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FP-INJECTIVE COMPLEXES AND FP-INJECTIVE DIMENSION OF COMPLEXES

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Abstract

In this paper we extend the notion of FP-injective modules to that of complexes and characterize such complexes. We show that some characterizations similar to those for injective complexes exist for FP-injective complexes. We also introduce and study the notion of an FP-injective dimension associated to every complex of left R-modules over an arbitrary ring. We show that there is a close connection between the FP-injective dimension of complexes and flat dimension.

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1. Introduction

In this paper, R denotes a ring with unity, R-Mod denotes the category of left R-modules and $\mathscr{C}(R)$ denotes the abelian category of complexes of left R-modules. A complex

$$\cdots \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} C_{-1} \xrightarrow{\delta_{-1}} \cdots$$

of left R-modules will be denoted by (C, δ) or C. Given a left R-module M, we use $D^n(M)$ to denote the complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\text{id}} M \longrightarrow 0 \longrightarrow \cdots$$

with M in the nth and $(n-1)$st positions. We also use $S^n(M)$ to denote the complex with M in the nth place and 0 in the other places.

Given a complex C and an integer i we use the notation $\Sigma^i C$ for the complex satisfying the condition that $(\Sigma^i C)_n = C_{n-i}$ and whose boundary operators are $(-1)^i \delta^C_{n-i}$.

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Given a complex C the nth homology module of C is the module

$$H_n(C) = Z_n(C)/B_n(C)$$

where $Z_n(C) = \mathrm{Ker}(\delta_n^C)$ and $B_n(C) = \mathrm{Im}(\delta_{n+1}^C)$. We set $H^n(C) = H_{-n}(C)$ and $C_n(C) = \mathrm{Coker}(\delta_{n+1}^C)$.

To every complex C we associate the numbers

$$\sup C = \sup\{i \mid C_i \neq 0\}$$

and

$$\inf C = \inf\{i \mid C_i \neq 0\}.$$

The complex C is said to be bounded above when $\sup C < \infty$. We say that C is bounded below when $\inf C > -\infty$ and bounded when it is both bounded below and bounded above.

Throughout this paper we use both the subscript notation for complexes and the superscript notation. When we use superscripts for a complex we will use subscripts to distinguish positions of the complexes. For example, if $(K^i)_{i\in I}$ is a family of complexes, then K_n^i denotes the nth component of the complex K^i.

For objects C and D of $\mathscr{C}(R)$ we let $\mathrm{Hom}(C, D)$ denote the abelian group of morphisms from C to D in $\mathscr{C}(R)$. We use the notation $\mathrm{Ext}^i(C, D)$ where $i \geq 1$ for the groups that arise from the right derived functor of Hom.

A homomorphism $\varphi : C \longrightarrow D$ of degree n is a family $(\varphi_i)_{i\in\mathbb{Z}}$ of homomorphisms of R-modules $\varphi_i : C_i \longrightarrow D_{n+i}$. The set of all such homomorphisms forms an abelian group which we denote by $\mathcal{H}\mathrm{om}(C, D)_n$. This group is clearly isomorphic to $\prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(C_i, D_{n+i})$. We let $\mathcal{H}\mathrm{om}(C, D)$ denote the complex of abelian groups with nth component $\mathcal{H}\mathrm{om}(C, D)_n$ and boundary operator

$$\delta_n((\varphi_i)_{i\in\mathbb{Z}}) = (\delta_{n+i}^D \varphi_i - (-1)^n \varphi_{i-1} \delta_i^C)_{i\in\mathbb{Z}}.$$

A homomorphism $\varphi \in \mathcal{H}\mathrm{om}(C, D)_n$ is called a chain map if $\delta(\varphi) = 0$, that is, if $\delta_{n+i}^D \varphi_i = (-1)^n \varphi_{i-1} \delta_i^C$ for all $i \in \mathbb{Z}$. A chain map of degree 0 is called a morphism. Homomorphisms φ and φ' in $\mathcal{H}\mathrm{om}(C, D)_n$ are called homotopic, denoted $\varphi \sim \varphi'$, if there exists a degree $n + 1$ homomorphism t, called a homotopy, such that $\delta(t) = \varphi - \varphi'$. A homotopy equivalence is a morphism $\varphi : C \longrightarrow D$ for which there exists a morphism $\psi : D \longrightarrow C$ such that $\varphi\psi \sim \mathrm{id}^D$ and $\psi\varphi \sim \mathrm{id}^C$. A morphism $\varphi : C \longrightarrow D$ is called a quasiisomorphism if the induced morphisms $H_n(\varphi) : H_n(C) \longrightarrow H_n(D)$ are isomorphisms for all $n \in \mathbb{Z}$.

It is easy to see that

$$\mathrm{Hom}(C, D) = Z_0(\mathcal{H}\mathrm{om}(C, D)).$$

We recall the notation introduced in [4]. Let

$$\underline{\mathrm{Hom}}(C, D) = Z(\mathcal{H}\mathrm{om}(C, D)).$$

One checks that $\underline{\mathrm{Hom}}(C, D)$ can be made into a complex in which $\underline{\mathrm{Hom}}(C, D)_n$ is the abelian group of morphisms from C to $\Sigma^{-n}D$ and whose boundary operator is given by

$$\delta_n(f) : C \longrightarrow \Sigma^{-(n-1)}D$$

where $\delta_n(f)_m = (-1)^n \delta^D f_m$ for all $m \in \mathbb{Z}$ and $f \in \underline{\mathrm{Hom}}(C, D)_n$. Note that the new functor $\underline{\mathrm{Hom}}(C, D)$ will have right derived functors whose values will be complexes. These values should certainly be denoted by $\underline{\mathrm{Ext}}^i(C, D)$. It is not hard to see that $\underline{\mathrm{Ext}}^i(C, D)$ is the complex

$$\cdots \longrightarrow \mathrm{Ext}^i(C, \Sigma^{-(n+1)}D) \longrightarrow \mathrm{Ext}^i(C, \Sigma^{-n}D) \longrightarrow \mathrm{Ext}^i(C, \Sigma^{-(n-1)}D) \longrightarrow \cdots$$

with boundary operators induced by the boundary operators of D. For a complex C we have

$$C^+ = \underline{\mathrm{Hom}}(C, D^1(\mathbb{Q}/\mathbb{Z})).$$

If X is a complex of right R-modules and Y is a complex of left R-modules, the tensor product of X and Y is the complex of abelian groups $X \otimes Y$ with

$$(X \otimes Y)_n = \bigoplus_{t \in \mathbb{Z}} (X_t \otimes_R Y_{n-t})$$

and

$$\delta(x \otimes y) = \delta_t^X(x) \otimes y + (-1)^t x \otimes \delta_{n-t}^Y(y)$$

for all $x \in X^t$ and $y \in Y^{n-t}$. We define

$$X \overline{\otimes} Y := \frac{X \otimes Y}{B(X \otimes Y)}$$

and endow this set with the maps

$$\frac{(X \otimes Y)_n}{B_n(X \otimes Y)} \longrightarrow \frac{(X \otimes Y)_{n-1}}{B_{n-1}(X \otimes Y)}, \quad x \otimes y \mapsto \delta_X(x) \otimes y.$$

Here $x \otimes y$ is used to denote a coset in

$$\frac{(X \otimes Y)_n}{B_n(X \otimes Y)},$$

and by this means we form a complex of abelian groups.

General background about complexes of R-modules can be found in [3–7, 9, 10]. In particular, it is well known that a complex (C, δ) is injective if and only if C is exact and $Z_n(C)$ is injective in R-Mod for all $n \in \mathbb{Z}$. Also we have $\mathrm{id}(C) \le m$ in $\mathscr{C}(R)$ if and only if C is exact and $\mathrm{id}(Z_n(C)) \le m$ in R-Mod for all $n \in \mathbb{Z}$ (see [5, Theorem 1.5]).

The main purpose of this article is to extend the notion of FP-injective modules to that of FP-injective complexes and to consider the relationships between an FP-injective complex C and FP-injective modules C_n or $Z_n(C)$ for all $n \in \mathbb{Z}$. We obtain

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some results similar to those for injective complexes. As an application, coherent rings are characterized by the FP-injective complexes. In addition, inspired by Avramov and Foxby's definition of injective dimension for every complex, we introduce and study a notion of FP-injective dimension associated to every complex of left R-modules over an arbitrary ring. We show that there is a close connection between the FP-injective dimension of complexes and their flat dimension.

