Canad. Math. Bull. Vol. **31** (1), 1988

THE NUMBER OF ROOTED CONVEX POLYHEDRA

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ABSTRACT. Let p_{ij} be the number of rooted convex polyhedra with $i + 1$ vertices and $j + 1$ faces. We express p_{ij} as a singly indexed summation whose terms decrease geometrically. From this we deduce that

$$p_{ij} \sim \frac{1}{3^5 ij}\binom{2i}{j+3}\binom{2j}{i+3}$$

uniformly as $\max(i, j) \to \infty$.

1. **Introduction.** Let p_{ij} be the number of rooted 3-connected planar maps with $i + 1$ vertices and $j + 1$ faces. When $i, j \geqq 3$, this is the same as the number of rooted convex polyhedra with $i + 1$ vertices and $j + 1$ faces by Steinitz's theorem. Our goal is to prove

THEOREM 1. *For $i, j \geqq 3$ we have*

$$p_{ij} = p_{ji} = \frac{1}{j(j-1)}\sum_k A_k\binom{-3}{k}\binom{2i-k-4}{j-2}\binom{2j}{i-k-1}$$

$$= \frac{1}{j(j-1)(j-2)(2j+1)}\sum_k B_k\binom{-4}{k}\binom{2i-k-5}{j-3}\binom{2j+1}{i-k-1}$$

where

$$A_k = j - kj - 2i + 2k + 2,$$

$$B_k = -6ki^2 - (k+2)id + 2(k-2)d^2$$
$$+ 2(3k^2 + 10k + 2)i - (3k^2 - 2k - 20)d - 6(k+2)^2,$$

$$d = 2i - j,$$

and k ranges from 0 *to* $\min(i - 1, 2i - j - 2)$.

Received by the editors May 26, 1986.
Research of the first author partially sponsored by the Office of Naval Research under Contract N00014-85-K-0495.
AMS-MOS Subject Classification (1980): 05C30.

THEOREM 2. *Uniformly as* $\max(i, j) \to \infty$

$$p_{ij} \sim \frac{1}{3^5 ij}\binom{2i}{j+3}\binom{2j}{i+3}$$

where the right hand side is zero when $p_{ij} = 0$ *and* $\max(i, j) \geqq 4$.

Theorem 1 is more efficient when $i \leqq j$ than when $i \geqq j$: there are fewer terms and they decrease geometrically in magnitude after the first few. When $d = 3$, there are only 2 terms which combine to give Tutte's formula [4] for the number of rooted triangulations. To calculate p_{ij} to n significant digits requires at most $O(\log n)$ terms in the latter summation in Theorem 1 independent of i and j provided $i \leqq j$. Theorem 2 simplifies and extends the range of Bender and Richmond's formula [1, Theorem 1]. It now follows from Bender and Wormald [2, Corollary 4.2] that the number of combinatorially distinct convex polyhedra is asymptotic to

$$\frac{1}{2^2 3^5 ij(i+j)}\binom{2i}{j+3}\binom{2j}{i+3}$$

uniformly as $\max(i, j) \to \infty$.

Unless otherwise noted, we shall assume that $i, j \geqq 3$ for the remainder of the paper.

2. **Proof of Theorem 1.** Mullin and Schellenberg [3, (6.24), (6.5)] obtained

$$\sum p_{ij} x^i y^j = xy\left(\frac{1}{1+x} + \frac{1}{1+y} - 1\right) - F$$

where

$$F = \frac{rs}{(1+r+s)^3}$$

and (r, s) is given implicity by $(r(0, 0), s(0, 0)) = (0, 0)$ and

$$(x, y) = (r/(1+s)^2, s/(1+r)^2).$$

They applied Lagrange inversion to this to obtain p_{ij} as a double summation. By arranging terms differently, we obtain a single summation. By Lagrange inversion, p_{ij} is the constant term in

$$\frac{-F}{x(r, s)^{i+1} y(r, s)^{j+1}} \begin{vmatrix} \partial x/\partial r & \partial x/\partial s \\ \partial y/\partial r & \partial y/\partial s \end{vmatrix} rs$$

