|}{|x-z|^{n+1}}\, dz &\lesssim |x-y|\, |B|^{-1/n} \sum_{j=1}^\infty 2^{-j} \frac{1}{|2^j B|} \int_{2^{j+1}B \setminus 2^j B} |f(z) - f_B|\, dz \\
&\lesssim |x-y|\, |B|^{-1/n} \sum_{j=1}^\infty 2^{-j} \sum_{k=1}^{j+1} \frac{1}{|2^k B|} \int_{2^k B} |f(z) - f_{2^k B}|\, dz.
\end{aligned}$$

The last expression is bounded by

$$C|x-y||B|^{-1/n} \sum_{j=1}^\infty 2^{-j} \sum_{k=1}^{j+1} \frac{w(2^k B)\Psi_{\theta_2}(2^k B)}{|2^k B|^{1-\beta/n}} \lesssim |x-y| \sum_{k=1}^\infty \frac{w(2^k B)}{|2^k B|^{1-\beta/n}} |2^k B|^{-1/n} 2^{k\theta_2}.$$

This implies that

$$\begin{aligned}
J_2 &\lesssim |x-y| \sum_{k=1}^\infty \frac{w(2^k B)}{(2^k|x-y|)^{n+1-\beta}} 2^{k\theta_2} \\
&\lesssim |x-y|^{\beta-n} w(B)\Psi_{\theta_1}(B) \sum_{k=1}^\infty 2^{-k(n+1-\beta-n\sigma-\theta_2)} \\
&\lesssim \int_B \frac{w(u)}{|x-u|^{n-\beta}}\, du = \mathcal{W}_\beta(x, |x-y|).
\end{aligned}$$

To deal with J_1, we write

$$\int_0^{\rho(x)^2}\int_{2B}|\nabla p_t(x,z)|\,|f(z)-f_B|\,dz\,\frac{dt}{\sqrt{t}}\lesssim\int_0^{\infty}\int_{2B}|x-z|\frac{e^{-|x-z|^2/Ct}}{t^{n/2+1}}|f(z)-f_B|\,dz\,\frac{dt}{\sqrt{t}}.$$

By using the change of variables $s=|x-z|^2/Ct$, the last expression is

$$C\int_{2B}\frac{|f(z)-f_B|}{|x-z|^n}\int_0^{\infty}e^{-s}s^{((n+1)/2)-1}\,ds\lesssim\int_{2B}\frac{|f(z)-f_B|}{|x-z|^n}\,dz.$$

Setting $B_j=2^{-j+1}B$,

$$\begin{aligned}\int_{2B}\frac{|f(z)-f_B|}{|x-z|^n}\,dz&\le\sum_{j=0}^{\infty}\int_{B_j\backslash B_{j+1}}\frac{|f(z)-f_B|}{|x-z|^n}\,dz\\&\lesssim\sum_{j=0}^{\infty}\frac{1}{|B_j|}\int_{B_j}|f(z)-f_B|\,dz.\end{aligned}$$

Hence,

$$\begin{aligned}J_1&\lesssim\sum_{j=0}^{\infty}\sum_{k=1}^{j}\frac{1}{|B_k|}\int_{B_k}|f(z)-f_{B_k}|\,dz\lesssim\sum_{j=0}^{\infty}\sum_{k=1}^{j}\frac{w(B_k)\Psi_{\theta_2}(B_k)}{|B_k|^{1-\beta/n}}\\&\lesssim\sum_{j=0}^{\infty}\Big(\frac{2^j}{|x-y|}\Big)^{n-\beta}w(B_j)\lesssim\sum_{j=0}^{\infty}\Big(\frac{2^j}{|x-y|}\Big)^{n-\beta}w(B_j\backslash B_{j+1})\\&\lesssim\sum_{j=0}^{\infty}\int_{B_j\backslash B_{j+1}}\frac{w(z)}{|x-z|^{n-\beta}}\,dz\lesssim\int_{2B}\frac{w(z)}{|x-z|^{n-\beta}}\,dz=\mathcal{W}_\beta(x,|x-y|),\end{aligned}$$

where we used $\Psi_{\theta_2}(B_k)\lesssim 1$ in the third inequality.