2. FP-injective complexes

A left R-module M is called FP-injective (or absolutely pure) if $\mathrm{Ext}^1(N, M) = 0$ for each finitely presented module N. General background material on FP-injective modules can be found in [1, 8, 11–16, 18].

It is well known that each injective module is FP-injective, but the converse is not true. In addition, injective complexes are the counterparts of injective modules in the category of complexes. In this section, as a generalization of injective complexes, FP-injective complexes are investigated and studied. We show that some characterizations similar to those for injective complexes hold for FP-injective complexes.

Definition 2.1 [6]. A complex C is called finitely generated if, in the case where we can write $C = \sum_{i\in I} D^i$ with $D^i \in \mathscr{C}(R)$ subcomplexes of C, there exists a finite subset $J \subset I$ such that $C = \sum_{i\in J} D^i$. A complex C is called finitely presented if C is finitely generated and for every exact sequence of complexes

$$0 \to K \to L \to C \to 0$$

with L finitely generated, K is also finitely generated.

Lemma 2.2 [6]. *A complex C is finitely generated if and only if C is bounded and C_n is finitely generated in R-*Mod *for all $n \in \mathbb{Z}$. A complex C is finitely presented if and only if C is bounded and C_n is finitely presented in R-*Mod *for all $n \in \mathbb{Z}$.*

It is easy to see that each finitely generated projective complex coincides with a finitely presented projective complex and that each finitely presented projective complex coincides with a finitely presented flat complex.

Remark 2.3. If M is a finitely generated module, then $S^i(M)$ and $D^i(M)$ are finitely generated complexes for each $i \in \mathbb{Z}$. If M is a finitely presented module, then $S^i(M)$ and $D^i(M)$ are finitely presented complexes for each $i \in \mathbb{Z}$.

Definition 2.4. A complex C is called FP-injective if $\mathrm{Ext}^1(F, C) = 0$ for every finitely presented complex F.

Remark 2.5. First, it is easy to see that the class of all FP-injective complexes is closed under extensions, direct products, direct sums and direct summands. Clearly each injective complex is FP-injective.

Moreover, if R is a left coherent ring and C is an FP-injective complex, then $\mathrm{Ext}^i(F, C) = 0$ for each finitely presented complex F and $i \geq 1$. In fact, for every exact sequence of complexes

$$0 \longrightarrow K \longrightarrow P \longrightarrow F \longrightarrow 0$$

with P a finitely presented projective complex, K is also finitely presented. So

$$0 = \mathrm{Ext}^1(K, C) \cong \mathrm{Ext}^2(F, C).$$

Applying this result inductively, we obtain $\mathrm{Ext}^i(F, C) = 0$ for all $i \geq 1$.

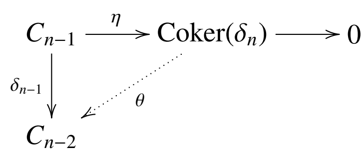
Proposition 2.6. *Let C be a complex. Then C is* FP-*injective if and only if C_n is* FP-*injective in R*-Mod *for all $n \in \mathbb{Z}$ and $\mathcal{H}\mathrm{om}(F, C)$ is exact for all finitely presented complexes F.*

Proof. Suppose that (C, δ) is FP-injective and let

$$0 \longrightarrow C_n \xrightarrow{\alpha} X \longrightarrow G \longrightarrow 0 \tag{2.1}$$

be an extension in R-Mod where G is a finitely presented module. We show that the sequence (2.1) splits.

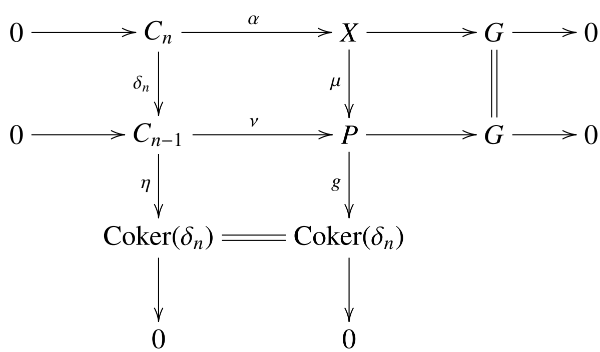
By the factor theorem (see [2, Theorem 3.6]) we have the following commutative diagram:



where

$$\eta : C_{n-1} \longrightarrow \mathrm{Coker}(\delta_n)$$

is the natural epimorphism. We form the pushout of $C_n \xrightarrow{\alpha} X$ and $C_n \xrightarrow{\delta_n} C_{n-1}$ to obtain a commutative diagram.



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So we have the commutative diagram

$$
\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n+2} & \xrightarrow{\mathrm{id}} & C_{n+2} & \longrightarrow & 0 & & \\
 & & \downarrow{\scriptstyle \delta_{n+2}} & & \downarrow{\scriptstyle \delta_{n+2}} & & \downarrow & & \\
0 & \longrightarrow & C_{n+1} & \xrightarrow{\mathrm{id}} & C_{n+1} & \longrightarrow & 0 & & \\
 & & \downarrow{\scriptstyle \delta_{n+1}} & & \downarrow{\scriptstyle \alpha\delta_{n+1}} & & \downarrow & & \\
0 & \longrightarrow & C_n & \xrightarrow{\alpha} & X & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta_n} & & \downarrow{\scriptstyle \mu} & & \| & & \\
0 & \longrightarrow & C_{n-1} & \xrightarrow{\nu} & P & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta_{n-1}} & & \downarrow{\scriptstyle \theta g} & & \downarrow & & \\
0 & \longrightarrow & C_{n-2} & \xrightarrow{\mathrm{id}} & C_{n-2} & \longrightarrow & 0 & & \\
 & & \downarrow{\scriptstyle \delta_{n-2}} & & \downarrow{\scriptstyle \delta_{n-2}} & & \downarrow & & \\
0 & \longrightarrow & C_{n-3} & \xrightarrow{\mathrm{id}} & C_{n-3} & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}
$$

and can form the complex

$$W = \cdots \longrightarrow C_{n+2} \longrightarrow C_{n+1} \longrightarrow X \longrightarrow P \longrightarrow C_{n-2} \longrightarrow \cdots .$$

Thus we have an exact sequence of complexes

$$0 \longrightarrow C \longrightarrow W \longrightarrow D^n(G) \longrightarrow 0. \tag{2.2}$$

Since G is a finitely presented module we may deduce from Lemma 2.2 that $D^n(G)$ is a finitely presented complex. By our hypothesis the sequence (2.2) splits in the category of complexes and so the sequence

$$0 \longrightarrow C_n \longrightarrow X \longrightarrow G \longrightarrow 0$$

splits in the category of modules. Therefore C_n is an FP-injective module.

For a finitely presented complex F we have that $\mathcal{H}\mathrm{om}(F, C)$ is exact if and only if for each n each morphism of complexes

$$f : F \longrightarrow \Sigma^{-n}C$$

is homotopic to 0. This is equivalent to the requirement that for each n and each morphism of complexes

$$f : F \longrightarrow \Sigma^{-n}C$$

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the sequence

$$0 \longrightarrow \Sigma^{-n}C \longrightarrow M(f) \longrightarrow \Sigma^{-1}F \longrightarrow 0$$

splits, or, equivalently, for each n and each morphism of complexes

$$f : F \longrightarrow \Sigma^{-n}C$$

the sequence

$$0 \longrightarrow C \longrightarrow \Sigma^{n}M(f) \longrightarrow \Sigma^{n-1}F \longrightarrow 0$$

splits where $M(f)$ denotes the mapping cone of f. Since F is finitely presented, so is $\Sigma^{n-1}F$. By hypothesis,

$$\mathrm{Ext}^1(\Sigma^{n-1}F, C) = 0.$$

Hence the sequence

$$0 \longrightarrow C \longrightarrow \Sigma^{n}M(f) \longrightarrow \Sigma^{n-1}F \longrightarrow 0$$

splits and $\mathcal{H}\mathrm{om}(F, C)$ is an exact complex.