$$= \frac{(1+s)^{2i-1}(1+r)^{2j-1}(3rs - r - s - 1)}{r^{i-1}s^{j-1}(1+r+s)^3}$$

$$= 3\frac{(1+s)^{2i-4}(1+r)^{2j-1}}{r^{i-2}s^{j-2}}\left(1 + \frac{r}{1+s}\right)^{-3}$$

https://doi.org/10.4153/CMB-1988-015-2 Published online by Cambridge University Press

$$-\frac{(1+s)^{2i-3}(1+r)^{2j-1}}{r^{i-1}s^{j-1}}\left(1+\frac{r}{1+s}\right)^{-2}.$$

By expanding the last factor in each of the terms and then extracting coefficients we obtain

$$\text{(2.1)}\qquad p_{ij}=\sum_{k\geqq 0}3\binom{-3}{k}\binom{2i-k-4}{j-2}\binom{2j-1}{i-k-2}$$
$$-\sum_{k\geqq 0}\binom{-2}{k}\binom{2i-k-3}{j-1}\binom{2j-1}{i-k-1}.$$

Write $K=k-1$ and

$$\binom{-2}{k}=\binom{-3}{k}+\binom{-3}{K}$$

in the second sum of (2.1), regroup terms by replacing K with k in the appropriate summation index, and perform a bit of algebra to obtain the first summation in Theorem 1. The second summation is obtained in a similar fashion after first writing

$$A_k\binom{-3}{k}=(j-2i+2)\left\{\binom{-4}{k}+\binom{-4}{K}\right\}+3(j-2)\binom{-4}{K}.$$

3. **Proof of Theorem 2.** We will use the latter summation in Theorem 1 and will assume, without loss of generality, that $j\geqq i$. By Euler's theorem,

$$d=2i-j\geqq 3$$

with equality for triangulations. For $\epsilon>0$ and $d=O(i^{1-\epsilon})$, it can be shown by straightforward calculations that the first two terms in the summation suffice to obtain Theorem 2 uniformly. We suppose $\epsilon<1/3$ and $d\geqq i^{1-\epsilon}$ for the remainder of the paper.

We have

$$\binom{2i-k-5}{j-3}\bigg/\binom{2i-5}{j-3}=\prod_{t=0}^{k-1}\left(\frac{d-2-t}{2i-5-t}\right)=\left(\frac{d}{2i}\right)^k(1-f)$$

where $0\leqq f$ and, for $k=O(i^\epsilon)$, $f=O(i^{3\epsilon-1})$ uniformly. Similarly,

$$\binom{2j+1}{i-k+1}\bigg/\binom{2j+1}{i+1}=\left(\frac{i}{2j-1}\right)^k(1-g)$$

where $0\leqq g$ and, for $k=O(i^\epsilon)$, $g=O(i^{3\epsilon-1})$ uniformly. Combining these results we obtain

$$p_{ij}\sim\frac{1}{2j^4}\binom{2i-5}{j-3}\binom{2j+1}{i-1}\left\{\sum(C+Dk)\binom{-4}{k}\rho^k+O(i^{1+3\epsilon})\right\}$$

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where $C = -4d^2 - 2id$, $D = 2d^2 - id - 6i^2$ and

$$\rho = \frac{d}{2(2j - i)} \leqq \frac{1}{2}.$$

By using

$$\sum \binom{-4}{k}\rho^k = (1 + \rho)^{-4} \text{ and } \sum k\binom{-4}{k}\rho^k = -4\rho(1 + \rho)^{-5}$$

and a bit of algebra,

$$p_{ij} \sim \frac{1}{2j^4}\binom{2i - 5}{j - 3}\binom{2j + 1}{i - 1}\frac{d(2i - d)}{(1 + \rho)^5}.$$

Theorem 2 follows with a bit of algebra.

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https://doi.org/10.4153/CMB-1988-015-2 Published online by Cambridge University Press