This and the estimate of J_2 imply that $J\lesssim\mathcal{W}_\beta(x,|x-y|)$, which completes the proof of the theorem. □

5.2. Littlewood–Paley square function. In this subsection, we consider the boundedness of the Littlewood–Paley square function $\mathcal{G}_{\mathcal{L}}$ defined in (1.1) on $\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}(w)$. We assume that $Q_t(x,y)$ is the kernel of the operator $t^2\mathcal{L}e^{-t^2\mathcal{L}}$. In what follows, we always set $V_r(x)=|B(x,r)|$ and $V(x,y)=|B(x,|x-y|)|$ for all $x,y\in\mathbb{R}^n$ and $r\in(0,\infty)$. By [21], we have the following lemma.

LEMMA 5.3. *Let $\mathcal{V}\in B_{n/2}$. Then there exist constants $\varepsilon\in(0,1]$, $\gamma,\delta_1\in(0,\infty)$, and $\delta_2\in(0,1)$ such that:*

(i) $$|Q_t(x,y)|\lesssim\frac{1}{V_t(x)+V(x,y)}\Big(\frac{t}{t+|x-y|}\Big)^{\gamma}\Big(\frac{\rho(x)}{t+\rho(x)}\Big)^{\delta_1};$$

(ii) $$|Q_t(x,y)-Q_t(x',y)|\lesssim\frac{1}{V_t(x)+V(x,y)}\Big(\frac{|x-x'|}{t+|x-y|}\Big)^{\varepsilon}\Big(\frac{t}{t+|x-y|}\Big)^{\gamma};$$

(iii) $$\Big|\int_{\mathbb{R}^n}Q_t(x,y)\,dy\Big|\lesssim\Big(\frac{t}{t+\rho(x)}\Big)^{\delta_2}.$$

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Next we prove the second main theorem.

THEOREM 5.4. *Assume that* $\mathcal{V} \in B_{n/2}$ *and* ε, γ, δ_1, δ_2 *are the constants defined as in Lemma 5.3. Set* $\Theta = \min(\varepsilon/3, \gamma, \delta_1, \delta_2/3)$. *Let* $p, \sigma \geq 1$, $\beta \in (0,1)$, θ, θ_1, $\theta_2 \geq 0$ *be such that*

$$\beta + \theta_1 + \theta_1 < \Theta, \quad \sigma < 1 + \frac{\Theta - \beta - \theta_1 - \theta_2}{n}$$

and $\phi := \theta_1(k_0 + 2) + \theta_2 + \theta(k_0 + 1)$, *with* k_0 *defined in Proposition 2.1. If* $w \in D_\sigma^{\mathcal{L},\theta_1} \cap A_p^{\mathcal{L},\theta}$, *then the Littlewood–Paley square function* $\mathcal{G}_{\mathcal{L}}$ *is bounded from* $\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}(w)$ *into* $\mathrm{BMO}_{\mathcal{L}}^{\beta,\phi}(w)$.

PROOF. Recall that

$$\mathcal{G}_{\mathcal{L}}(f)(x) = \Big(\int_0^\infty |Q_t(f)(x)|^2 \,\frac{dt}{t}\Big)^{1/2},$$

where $Q_t f(x) = t^2 \mathcal{L} e^{-t^2\mathcal{L}} f(x)$.

We may assume that $\|f\|_{\mathrm{BMO}_{\mathcal{L}}^{\beta,\theta_2}(w)} = 1$. We first prove that for all balls $B = B(x_0, r)\,(r \geq \rho(x_0))$,

$$\int_B |\mathcal{G}_{\mathcal{L}}(f)(x)| \lesssim |B|^{\beta/n} w(B)\Psi_\phi(B). \tag{5.6}$$