Suppose C_n is an FP-injective module for all $n \in \mathbb{Z}$ and $\mathcal{H}\mathrm{om}(F, C)$ is exact for every finitely presented complex F. An exact sequence

$$0 \longrightarrow C \longrightarrow W \longrightarrow F \longrightarrow 0$$

of complexes with F finitely presented splits at the module level. So this sequence is isomorphic to

$$0 \longrightarrow C \longrightarrow M(f) \longrightarrow F \longrightarrow 0$$

where

$$f : \Sigma^{1}F \longrightarrow C$$

is a morphism of complexes. Since $\mathcal{H}\mathrm{om}(\Sigma^{1}F, C)$ is exact the sequence

$$0 \longrightarrow C \longrightarrow M(f) \longrightarrow F \longrightarrow 0$$

splits by [9, Lemma 2.3.2], so

$$0 \longrightarrow C \longrightarrow W \longrightarrow F \longrightarrow 0$$

also splits. □

Proposition 2.7. *If C is an exact complex and $Z_n(C)$ is* FP*-injective in R-*Mod *for all $n \in \mathbb{Z}$, then $\mathcal{H}\mathrm{om}(F, C)$ is exact for each finitely presented complex F.*

Proof. If F is a finitely presented complex, then F is bounded by Lemma 2.2. Hence we can assume that

$$F = \cdots \longrightarrow 0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0 \longrightarrow \cdots .$$

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It is well known that $\mathcal{H}om(F, C)$ is a complex and

$$\mathcal{H}om(F, C) = \cdots \xrightarrow{\delta_{n+1}} \prod_{i\in\mathbb{Z}} \operatorname{Hom}(F_i, C_{n+i}) \xrightarrow{\delta_n} \prod_{i\in\mathbb{Z}} \operatorname{Hom}(F_i, C_{n-1+i}) \xrightarrow{\delta_{n-1}} \cdots .$$

It is enough to prove that $\operatorname{Ker}(\delta_{n-1}) \subseteq \operatorname{Im}(\delta_n)$ for each $n \in \mathbb{Z}$.

Let $g \in \operatorname{Ker}(\delta_{n-1})$. Then

$$\delta_{n-1}(g) = (\delta^C_{n-1+t} g_t - (-1)^{n-1} g_{t-1}\delta^F_t)_{t\in\mathbb{Z}} = 0.$$

In the following procedure we are going to construct a morphism f satisfying

$$f \in \mathcal{H}om(F, C)_n = \prod_{i\in\mathbb{Z}} \operatorname{Hom}(F_i, C_{n+i})$$

and

$$\delta_n(f) = (\delta^C_{n+t} f_t - (-1)^n f_{t-1}\delta^F_t)_{t\in\mathbb{Z}} = (g_t)_{t\in\mathbb{Z}}.$$

Since $g_t = 0$ for $t \leq -1$ we take $f_t = 0$ if $t \leq -1$. If $t = 0$, then $\delta^C_{n-1} g_0 = 0$ and so

$$\operatorname{Im}(g_0) \subseteq \operatorname{Ker}(\delta^C_{n-1}) = \operatorname{Im}(\delta^C_n).$$

Since $Z_n(C)$ is FP-injective and F_0 is finitely presented there exists a homomorphism $f_0 : F_0 \to C_n$ such that $\delta^C_n f_0 = g_0$. That is, the diagram

$$\begin{array}{ccccccccc} & & & & & & F_0 & & \\ & & & & & \swarrow^{f_0} & \downarrow{\scriptstyle g_0} & & \\ 0 & \longrightarrow & Z_n(C) & \longrightarrow & C_n & \xrightarrow[\delta^C_n]{} & Z_{n-1}(C) & \longrightarrow & 0 \end{array}$$

commutes.

If $t = 1$, then

$$\delta^C_n(g_1 - (-1)^{n-1} f_0\delta^F_1) = \delta^C_n g_1 - (-1)^{n-1}\delta^C_n f_0\delta^F_1 = 0$$

and so

$$\operatorname{Im}(g_1 - (-1)^{n-1} f_0\delta^F_1) \subseteq \operatorname{Ker}(\delta^C_n).$$

Set $h_1 = g_1 - (-1)^{n+1} f_0\delta^F_1$. Since $Z_{n+1}(C)$ is FP-injective and F_1 is finitely presented we have the following commutative diagram.

$$\begin{array}{ccccccccc} & & & & & & F_1 & & \\ & & & & & \swarrow^{f_1} & \downarrow{\scriptstyle h_1} & & \\ 0 & \longrightarrow & Z_{n+1}(C) & \longrightarrow & C_{n+1} & \xrightarrow[\delta^C_{n-1}]{} & Z_n(C) & \longrightarrow & 0 \end{array}$$

That is, $g_1 = \delta^C_{n-1} f_1 - (-1)^n f_0\delta^F_1$. Repeating this procedure, we deduce that $f \in \operatorname{Im}(\delta_n)$ and $\delta_n f = g$. □

Proposition 2.8. *If C is an exact complex and $Z_n(C)$ is* FP*-injective in R*-Mod *for all $n \in \mathbb{Z}$, then C is an* FP*-injective complex.*

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Proof. Since C is exact we have an exact sequence

$$0 \to Z_n(C) \to C_n \to Z_{n-1}(C) \to 0$$

for each $n \in \mathbb{Z}$. Now $Z_n(C)$ and $Z_{n-1}(C)$ are FP-injective which implies that C_n is FP-injective. The result now follows from Propositions 2.6 and 2.7. □

Proposition 2.9. *If C is an* FP-*injective complex, then C is exact and $Z_n(C)$ is* FP-*injective in R*-Mod *for all $n \in \mathbb{Z}$.*

Proof. Suppose that C is an FP-injective complex. Since $H^i(C) \cong \mathrm{Ext}^1(S^{1-i}(R), C)$ for all $i \in \mathbb{Z}$ and $S^{1-i}(R)$ is finitely presented, then C is exact. We only need to prove that $\mathrm{Ext}^1(G, Z_n(C)) = 0$ for every finitely presented module G.

Consider the exact sequence

$$0 \to K \to P \to G \to 0$$

with P a finitely generated projective module. It yields an exact sequence of complexes

$$0 \longrightarrow S^n(K) \longrightarrow S^n(P) \longrightarrow S^n(G) \longrightarrow 0.$$

By the hypothesis $\mathrm{Ext}^1(S^n(G), C) = 0$. So

$$\mathrm{Hom}(S^n(P), C) \longrightarrow \mathrm{Hom}(S^n(K), C) \longrightarrow 0$$

is exact.

Let $f : K \longrightarrow Z_n(C)$ be an R-homomorphism. We define $\alpha_n : K \longrightarrow C_n$ by $\alpha_n = \lambda f$ where λ is the inclusion map and $\alpha_i = 0$ for $i \neq n$. In this way we obtain a morphism of complexes $\alpha : S^n(K) \longrightarrow C$. Then there exists $\beta : S^n(P) \longrightarrow C$ such that the diagram

$$\begin{array}{ccc} S^n(K) & \longrightarrow & S^n(P) \\ {\scriptstyle\alpha}\downarrow & \swarrow {\scriptstyle\beta} & \\ C & & \end{array}$$

commutes. Hence we have the commutative diagram

$$\begin{array}{ccc} K & \longrightarrow & P \\ {\scriptstyle\lambda f}\downarrow & \swarrow {\scriptstyle\beta_n} & \\ C_n & & \end{array}$$

and $\delta_n \beta_n = 0$, which implies that $\mathrm{Im}(\beta_n) \subseteq \mathrm{Ker}(\delta_n)$. So we define $g : P \to \mathrm{Ker}(\delta_n)$ by $g = \beta_n$. Thus the sequence

$$\mathrm{Hom}(P, Z_n(C)) \longrightarrow \mathrm{Hom}(K, Z_n(C)) \longrightarrow 0$$

is exact.

On the other hand, we have an exact sequence

$$\mathrm{Hom}(P, Z_n(C)) \longrightarrow \mathrm{Hom}(K, Z_n(C)) \longrightarrow \mathrm{Ext}^1(G, Z_n(C)) \longrightarrow 0.$$

Therefore $\mathrm{Ext}^1(G, Z_n(C)) = 0$ and we have established our result. □

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Based on the above results we may deduce the following theorem.

THEOREM 2.10. *Let C be a complex. Then the following statements are equivalent.*

(1) *C is* FP*-injective.*
(2) *C is exact and $Z_n(C)$ is* FP*-injective in R-*Mod *for all $n \in \mathbb{Z}$.*
(3) *C_n is* FP*-injective in R-*Mod *for all $n \in \mathbb{Z}$ and $\mathcal{H}\mathrm{om}(F, C)$ is exact for each finitely presented complex F.*

COROLLARY 2.11. *A module M is* FP*-injective if and only if the complex $D^i(M)$ is* FP*-injective complex for each $i \in \mathbb{Z}$.*

EXAMPLE 2.12. If R is not Noetherian, then we can form a direct sum $\bigoplus_I M_i$ of injective R-modules $(M_i)_I$ which is not injective, but which is necessarily FP-injective. Hence $D^0(\bigoplus_I M_i)$ is an FP-injective complex but is not an injective complex.

It is well known that a complex C is flat (injective, projective respectively) if and only if C is exact and $Z_n(C)$ is flat (injective, projective respectively) in R-Mod for all $n \in \mathbb{Z}$. So we have the following corollaries.

COROLLARY 2.13. *Let C be a complex. Then the following statements are equivalent.*

(1) *C^+ is* FP*-injective.*
(2) *C is flat.*
(3) *C^+ is injective.*

COROLLARY 2.14. *Let R be a ring and let C be a complex. Then the following assertions are equivalent.*

(1) *R is a left coherent ring.*
(2) *C is an* FP*-injective complex if and only if C^+ is a flat complex.*
(3) *C is an* FP*-injective complex if and only if C^{++} is an injective complex.*
(4) *The complex C of right R-modules is flat if and only if C^{++} is flat.*

COROLLARY 2.15. *Let R be a left coherent ring and $0 \to A \to B \to C \to 0$ be exact in $\mathscr{C}(R)$ with A* FP*-injective. Then B is* FP*-injective if and only if C is* FP*-injective.*

COROLLARY 2.16. *Let R be left coherent and let C be a bounded above complex. Then C is* FP*-injective if and only if C is exact and C_n is* FP*-injective for all $n \in \mathbb{Z}$.*

COROLLARY 2.17. *Let R be a ring. Then the following assertions are equivalent.*

(1) *R is a left coherent and right perfect ring.*
(2) *C is an* FP*-injective complex if and only if C^+ is a projective complex.*

According to [9], a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathscr{C}(R)$ is called pure if the sequence

$$0 \longrightarrow F \overline{\otimes} A \longrightarrow F \overline{\otimes} B$$

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is exact for every (or every finitely presented) complex F of right R-modules. Equivalently,

$$\underline{\mathrm{Hom}}(F, B) \longrightarrow \underline{\mathrm{Hom}}(F, C) \longrightarrow 0$$

is surjective for all finitely presented complexes F of left R-modules. A subcomplex A of complex B is pure if

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$$

is a pure exact sequence.

Definition 2.18. A complex C is called absolutely pure if it is pure in every complex that contains it.

Proposition 2.19. *Let C be a complex. Then the following assertions are equivalent.*

(1) *C is absolutely pure.*
(2) $\underline{\mathrm{Ext}}^1(F, C) = 0$ *for every finitely presented complex F.*
(3) *C is* FP*-injective.*

Proof. We show that (1) implies (2). For a complex C we have an exact sequence

$$0 \to C \to E \to L \to 0$$

with E an injective complex. By (1),

$$\underline{\mathrm{Hom}}(F, E) \to \underline{\mathrm{Hom}}(F, L) \to 0$$

is exact for every finitely presented complex F. On the other hand, the sequence

$$\underline{\mathrm{Hom}}(F, E) \to \underline{\mathrm{Hom}}(F, L) \to \underline{\mathrm{Ext}}^1(F, C) \to 0$$

is exact. So $\underline{\mathrm{Ext}}^1(F, C) = 0$.

The implications (2) implies (1) and (2) implies (3) are trivial.

To see that (3) implies (2), suppose that F is a finitely presented complex. Then $\Sigma^n F$ is finitely presented for each $n \in \mathbb{Z}$. But C is FP-injective and

$$\mathrm{Ext}^1(\Sigma^n F, C) \cong \mathrm{Ext}^1(F, \Sigma^{-n} C).$$

It follows that $\mathrm{Ext}^1(F, \Sigma^{-n} C) = 0$ which implies that $\underline{\mathrm{Ext}}^1(F, C) = 0$. □

Proposition 2.20. *Let C be a complex. Then the following assertions are equivalent.*

(1) *C is absolutely pure.*
(2) *C is* FP*-injective.*
(3) *Every exact sequence*

$$0 \to C \to L \to F \to 0$$

with F finitely presented splits.

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(4) *C is injective with respect to every exact sequence*

$$0 \to U \to V \to F \to 0$$

in $\mathscr{C}(R)$ with F finitely presented.

(5) *C is a pure subcomplex of an* (FP-) *injective complex.*

(6) *Every diagram*

$$\begin{array}{ccc} S & \longrightarrow & T \\ \downarrow & \swarrow & \\ C & & \end{array}$$

with T finitely generated and projective (free) and S a finitely generated subcomplex of T can be completed to a commutative diagram.

PROOF. The implications (2) if and only if (3), (2) if and only if (6), (1) implies (5), (2) implies (4) and (4) implies (3) are obvious.

We show that (3) implies (4). Let $0 \to U \to V \to F \to 0$ be an exact sequence with F finitely presented. For a morphism $\alpha : U \to C$ we form the following pushout diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{f} & V & \longrightarrow & F & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & \overset{\theta}{\swarrow} & \downarrow & \overset{\gamma}{\swarrow} & \| & & \\ 0 & \longrightarrow & C & \longrightarrow & Q & \xrightarrow[\beta]{} & F & \longrightarrow & 0 \end{array}$$

By (3), the sequence

$$0 \to C \to Q \to F \to 0$$

splits and so there exists γ such that $\beta\gamma = 1$. Thus there exists θ such that $\theta f = \alpha$ by the homotopy lemma.

Now we show that (5) implies (3). Let $0 \to C \to L \to F \to 0$ be an exact sequence with F finitely presented and let C be a pure subcomplex of the FP-injective complex V. Since V is FP-injective we get the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \xrightarrow{f} & L & \longrightarrow & F & \longrightarrow & 0 \\ & & \| & \overset{\theta}{\swarrow} & \downarrow & \overset{\gamma}{\swarrow} & \downarrow{\scriptstyle\alpha} & & \\ 0 & \longrightarrow & C & \longrightarrow & V & \xrightarrow[\beta]{} & V/C & \longrightarrow & 0 \end{array}$$

Since C is pure in V there exists γ such that $\beta\gamma = \alpha$. Thus there exists θ such that $\theta f = 1$ by the homotopy lemma. That is, the sequence $0 \to C \to L \to F \to 0$ splits. □

LEMMA 2.21. *If $0 \to A \to B \to C \to 0$ is pure exact in $\mathscr{C}(R)$ and A or B is exact, then the sequence*

$$0 \to Z_n(A) \to Z_n(B) \to Z_n(C) \to 0$$

*is pure exact in R-*Mod *for all $n \in \mathbb{Z}$.*

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Proof. By the hypothesis we have an exact sequence

$$0 \to Z_n(A) \to Z_n(B) \to Z_n(C) \to 0$$

in R-Mod. Let P be a finitely presented module and $f : P \to Z_n(C)$ be an R-homomorphism. We define $\alpha : S^n(P) \longrightarrow C$ by

$$\begin{array}{ccccccccc}
S^n(P) = \cdots & \longrightarrow & 0 & \longrightarrow & P & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \lambda f} & & \downarrow & & \\
C = \cdots & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow & \cdots
\end{array}$$

where $\lambda : Z_n(C) \longrightarrow C_n$ is the natural inclusion. Since $S^n(P)$ is a finitely presented complex there exists $\beta : S^n(P) \longrightarrow B$ such that the diagram

$$\begin{array}{ccccccccc}
 & & & & & & S^n(P) & & \\
 & & & & & \swarrow{\scriptstyle \beta} & \downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

commutes. Thus

$$\begin{array}{ccccc}
 & & P & & \\
 & \swarrow{\scriptstyle \beta_n} & \downarrow{\scriptstyle \lambda f} & & \\
B_n & \longrightarrow & C_n & \longrightarrow & 0
\end{array}$$

commutes and $\mathrm{Im}(\beta_n) \subseteq Z_n(B)$. This implies that $\beta_n : P \longrightarrow Z_n(B)$ and the diagram

$$\begin{array}{ccccc}
 & & P & & \\
 & \swarrow{\scriptstyle \beta_n} & \downarrow{\scriptstyle f} & & \\
Z_n(B) & \longrightarrow & Z_n(C) & \longrightarrow & 0
\end{array}$$

commutes. □

Proposition 2.22. *If S is a pure subcomplex of an* FP-*injective complex C, then S is* FP-*injective.*

Proof. Suppose that C is an FP-injective complex and S is a pure subcomplex of C. Then we have a pure exact sequence

$$0 \longrightarrow S \longrightarrow C \longrightarrow C/S \longrightarrow 0$$

and so the sequence

$$0 \longrightarrow (C/S)^+ \longrightarrow C^+ \longrightarrow S^+ \longrightarrow 0$$

splits. Thus S^+ is isomorphic to some direct summand of C^+.