For any $x \in B$, we write

$$\begin{aligned}\mathcal{G}_{\mathcal{L}}(f)(x) &= \Big(\int_0^{8r} |Q_t(f)(x)|^2 \,\frac{dt}{t} + \int_{8r}^\infty |Q_t(f)(x)|^2 \,\frac{dt}{t}\Big)^{1/2}\\ &\leq \Big(\int_0^{8r} |Q_t(f)(x)|^2 \,\frac{dt}{t}\Big)^{1/2} + \Big(\int_{8r}^\infty |Q_t(f)(x)|^2 \,\frac{dt}{t}\Big)^{1/2}\\ &= \mathcal{G}_{\mathcal{L},1}(f)(x) + \mathcal{G}_{\mathcal{L},2}(f)(x).\end{aligned}$$

From the $L^{p'}(w^{1-p'})$ boundedness of $\mathcal{G}_{\mathcal{L}}$ (see [3, Theorem 5]) and Proposition 3.5,

$$\begin{aligned}\int_B \mathcal{G}_{\mathcal{L},1}(f\chi_B) &\lesssim w(B)^{1/p}\Big(\int_B \mathcal{G}_1(f\chi_B)^{p'} w^{1-p'}\Big)^{1/p'}\\ &\lesssim w(B)^{1/p}\Big(\int_B |f|^{p'} w^{1-p'}\Big)^{1/p'}\\ &\lesssim w(B)|B|^{\beta/n}\Psi_\phi(B).\end{aligned} \tag{5.7}$$

Set $B_x = B(x, r)$. For any $x \in B$ and $t \in (0, 8r)$, by (i) of Lemma 5.3, and $2^{j+1}B \subset 2^{j+2}B_x \subset 2^{j+3}B$,

$$\begin{aligned}|Q_t(f\chi_{B^c})(x)| &\lesssim \int_{B^c} \frac{1}{V_t(x) + V(x,y)}\Big(\frac{t}{t + |x-y|}\Big)^\gamma |f(y)|\,dy\\ &\lesssim \Big(\frac{t}{r}\Big)^\gamma \sum_{j=0}^\infty \frac{2^{-j\gamma}}{V_{2^{j-1}r}(x)} \int_{2^jB_x} |f(y)|\,dy\\ &\lesssim \Big(\frac{t}{r}\Big)^\gamma \sum_{j=0}^\infty 2^{-j(\gamma-\theta_2)}|2^jB_x|^{\beta/n-1} w(2^jB_x).\end{aligned}$$

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By $w \in D_\sigma^{\mathcal{L},\theta_1}$ and $\gamma < 1 + ((\Theta - \beta - \theta_2)/n)$,

$$|Q_t(f\chi_{B^c})(x)| \lesssim \frac{w(B)|B|^{\beta/n}}{|B|}\Big(\frac{t}{r}\Big)^{\gamma} \sum_{j=0}^{\infty} 2^{-j(\gamma-n\sigma-\beta+n-\theta_2)} \lesssim \frac{w(B)|B|^{\beta/n}}{|B|}\Big(\frac{t}{r}\Big)^{\gamma}.$$

From the above inequality, we arrive at

$$\int_B \mathcal{G}_{\mathcal{L},1}(f\chi_{B^c})(x)\,dx \lesssim \frac{w(B)|B|^{\beta/n}}{|B|}\int_B \Big[\int_0^{8r}\Big(\frac{t}{r}\Big)^{2\gamma}\frac{dt}{t}\Big]^{1/2} dx \lesssim |B|^{\beta/n} w(B). \tag{5.8}$$

Combining (5.7) and (5.8),

$$\int_B \mathcal{G}_{\mathcal{L},1}(f)(x)\,dx \lesssim |B|^{\beta/n} w(B)\Psi_\phi(B). \tag{5.9}$$

Put $B_{x,t} := B(x,t)$. To prove (5.6) with $\mathcal{G}_{\mathcal{L},2}$, we first notice that for all $x \in B$ and $t \geq 8r$,

$$\begin{aligned}|Q_t(f)(x)| &\lesssim \int_{\mathbb{R}^n} \frac{1}{V_t(x)+V(x,y)}\Big(\frac{t}{t+|x-y|}\Big)^{\gamma}\Big(\frac{\rho(x)}{t+\rho(x)}\Big)^{\delta_1}|f(y)|\,dy \\ &\lesssim \Big(\frac{\rho(x)}{t}\Big)^{\delta_1}\sum_{j=0}^{\infty} 2^{-j\gamma}\frac{1}{V_{2^{j-1}t}(x)}\int_{2^jB_{x,t}}|f(y)|\,dy \\ &\lesssim \Big(\frac{\rho(x)}{t}\Big)^{\delta_1-\theta_2}\sum_{j=1}^{\infty} 2^{-j(\gamma-\theta_2)}|2^jB_{x,t}|^{\beta/n-1}w(2^jB_{x,t}).\end{aligned}$$

By $w \in D_\sigma^{\mathcal{L},\theta_1}$,

$$\begin{aligned}|Q_t(f)(x)| &\lesssim |B|^{\beta/n-1}w(B)\Big(\frac{\rho(x)}{t}\Big)^{\delta_1-\theta_2-n\sigma+n-\beta}\sum_{j=1}^{\infty}2^{-j(\gamma-\theta_2-n\sigma+n-\beta)} \\ &\lesssim |B|^{\beta/n-1}w(B)\Big(\frac{\rho(x)}{t}\Big)^{\delta_1-\theta_2-n\sigma+n-\beta}.\end{aligned}$$

Then

$$\int_B \mathcal{G}_2(f)(x)|\,dx \lesssim |B|^{\beta/n-1}w(B)\int_B\Big[\int_{8r}^{\infty}\Big(\frac{\rho(x)}{t}\Big)^{2(\delta_1-\theta_2-n\sigma+n-\beta)}\Big]^{1/2}dx \lesssim |B|^{\beta/n-1}w(B),$$

which together with (5.9) gives (5.6).

For $B = B(x_0, r)$ with $r < \rho(x_0)$, we may assume that $r < \rho(x_0)/8$. For all $x \in B$, we write

$$\begin{aligned}\mathcal{G}_{\mathcal{L}}(f)(x) &= \Big[\int_0^{8r}|Q_t(f)(x)|^2\frac{dt}{t} + \int_{8r}^{\infty}|Q_t(f)(x)|^2\frac{dt}{t}\Big]^{1/2} \\ &\leq \Big[\int_0^{8r}|Q_t(f)(x)|^2\frac{dt}{t}\Big]^{1/2} + \Big[\int_{8r}^{\infty}|Q_t(f)(x)|^2\frac{dt}{t}\Big]^{1/2} \\ &= \mathcal{G}_{\mathcal{L},r}(f)(x) + \mathcal{G}_{\mathcal{L},\infty}(f)(x).\end{aligned}$$

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It suffices to show that for almost every $y \in B$,

$$\int_B |\mathcal{G}_{\mathcal{L}}(f)(x) - \mathcal{G}_{\mathcal{L},\infty}(f)(y)|\, dx \lesssim |B|^{\beta/n} w(B).$$

Observe that for almost every $y \in B$,

$$\int_B |\mathcal{G}_{\mathcal{L}}(f)(x) - \mathcal{G}_{\mathcal{L},\infty}(f)(y)|\, dx \leq \int_B |\mathcal{G}_{\mathcal{L},r}(f)(x) + \mathcal{G}_{\mathcal{L},\infty}(f)(x) - \mathcal{G}_{\mathcal{L},\infty}(f)(y)|\, dx.$$

We first prove that

$$\int_B \mathcal{G}_r(f)(x)\, dx \lesssim |B|^{\beta/n} w(B). \tag{5.10}$$

We set $f = f_1 + f_2 + f_3$, where $f_1 = (f - f_B)\chi_{2B}$ and $f_2 = (f - f_B)\chi_{(2B)^c}$. By the $L^{p'}(w^{1-p'})$-boundedness of $\mathcal{G}_{\mathcal{L}}$ (see [3, Theorem 5]),

$$\begin{aligned}\int_B \mathcal{G}_r(f_1)(x)\, dx &\lesssim w(B)^{1/p}\Big(\int_B [\mathcal{G}_r(f_1)(x)]^{p'} w^{1-p'}\, dx\Big)^{1/p'}\\ &\lesssim w(B)^{1/p}\Big(\int_B |f - f_B|^{p'} w^{1-p'}\Big)^{1/p'}\\ &\lesssim |B|^{\beta/n} w(B). \end{aligned}\tag{5.11}$$