Since C is FP-injective, C is exact and $Z_n(C)$ is FP-injective in R-Mod for all $n \in \mathbb{Z}$. Therefore, C^+ is exact which implies that S^+ is exact. Furthermore, S is exact.

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On the other hand, we obtain a pure exact sequence

$$0 \longrightarrow Z_n(S) \longrightarrow Z_n(C) \longrightarrow Z_n(C/S) \longrightarrow 0$$

for each $n \in \mathbb{Z}$ by Lemma 2.21. Since $Z_n(C)$ is FP-injective, $Z_n(S)$ is FP-injective. By Theorem 2.10 S is an FP-injective complex and our result is established. □

Let $\mathcal{FI}$ stand for the class of all FP-injective modules and let $\widetilde{\mathcal{FI}}$ denote the class of all FP-injective complexes.

Proposition 2.23. *Let R be a ring. Then the following assertions are equivalent.*

(1) *R is a left coherent ring.*
(2) *$\widetilde{\mathcal{FI}}$ is injectively coresolving.*
(3) *Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence in $\mathscr{C}(R)$. If $A, B \in \widetilde{\mathcal{FI}}$, then $C \in \widetilde{\mathcal{FI}}$.
(4) *If $C \in \widetilde{\mathcal{FI}}$ and S is a pure subcomplex of C, then $C/S \in \widetilde{\mathcal{FI}}$.*

Proof. The implications (1) implies (2) and (2) if and only if (3) are obvious.

To show that (3) implies (1), we note that R is a left coherent ring if and only if every factor module of an FP-injective module by a pure submodule is FP-injective (see [18]).

Now we show that (3) implies (4). Let

$$0 \longrightarrow S \longrightarrow C \longrightarrow C/S \longrightarrow 0$$

be a pure exact sequence and $C \in \widetilde{\mathcal{FI}}$. Then $S \in \widetilde{\mathcal{FI}}$ by Proposition 2.22, which implies that $C/S \in \widetilde{\mathcal{FI}}$.

To show that (4) implies (3), let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence in $\mathscr{C}(R)$ and let A and B be FP-injective complexes. Since A is FP-injective the sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is pure. By (4) we have $C \in \widetilde{\mathcal{FI}}$. □

Definition 2.24. Let C be a complex. The FP-injective dimension FP-id(C) of C is defined to be the smallest nonnegative integer n such that $\mathrm{Ext}^{n+1}(F, C) = 0$ for every finitely presented complex F. If no such n exists, then we set FP-id$(C) = \infty$.

The FP-injective dimension FP-id(M) of a left R-module M is similarly defined. Details and results on the FP-injective dimension of modules appeared in [8, 16].

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Lemma 2.25. *Let R be a left coherent ring. For a complex C of left R-modules the following assertions are equivalent.*

(1) $\text{FP-id}(C) \leq m$.
(2) $\text{Ext}^{m+1}(F, C) = 0$ *for all finitely presented complexes F.*
(3) $\text{Ext}^{m+k}(F, C) = 0$ *for all finitely presented complexes F and all $k \geq 1$.*
(4) *If the sequence*

$$0 \to C \to E^0 \to \cdots \to E^m \to 0$$

is exact with $E^0, \ldots, E^{m-1}$ FP-injective, then E^m is also FP-injective.

Proof. This is a straightforward application of the definition. □

By [9, Theorem 3.1.3] we have $\text{id}(C) \leq m$ in $\mathscr{C}(R)$ if and only if C is exact and $\text{id}(Z_n(C)) \leq m$ in R-Mod for all $n \in \mathbb{Z}$. Similar results hold for FP-injective dimensions.

Theorem 2.26. *Let R be a left coherent ring and C be a complex. Then $\text{FP-id}(C) \leq m$ in $\mathscr{C}(R)$ if and only if C is exact and $\text{FP-id}(Z_n(C)) \leq m$ in R-Mod for all $n \in \mathbb{Z}$.*

Proof. Suppose that $\text{FP-id}(C) \leq m$. Let

$$0 \to C \to E^0 \to \cdots \to E^m \to 0$$

be an FP-injective resolution of C. Then the E^i are exact and so C is exact.

On the other hand, we have the exact sequence

$$0 \longrightarrow Z_n(C) \longrightarrow Z_n(E^0) \longrightarrow \cdots \longrightarrow Z_n(E^m) \longrightarrow 0$$

where the $Z_n(E^i)$ are FP-injective modules. Hence $\text{FP-id}(Z_n(C)) \leq m$.

Suppose that the sequence

$$0 \longrightarrow C \longrightarrow E^0 \longrightarrow \cdots \longrightarrow E^{m-1} \longrightarrow K^m \longrightarrow 0$$

is exact and the E^i are FP-injective. We only need to show that K^m is FP-injective. Then we get an exact sequence,

$$0 \longrightarrow Z_n(C) \longrightarrow Z_n(E^0) \longrightarrow \cdots \longrightarrow Z_n(K^m) \longrightarrow 0.$$

Since $\text{FP-id}(Z_n(C)) \leq m$ it follows that each $Z_n(K^m)$ is FP-injective. Since the C, E^i are exact, K^m is exact. By Theorem 2.10, K^m is FP-injective. □

According to [9, Theorem 5.4.1] we have $\text{fd}(C) \leq m$ in $\mathscr{C}(R)$ if and only if C is exact and $\text{fd}(Z_n(C)) \leq m$ in R-Mod for all $n \in \mathbb{Z}$. In [8] it is shown that, for a left R-module M,

$$\text{fd}(M) = \text{id}(M^+) = \text{FP-id}(M^+)$$

and, over a left coherent ring R, we have $\text{fd}(M^+) = \text{FP-id}(M)$. Thus we get the following corollaries.

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COROLLARY 2.27. *Let R be a left coherent ring and let C be a complex. Then*

$$\mathrm{id}(C^+) = \mathrm{fd}(C) = \text{FP-id}(C^+).$$

COROLLARY 2.28. *Let R be a left coherent ring and let C be a complex. Then*

$$\mathrm{fd}(C^+) = \text{FP-id}(C).$$

PROPOSITION 2.29. *Let C be an exact complex. Then*

$$\text{FP-id}(C) = \sup\{\text{FP-id}(Z_n(C)) \mid n \in \mathbb{Z}\}.$$

PROOF. If

$$\sup\{\text{FP-id}(Z_n(C)) \mid n \in \mathbb{Z}\} = \infty,$$

then

$$\text{FP-id}(C) \leq \sup\{\text{FP-id}(Z_n(C)) \mid n \in \mathbb{Z}\}.$$

So naturally we may assume that

$$\sup\{\text{FP-id}(Z_n(C)) \mid n \in \mathbb{Z}\} = m$$

is finite.

Consider an FP-injective resolution

$$0 \longrightarrow C \longrightarrow E^0 \longrightarrow \cdots \longrightarrow E^{m-1} \longrightarrow K^m \longrightarrow 0$$

of C where each E^i is FP-injective. Then we get an exact sequence

$$0 \longrightarrow Z_n(C) \longrightarrow Z_n(E^0) \longrightarrow \cdots \longrightarrow Z_n(K^m) \longrightarrow 0$$

for all $n \in \mathbb{Z}$. Since $\text{FP-id}(Z_n(C)) \leq m$ we have that $Z_n(K^m)$ is FP-injective for all $n \in \mathbb{Z}$. By Theorem 2.10 K^m is FP-injective. This shows that $\text{FP-id}(C) \leq m$ and so

$$\text{FP-id}(C) \leq \sup\{\text{FP-id}(Z_n(C)) \mid n \in \mathbb{Z}\}.$$

Now it is enough to show that

$$\sup\{\text{FP-id}(Z_n(C)) \mid n \in \mathbb{Z}\} \leq \text{FP-id}(C).$$

Naturally we may assume that $\text{FP-id}(C) = m$ is finite. Then there exists an exact sequence

$$0 \longrightarrow C \longrightarrow E^0 \longrightarrow \cdots \longrightarrow E^{m-1} \longrightarrow E^m \longrightarrow 0$$

with each E^i FP-injective. So we get an exact sequence

$$0 \longrightarrow Z_n(C) \longrightarrow Z_n(E^0) \longrightarrow \cdots \longrightarrow Z_n(E^m) \longrightarrow 0.$$

Now $\text{FP-id}(Z_n(C)) \leq m$ because $Z_n(E^i)$ is FP-injective for all $n \in \mathbb{Z}$ and $i = 0, 1, \ldots, m$, and so

$$\sup\{\text{FP-id}(Z_n(C)) \mid n \in \mathbb{Z}\} \leq m = \text{FP-id}(C).$$

This concludes the proof. □

Proposition 2.30. *The following are equivalent for a ring R.*

(1) *R is left coherent.*
(2) *Every direct limit of* FP*-injective complexes of left R-modules is* FP*-injective.*
(3) *The map*

$$\varinjlim \operatorname{Ext}^1(F, C_i) \to \operatorname{Ext}^1(F, \varinjlim C_i)$$

is an isomorphism for every finitely presented F and direct system $(C_i)_I$.
(4) *The map*

$$\varinjlim \operatorname{Ext}^n(F, C_i) \to \operatorname{Ext}^n(F, \varinjlim C_i)$$

is an isomorphism for every finitely presented F and direct system $(C_i)_I$.
(5) *Every direct limit of complexes of* FP*-injective dimension less than or equal n has* FP*-injective dimension less than or equal n.*

Proof. The implications (4) implies (3), (3) implies (2), (4) implies (5) and (5) implies (2) are obvious.

We show that (1) implies (4). Suppose that R is a left coherent ring. Then every finitely presented complex of left R-modules has a projective resolution consisting of finitely presented complexes and so the result follows by [17, Proposition 3.4].

To see that (2) implies (1), let $\{M_i\}_I$ be a direct system of FP-injective left R-modules. Then $\{D^0(M_i)\}_I$ is a direct system of FP-injective complexes. By (2), $\varinjlim D^0(M_i)$ is FP-injective. However,

$$\varinjlim D^0(M_i) = D^0(\varinjlim M_i),$$

and so $D^0(\varinjlim M_i)$ is an FP-injective complex and $\varinjlim M_i$ is an FP-injective module. Thus R is a left coherent ring. □

3. FP-injective dimension of complexes

In [3] Avramov and Foxby defined injective, projective and flat dimensions for arbitrary complexes of left R-modules over associative rings in terms of DG-injective, DG-projective and DG-flat complexes respectively. A complex I is DG-injective if each I_n is injective and $\mathcal{H}\mathrm{om}(E, I)$ is exact for every exact complex E. For example, every bounded above complex of injective modules is DG-injective (see [3, Remark 1.1I]). By [7, Remark, p. 31] the class of DG-injective complexes is injectively coresolving, that is, if

$$0 \longrightarrow I' \longrightarrow I \longrightarrow I'' \longrightarrow 0$$

is a short exact sequence of complexes with I' DG-injective, then I is DG-injective if and only if I'' is DG-injective.

A DG-injective resolution of X is a quasiisomorphism $X \longrightarrow I$ for which I is DG-injective. By [7, Corollary 3.10] every complex X has an injective DG-injective resolution $X \longrightarrow I$. If $H(X)$ is bounded above, then I can be chosen so that $\sup I = \sup H(X)$. A complex P is DG-projective if each P_n is projective and $\mathcal{H}\mathrm{om}(P,\ E)$

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is exact for every exact complex E. For example, every bounded below complex of projective modules is a DG-projective complex (see [3, Remark 1.1P]).

By [7, Remark, p. 31] the class of DG-projective complexes is projectively resolving. That is, if

$$0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0$$

is a short exact sequence of complexes with P'' DG-projective, then P' is DG-projective if and only if P is DG-projective. A quasiisomorphism $P \longrightarrow X$ with P DG-projective is called a DG-projective resolution of X. By [7, Corollary 3.10] every complex X has a surjective DG-projective resolution $P \longrightarrow X$. If $H(X)$ is bounded below, then P can be chosen so that $\inf P = \inf H(X)$.

A complex F of left R-modules is DG-flat if each F_n is flat and the complex $E \otimes F$ is exact for every exact complex E of right R-modules. Since $(E \otimes F)^+ \cong \mathcal{H}\mathrm{om}(E, F^+)$ we have that F is DG-flat if and only if F^+ is DG-injective. Thus every bounded below complex of flat modules is DG-flat.

The class of DG-flat complexes is projectively resolving. A DG-flat resolution of a complex X is a quasiisomorphism $F \longrightarrow X$ with F a DG-flat complex. Since every DG-projective complex is DG-flat, every complex has a surjective DG-flat resolution. By [7, Proposition 3.7], a complex is projective (injective, flat respectively) if and only if it is exact and DG-projective (DG-injective, DG-flat respectively).

We now define the injective and flat dimensions of a complex X. Let $\mathrm{DG}_{\mathrm{Inj}}(X)$ denote the class of all DG-injective complexes I such that $X \simeq I$ and let $\mathrm{DG}_{\mathrm{Flat}}(X)$ denote the class of all DG-flat complexes F such that $X \simeq F$. The injective dimension of the complex X is defined by

$$\mathrm{id}_R(X) = \inf\{\sup\{-n \in \mathbb{Z} \mid I_n \neq 0\} \mid I \in \mathrm{DG}_{\mathrm{Inj}}(X)\},$$

and the flat dimension of X is defined by

$$\mathrm{fd}_R(X) = \inf\{\sup\{n \in \mathbb{Z} \mid F_n \neq 0\} \mid F \in \mathrm{DG}_{\mathrm{Flat}}(X)\}.$$

In [10], Gillespie introduced the following definition.

Definition 3.1 [10, Definition 3.3]. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair on an abelian category $\mathscr{C}$. Let X be a complex. Then we have the following definitions.

(1) X is called an $\mathcal{A}$ complex if it is exact and $Z_n(X) \in \mathcal{A}$ for all n.
(2) X is called a $\mathcal{B}$ complex if it is exact and $Z_n(X) \in \mathcal{B}$ for all n.
(3) X is called a dg-$\mathcal{A}$ complex if $X_n \in \mathcal{A}$ for each n and $\mathcal{H}\mathrm{om}(X, B)$ is exact whenever B is a $\mathcal{B}$ complex.
(4) X is called a dg-$\mathcal{B}$ complex if $X_n \in \mathcal{B}$ for each n and $\mathcal{H}\mathrm{om}(A, X)$ is exact whenever A is an $\mathcal{A}$ complex.

We denote the class of $\mathcal{A}$ complexes by $\widetilde{\mathcal{A}}$ and the class of dg-$\mathcal{A}$ complexes by $\mathrm{dg}\,\widetilde{\mathcal{A}}$. Similarly the class of $\mathcal{B}$ complexes are denoted by $\widetilde{\mathcal{B}}$ and the class of dg-$\mathcal{B}$ complexes is denoted by $\mathrm{dg}\,\widetilde{\mathcal{B}}$.

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Lemma 3.2 [10]. *Let* $(\mathcal{A}, \mathcal{B})$ *be a cotorsion pair in* R-Mod.

(1) *Complexes that are bounded below and have components in* $\mathcal{A}$ *are* dg-$\mathcal{A}$ *complexes (see* [10, Lemma 3.4]*). Complexes that are bounded above and have components in* $\mathcal{B}$ *are* dg-$\mathcal{B}$ *complexes.*
(2) $(\widetilde{\mathcal{A}}, \mathrm{dg}\,\widetilde{\mathcal{B}})$ *and* $(\mathrm{dg}\,\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ *are cotorsion pairs in* $\mathscr{C}(R)$ *(see* [10, Proposition 3.6]*).*
(3) *If* $(\mathcal{A}, \mathcal{B})$ *is hereditary, then* $\mathrm{dg}\,\widetilde{\mathcal{A}} \cap \mathcal{E} = \widetilde{\mathcal{A}}$ *and* $\mathrm{dg}\,\widetilde{\mathcal{B}} \cap \mathcal{E} = \widetilde{\mathcal{B}}$ *where* $\mathcal{E}$ *denotes the class of exact complexes (see* [10, Theorem 3.12]*).*
(4) *If* $(\mathcal{A}, \mathcal{B})$ *is hereditary, then* $(\widetilde{\mathcal{A}}, \mathrm{dg}\,\widetilde{\mathcal{B}})$ *and* $(\mathrm{dg}\,\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ *are hereditary in* $\mathscr{C}(R)$ *(see* [10, Corollary 3.13]*).*

It is well known that over a left coherent ring $({}^{\perp}\mathcal{FI}, \mathcal{FI})$ there is a hereditary cotorsion pair. According to Lemma 3.2, complexes of FP-injective modules that are bounded above are dg-FP-injective complexes, $\mathrm{dg}\,\widetilde{\mathcal{FI}} \cap \mathcal{E} = \widetilde{\mathcal{FI}}$ and $\mathrm{dg}\,\widetilde{\mathcal{FI}}$ is injectively coresolving.