For all $x \in B$, by (i) of Lemma 5.3, $\Psi_{\theta_1+\theta_2}(B) \lesssim 1$, and $2^{j+1}B \subset 2^{j+2}B_{x,r} \subset 2^{j+3}B$,

$$\begin{aligned}|Q_t(f_2)(x)| &\lesssim \int_{(2B)^c} \frac{1}{V_t(x) + V(x,y)}\Big(\frac{t}{t + |x - y|}\Big)^{\gamma} |f(z) - f_B|\, dz\\ &\lesssim \sum_{j=1}^{\infty}\Big(\frac{t}{2^{j-1}r}\Big)^{\gamma}\Big(\frac{1}{V_{2^{j-1}r}(x)}\int_{2^{j+1}B} |f(z) - f_B|\, dz\Big)\\ &\lesssim |B|^{\beta/n-1} w(B)\Big(\frac{t}{r}\Big)^{\gamma} \Psi_{\theta_1+\theta_2}(B)\sum_{j=1}^{\infty}(j+2)2^{-j(\gamma-n\sigma-\beta-\theta_2+n)}\\ &\lesssim |B|^{\beta/n-1} w(B)\Big(\frac{t}{r}\Big)^{\gamma},\end{aligned}$$

which further implies that

$$\begin{aligned}\int_B \mathcal{G}_r(f_2)(x)\, dx &\lesssim |B|^{\beta/n-1} w(B)\int_B\Big[\int_0^{8r}\Big(\frac{t}{r}\Big)^{2\gamma}\frac{dt}{t}\Big]^{1/2} dx\\ &\lesssim |B|^{\beta/n} w(B).\end{aligned}\tag{5.12}$$

By (5.11) and (5.12), to prove (5.10), it remains to show that

$$\int_B \mathcal{G}_r(f_B)(x)\, dx \lesssim |B|^{\beta/n} w(B). \tag{5.13}$$

We consider the case $\beta > 0$ or $\sigma > 1$. By Proposition 3.5, we observe that

$$|f_B| \lesssim |B|^{\beta/n-1} w(B)\Big(\frac{\rho(x_0)}{r}\Big)^{n\sigma-n+\beta}.$$

For all $x \in B$ and $t \in (0, 8r)$, from (iii) of Lemma 5.3 and the fact that $r < \rho(x_0)/8$, it follows that

$$|Q_t(f_B)(x)| \lesssim \left(\frac{t}{\rho(x)}\right)^{\delta_2} |f_B| \lesssim B|^{\beta/n-1} w(B)\left(\frac{\rho(x_0)}{r}\right)^{n\sigma-n+\beta}\left(\frac{t}{\rho(x_0)}\right)^{\delta_2},$$

which along with $t \leq 8r < \rho(x_0)$ yields (5.13).

For the case of $\beta = 0$ and $\sigma = 1$, by the same argument and using the inequality $1 + \log_2 s \lesssim s^{u/2}$ for $s > 1/8$ and sufficiently small u, we also arrive at (5.13).

For $x, y \in B$,

$$|\mathcal{G}_{\mathcal{L},\infty}(f)(x) - \mathcal{G}_{\mathcal{L},\infty}(f)(y)| \leq \left(\int_{8r}^{\infty} |Q_t(f)(x) - Q_t(f)(y)|^2 \frac{dt}{t}\right)^{1/2}.$$

For $t \in [8r, \infty)$ and $x, y \in B$, we write

$$\begin{aligned} |Q_t(f)(x) - Q_t(f)(y)| &\leq \left|\int_{\mathbb{R}^n} [Q_t(x,z) - Q_t(y,z)][f(z) - f_{B_{x,t}}]\,dz\right| \\ &\quad + |f_{B_{x,t}}|\left|\int_{\mathbb{R}^n} [Q_t(x,z) - Q_t(y,z)]\,dz\right| \\ &= J_1 + J_2. \end{aligned}$$