Inspired by Avramov and Foxby's definition of injective dimension for every complex, we now introduce and study a notion of FP-injective dimension associated to every complex of left R-modules over an arbitrary ring.

Definition 3.3. A morphism $X \longrightarrow F$ is called a dg-FP-injective resolution of X if $X \longrightarrow F$ is a quasiisomorphism and F is a dg-FP-injective complex.

Since every DG-injective complex is dg-FP-injective and every complex has an injective DG-injective resolution every complex has an injective dg-FP-injective resolution.

Definition 3.4. Let X be a complex of left R-modules. The FP-injective dimension of X is defined by $\text{FP-id}_R(X) \leq n$ if there is a dg-FP-injective resolution $X \longrightarrow F$ such that $\inf H(X) \geq -n$ and $Z_{-n}(F)$ is an FP-injective module. If $\text{FP-id}_R(X) \leq n$, but $\text{FP-id}_R(X) \leq n-1$ does not hold, then $\text{FP-id}_R(X) := n$. If $\text{FP-id}_R(X) \leq n$ for each n, then $\text{FP-id}_R(X) := -\infty$. If $\text{FP-id}_R(X) \leq n$ does not hold for each n, then $\text{FP-id}_R(X) := \infty$.

This definition is quite different from Definition 2.24. The reason is that $\text{FP-id}_R(X)$ defines an R-dimension while Definition 2.24 defines a $\mathscr{C}(R)$-dimension.

Remark 3.5. First, $\text{FP-id}_R(X) = -\infty$ if and only if X is exact.

Next, for each $k \in \mathbb{Z}$

$$\text{FP-id}_R(\Sigma^k X) = \text{FP-id}_R(X) + k.$$

Finally, $\text{FP-id}_R(X) \leq \mathrm{id}_R(X)$. If R is a left Noetherian ring, then $\text{FP-id}_R(X) = \mathrm{id}_R(X)$.

Proposition 3.6. *Let R be a left coherent ring and let X be a complex. If X has a* dg-FP-*injective resolution* $X \longrightarrow F$ *such that* $\inf H(F) \geq -n$ *and* $Z_{-n}(F)$ *is* FP-*injective, then for every* DG-*injective resolution* $X \longrightarrow I$ *we have* $\inf H(I) \geq -n$ *and* $Z_{-n}(I)$ *is an* FP-*injective module.*

Proof. If $X \longrightarrow I$ is a DG-injective resolution, then

$$\inf H(I) = \inf H(X) = \inf H(F) \geq -n.$$

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We can assume, without loss of generality, that $X \longrightarrow F$ is an injective dg-FP-injective resolution. Then there exists an exact sequence

$$0 \longrightarrow X \longrightarrow F \longrightarrow L \longrightarrow 0$$

with L exact. This yields an exact sequence

$$0 \longrightarrow \mathrm{Hom}(L, I) \longrightarrow \mathrm{Hom}(F, I) \longrightarrow \mathrm{Hom}(X, I) \longrightarrow \mathrm{Ext}^1(L, I) = 0.$$

Hence there is a morphism of complexes $F \longrightarrow I$ such that the diagram

$$\begin{array}{ccc} X & \longrightarrow & F \\ \downarrow & \swarrow & \\ I & & \end{array}$$

commutes. Since both $X \longrightarrow F$ and $X \longrightarrow I$ are quasiisomorphisms, so is $F \longrightarrow I$.

We can assume that $F \longrightarrow I$ is an injective quasiisomorphism (if not, let $F \longrightarrow \bar{I}$ be injective with $\bar{I}$ an injective complex, then $F \longrightarrow I \oplus \bar{I}$ is an injective quasiisomorphism). Then there exists an exact sequence

$$0 \longrightarrow F \longrightarrow I \longrightarrow V \longrightarrow 0$$

with V an exact complex. Both F and I are dg-FP-injective complexes and so V is a dg-FP-injective complex. Thus V is exact and dg-FP-injective and so V is FP-injective.

On the other hand, we have an exact sequence

$$0 \longrightarrow Z_{-n}(F) \longrightarrow Z_{-n}(I) \longrightarrow Z_{-n}(V) \longrightarrow 0$$

with $Z_{-n}(V)$ and $Z_{-n}(F)$ FP-injective modules. It follows that $Z_{-n}(I)$ is FP-injective and our result is established. □

THEOREM 3.7. *Let R be a left coherent ring and let X a be complex. Then the following assertions are equivalent.*

(1) FP-$\mathrm{id}_R(X) \leq n$.

(2) $\inf H(X) \geq -n$ *and* $Z_{-n}(I)$ *is* FP-*injective for each* DG-*injective resolution* $X \longrightarrow I$.

(2)′ $\inf H(X) \geq -n$ *and* $Z_j(I)$ *is* FP-*injective for every* $j \leq -n$ *for each* DG-*injective resolution* $X \longrightarrow I$.

(3) *There exists a* DG-*injective resolution* $X \longrightarrow I'$ *such that* $H_j(I') = 0$ *for every* $j \leq -n-1$ *and* $Z_{-n}(I')$ *is* FP-*injective.*

(3)′ *There exists a* DG-*injective resolution* $X \longrightarrow I'$ *such that* $H_j(I') = 0$ *for every* $j \leq -n-1$ *and* $Z_j(I')$ *is* FP-*injective for every* $j \leq -n$.

PROOF. It follows from Proposition 3.6 that (1) implies (2). The implication (2) implies (3) is obvious, and (3) implies (1) by definition, since every DG-injective resolution is a dg-FP-injective resolution. The implications (2) implies (2)′ and (3) implies (3)′ are clear, since over a left coherent ring the class of all FP-injective modules is injectively coresolving. □

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REMARK 3.8. We have proved that $\text{FP-id}_R(X) \le n$ is equivalent to $\inf H(X) \ge -n$ and $Z_{-n}(I)$ is FP-injective for each DG-injective resolution $X \longrightarrow I$. However, we do not know whether $\text{FP-id}_R(X) \le n$ is equivalent to $\inf H(X) \ge -n$ and $Z_{-n}(F)$ is FP-injective for each dg-FP-injective resolution $X \longrightarrow F$ and whether $\text{FP-id}_R(X)$ can be expressed in the form

$$\inf\{\sup\{-n \in \mathbb{Z} \mid F_n \neq 0\} \mid F \in \text{dg-FP}_{\text{Inj}}(X)\}$$

where $\text{dg-FP}_{\text{Inj}}(X)$ denotes the class of all dg-FP-injective complexes F such that $X \simeq F$.

The following result shows that for coherent rings the FP-injective dimension of complexes is a generalization of the FP-injective dimension of modules.

PROPOSITION 3.9. *Let R be a left coherent ring and let M be an R-module. Then*

$$\text{FP-id}(M) = \text{FP-id}_R(S^0(M)).$$

PROOF. Let

$$0 \longrightarrow M \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow \cdots$$

be an injective resolution of M. Then $S^0(M) \longrightarrow F$ is a DG-injective resolution where

$$F = \cdots \longrightarrow 0 \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow \cdots .$$

If $\text{FP-id}(M) = \infty$ and

$$\text{FP-id}_R(S^0(M)) = n < \infty,$$

then $Z_j(F)$ is FP-injective for some $j \le -n$ by Theorem 3.7.

Since

$$0 \longrightarrow M \longrightarrow F_0 \longrightarrow \cdots \longrightarrow F_{-n-1} \longrightarrow Z_{-n}(F) \longrightarrow 0$$

is exact for FP-injective modules $Z_{-n}(F)$ and F_j it follows that $\text{FP-id}(M) \le n$. This contradicts the fact that $\text{FP-id}(M) = \infty$. So $\text{FP-id}_R(S^0(M)) = \infty$.

If $\text{FP-id}(M) = n < \infty$, then $Z_{-n}(F)$ is FP-injective and so $Z_j(F)$ is FP-injective for every $j \le -n$. Thus $S^0(M) \longrightarrow F$ is a dg-FP-injective resolution with $Z_j(F)$ FP-injective for all $j \le -n$ and $H_j(F) = 0$ for every $j \le -n-1$.

By definition we get $\text{FP-id}_R(S^0(M)) \le n$. Suppose that $\text{FP-id}_R(S^0(M)) \le n-1$. Then $\text{FP-id}(M) \le n-1$. This contradicts the fact that $\text{FP-id}(M) = n$. Therefore, $\text{FP-id}_R(S^0(M)) = n$. □

Let R be a left coherent ring and let X be a homologically bounded above complex. In the following proposition we obtain a description of $\text{FP-id}_R(X)$ in terms of the class $\text{BA}(X)$ of bounded above complexes Q of FP-injective modules which satisfy the condition that $X \simeq Q$.