Note that $|x - y| \leq 2r < t/2$ for all $x, y \in B$. We deduce from (ii) in Lemma 5.3 that

$$\begin{aligned} J_1 &\lesssim \int_{\mathbb{R}^n} \frac{1}{V_t(x) + V(x,z)}\left[\frac{|x-y|}{t + |x-z|}\right]^{\varepsilon}\left[\frac{t}{t + |x-z|}\right]^{\gamma} |f(z) - f_{B_{x,t}}|\,dz \\ &\lesssim \left(\frac{r}{t}\right)^{\varepsilon} \sum_{j=0}^{\infty} \frac{2^{-j\gamma}}{V_{2^{j-1}t}(x)} \int_{2^{j+1}B_{x,t}} |f(z) - f_{B_{x,t}}|\,dz \\ &\lesssim \left(\frac{r}{t}\right)^{\varepsilon} \Psi_{\theta_1+\theta_2}(B_{x,t}) \sum_{j=0}^{\infty} \frac{|B_{x,t}|^{\beta/n} w(B_{x,t})}{|B_{x,t}|}(j+2)2^{-j(\gamma+n-n\sigma-\beta-\theta_2)}. \end{aligned}$$

By $B_{x,t} \subset B_{x_0,2t} \subset B_{x,3t}$ and $t/\rho(x) \lesssim t/r$,

$$J_1 \lesssim \left(\frac{r}{t}\right)^{\varepsilon-\theta_1-\theta_2} \sum_{j=0}^{\infty} \frac{|B_{x_0,t}|^{\beta/n} w(B_{x_0,t})}{|B_{x_0,t}|}(j+2)2^{-j(\gamma+n-n\sigma-\beta-\theta_2)}.$$

Using $w \in D_{\sigma}^{\mathcal{L},\theta_1}$,

$$w(B_{x_0,t}) \lesssim \left(\frac{t}{r}\right)^{n\sigma} \Psi_{\theta_1}(B) \lesssim \left(\frac{t}{r}\right)^{n\sigma}.$$

Thus,

$$\begin{aligned} J_1 &\lesssim |B|^{\beta/n-1} w(B)\left(\frac{r}{t}\right)^{\varepsilon-\theta_1-\theta_2-n\sigma-\beta+n} \sum_{j=1}^{\infty} (j+2)2^{-j(\gamma+n-n\sigma-\beta-\theta_2)} \\ &\lesssim \frac{|B|^{n} w(B)}{|B|}\left(\frac{r}{t}\right)^{\varepsilon-\theta_1-\theta_2-n\sigma-\beta+n} \end{aligned} \tag{5.14}$$

as long as $r < t$.

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We also have

$$\begin{aligned}\Big|\int_{\mathbb{R}^n}[Q_t(x,z)-Q_t(y,z)]\,dz\Big| &\lesssim \int_{\mathbb{R}^n}\frac{1}{V_t(x)+V(x,z)}\Big[\frac{|x-y|}{t+|x-z|}\Big]^{\varepsilon}\Big[\frac{t}{t+|x-z|}\Big]^{\gamma}dz\\ &\lesssim \Big(\frac{r}{t}\Big)^{\varepsilon}\sum_{j=0}^{\infty}2^{-j\gamma}\lesssim\Big(\frac{r}{t}\Big)^{\varepsilon}. \end{aligned}\tag{5.15}$$

By (iii) in Lemma 5.3,

$$\Big|\int_{\mathbb{R}^n}[Q_t(x,z)-Q_t(y,z)]\,dz\Big|\lesssim\Big(\frac{t}{t+\rho(x_0)}\Big)^{\delta_2}.\tag{5.16}$$

We consider the case of $\beta>0$ or $\eta>1$. Combining (5.15), (5.16), and Proposition 3.5,

$$\begin{aligned}J_2 &\lesssim |f_{B_{x,t}}|\Big|\int_{\mathbb{R}^n}[Q_t(x,z)-Q_t(y,z)]\,dz\Big|^{2/3}\Big(\frac{t}{t+\rho(x_0)}\Big)^{\delta_2/3}\\ &\lesssim |B_{x,t}|^{\beta/n-1}w(B_{x,t})\Psi_{\theta_2}(B_{x,t})\max\Big\{1,\Big[\frac{\rho(x)}{t}\Big]^{n\sigma-n+\beta}\Big\}\\ &\quad\times\Big(\frac{r}{t}\Big)^{\varepsilon/3}\Big(\frac{r}{t}\Big)^{\min(\varepsilon/3,\delta_2/3)}\Big(\frac{t}{t+\rho(x_0)}\Big)^{\min(\varepsilon/3,\delta_2/3)}\\ &\lesssim |B_{x_0,t}|^{\beta/n-1}w(2B_{x_0,t})\Big(\frac{t}{r}\Big)^{\theta_2}\Big(1+\Big[\frac{\rho(x)}{t}\Big]^{n\sigma-\beta+n}\Big)\Big(\frac{r}{t}\Big)^{\varepsilon/3}\Big(\frac{r}{\rho(x_0)}\Big)^{\min(\varepsilon/3,\delta_2/3)}.\end{aligned}$$

Using $w\in D_{\eta}^{\theta_1}$,

$$\begin{aligned}J_2 &\lesssim |B|^{\beta/n-1}w(B)\Big(\frac{r}{t}\Big)^{\varepsilon/3+n-n\sigma-\beta-\theta_2}\Big[1+\Big(\frac{\rho(x_0)}{r}\Big)^{n\sigma+\beta-n}\Big]\Big(\frac{r}{\rho(x_0)}\Big)^{\min(\varepsilon/3,\delta_2/3)}\\ &\lesssim |B|^{\beta/n-1}w(B)\Big(\frac{r}{t}\Big)^{\varepsilon/3+n-n\sigma-\beta-\theta_2}\Big(\frac{r}{\rho(x_0)}\Big)^{\min(\varepsilon/3,\delta_2/3)-n\sigma-\beta+n}\\ &\lesssim |B|^{\beta/n-1}w(B)\Big(\frac{r}{t}\Big)^{\varepsilon/3+n-n\sigma-\beta-\theta_2}.\end{aligned}\tag{5.17}$$

For the case of $\beta=0$ and $\eta=1$, by the inequality $1+\log_2 s\lesssim s^{u/2}$, for $s>1/8$ and u small enough, we also have that

$$J_2\lesssim |B|^{\beta/n-1}w(B)\Big(\frac{r}{t}\Big)^{\varepsilon/3+n-n\sigma-\beta-\theta_2}.$$

By (5.14) and (5.17),

$$J_1\lesssim |B|^{\beta/n-1}w(B)\Big(\frac{r}{t}\Big)^{\Omega_1},\quad J_2\lesssim |B|^{\beta/n-1}w(B)\Big(\frac{r}{t}\Big)^{\Omega_2},$$

where $\Omega_1=\varepsilon-\theta_1-\theta_2-n\sigma-\beta+n,\Omega_2=\varepsilon/3+n-n\sigma-\beta-\theta_2$.

Therefore,

$$|Q_t(f)(x)-Q_t(f)(y)|\lesssim |B|^{\beta/n-1}w(B)\Big(\frac{r}{t}\Big)^{\min(\Omega_1,\Omega_2)}$$

for all $x,y\in B$ and $t\geq 8r$.

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This implies that

$$|\mathcal{G}_{\mathcal{L},\infty}(f)(x) - \mathcal{G}_{\mathcal{L},\infty}(f)(y)| \lesssim |B|^{\beta/n-1} w(B)\Big[\int_{8r}^{\infty}\Big(\frac{r}{t}\Big)^{2\min(\Omega_1,\Omega_2)}\frac{dt}{t}\Big]^{1/2} \lesssim |B|^{\beta/n-1} w(B).$$

Thus,

$$\int_B |\mathcal{G}_{\mathcal{L},\infty}(f)(x) - \mathcal{G}_{\mathcal{L},\infty}(f)(y)|\,dx \lesssim |B|^{\beta/n} w(B).$$

This together with (5.10) leads to

$$\int_B |\mathcal{G}_{\mathcal{L}}(f)(x) - \mathcal{G}_{\mathcal{L},\infty}(f)(y)|\,dx \lesssim |B|^{\beta/n} w(B)$$

for almost every $y \in B$. This completes our proof. □

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