PROPOSITION 3.10. *Let R be a left coherent ring and let X be a homologically bounded above complex. Then*

$$\text{FP-id}_R(X) = \inf\{\sup\{l \in \mathbb{Z} \mid Q_{-l} \neq 0\} \mid Q \in \text{BA}(X)\}.$$

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Proof. Since X is a homologically bounded above complex we can assume that $\sup H(X) = 0$. Let $X \longrightarrow I$ be a DG-injective resolution of X such that $\sup I = 0$. Set

$$\Omega = \inf\{\sup\{l \in \mathbb{Z} \mid Q_{-l} \neq 0\} \mid Q \in \mathrm{BA}(X)\}.$$

If $\text{FP-id}_R(X) = n$, then $Z_{-n}(I)$ is FP-injective and $\inf H(X) \geq -n$ by Theorem 3.7. Let

$$I' = 0 \longrightarrow I_0 \longrightarrow I_{-1} \longrightarrow \cdots \longrightarrow Z_{-n}(I) \longrightarrow 0,$$

and

$$X' = \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow \cdots \longrightarrow X_{-n-1} \longrightarrow Z_{-n}(I) \longrightarrow 0.$$

Since $X \longrightarrow I$ is a quasiisomorphism $X' \longrightarrow I'$ is also a quasiisomorphism. However, $X \simeq X'$ and so we get $X \simeq I'$. Each component of I' is an FP-injective module, which implies that $\Omega \leq n$.

Now suppose $\Omega = n < \infty$. We will show that $\text{FP-id}_R(X) \leq n$. By the hypothesis there exists a complex

$$Q = 0 \longrightarrow Q_0 \longrightarrow Q_{-1} \longrightarrow \cdots \longrightarrow Q_{-n-1} \longrightarrow Q_{-n} \longrightarrow 0$$

of FP-injective modules such that $X \simeq Q$. Since $Q \simeq X \simeq I$ and I is a DG-injective complex there is a quasiisomorphism $Q \longrightarrow I$. In addition, Q is bounded above and so there is an injective morphism $Q \longrightarrow I^*$ with I^* a bounded above injective complex. Thus $Q \longrightarrow I \oplus I^*$ is an injective quasiisomorphism, and we have an exact sequence

$$0 \longrightarrow Q \longrightarrow I \oplus I^* \longrightarrow W \longrightarrow 0$$

with W exact, which implies that there is an exact sequence

$$0 \longrightarrow Q_j \longrightarrow I_j \oplus I_j^* \longrightarrow W_j \longrightarrow 0$$

in R-Mod for all $j \in \mathbb{Z}$. Since $I_j \oplus I_j^*$ and Q_j are FP-injective it follows that W_j is FP-injective. Thus W is a bounded above exact complex of FP-injective modules. That is, W is an FP-injective complex and so $Z_j(W)$ is an FP-injective module.

In the exact sequence

$$0 \longrightarrow Z_{-n}(Q) \longrightarrow Z_{-n}(I) \oplus Z_{-n}(I^*) \longrightarrow Z_{-n}(W) \longrightarrow 0$$

we have that $Z_{-n}(Q) = Q_{-n}$ and $Z_{-n}(W)$ are FP-injective and so $Z_{-n}(I) \oplus Z_{-n}(I^*)$ is FP-injective which implies that $Z_{-n}(I)$ is FP-injective. Since $X \longrightarrow I$ is a DG-injective resolution with $\inf H(I) \geq -n$ and $Z_{-n}(I)$ FP-injective it follows that $\text{FP-id}_R(X) \leq n$. From the above, we have $\text{FP-id}_R(X) = \infty$ if and only if $\Omega = \infty$. Note that $\text{FP-id}_R(X) = -\infty$ if and only if X is exact if and only if $\Omega = -\infty$ and we have established our result. □

Lemma 3.11 (Horseshoe lemma). *For every exact sequence of complexes*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

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there exists a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^X & \longrightarrow & I^Y & \longrightarrow & I^Z & \longrightarrow & 0 \end{array}$$

in which the columns are injective DG-injective resolutions.

Proposition 3.12. *Let R be a left coherent ring and let*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

be an exact sequence of complexes of R-modules. If any two of the three complexes X, Y and Z have finite FP*-injective dimension, then so does the third.*

Proof. By Lemma 3.11 there is an exact sequence of complexes

$$0 \longrightarrow I^X \longrightarrow I^Y \longrightarrow I^Z \longrightarrow 0$$

with $X \longrightarrow I^X$, $Y \longrightarrow I^Y$ and $Z \longrightarrow I^Z$ DG-injective resolutions. If two of the complexes X, Y and Z have finite FP-injective dimension, then there is $n \in \mathbb{Z}$ such that

$$H_j(I^X) = H_j(I^Y) = H_j(I^Z) = 0$$

for all $j \leq -n$.

For each $j \leq -n$ we have an exact sequence

$$0 \longrightarrow Z_j(I^X) \longrightarrow Z_j(I^Y) \longrightarrow Z_j(I^Z) \longrightarrow 0$$

in R-Mod. If $Z_j(I^X)$ is FP-injective, then $Z_j(I^Y)$ is FP-injective if and only if $Z_j(I^Z)$ is FP-injective. If both $Z_j(I^Y)$ and $Z_j(I^Z)$ are FP-injective, then FP-id$(Z_j(I^X)) \leq 1$ and so $Z_{j-1}(I^X)$ is FP-injective. □

Proposition 3.13. *Let R be a left coherent ring. Then the following equalities hold.*

(1) $\mathrm{fd}_R(X) = \text{FP-id}_R(X^+)$.
(2) $\mathrm{fd}_R(X) = \mathrm{id}_R(X^+)$.

Proof. We show (1) holds. If X is exact, then X^+ is also exact, so

$$\text{FP-id}_R(X^+) = -\infty = \mathrm{fd}_R(X).$$

If $\mathrm{fd}_R(X) = n < \infty$, then there exists a DG-flat resolution $F \longrightarrow X$ with the properties that $\sup H(F) \leq n$ and $C_j(F)$ is flat for all $j \geq n$. It follows that $X^+ \longrightarrow F^+$ is a DG-injective resolution, and $\inf H(F^+) \geq -n$.

The exact sequence

$$\cdots \longrightarrow F_{n+1} \xrightarrow{\delta_{n+1}} F_n \xrightarrow{\pi} C_n(F) \longrightarrow 0$$

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yields an exact sequence

$$0 \longrightarrow C_n(F)^+ \xrightarrow{\pi^+} F_n^+ \xrightarrow{\delta_{n+1}^+} F_{n+1}^+ \longrightarrow \cdots .$$

Thus

$$\operatorname{Ker}(\delta_{n+1}^+) = \operatorname{Im}(\pi^+) \cong C_n(F)^+.$$

Since $C_n(F)$ is flat, $C_n(F)^+$ is FP-injective. Hence

$$0 \longrightarrow \operatorname{Ker}(\delta_{n+1}^+) \longrightarrow F_n^+ \xrightarrow{\delta_{n+1}^+} F_{n+1}^+ \longrightarrow \cdots$$

is an exact sequence of FP-injective modules, which implies that $Z_j(F^+) = \operatorname{Ker}(\delta_j^+)$ is FP-injective for all $j \geq n+1$. Therefore $\text{FP-id}_R(X^+) \leq n$.

Suppose that $\text{FP-id}_R(X^+) = n \leq \infty$, and let $F \longrightarrow X$ be a DG-flat resolution. Then $X^+ \longrightarrow F^+$ is a DG-injective resolution. Since $\text{FP-id}_R(X^+) = n$, it is quite clear that $H_j(F^+) = 0$ for all $j \leq -n-1$ and $Z_j(F^+)$ is FP-injective for all $j \geq n+1$. Since $H_j(F^+) \cong H_j(F)^+ = 0$, we obtain $H_j(F) = 0$ for every $j \geq n+1$. Thus the sequence

$$\cdots \longrightarrow F_{n+1} \xrightarrow{\delta_{n+1}} F_n \longrightarrow C_n(F) \longrightarrow 0$$

is exact and hence yields the sequence

$$0 \longrightarrow C_n(F)^+ \longrightarrow F_n^+ \xrightarrow{\delta_{n+1}^+} F_{n+1}^+ \longrightarrow \cdots ,$$

which is also exact. Therefore $\operatorname{Ker}(\delta_{n+1}^+) \cong C_n(F)^+$ is FP-injective. It follows that $C_n(F)$ is flat and so $\text{fd}_R(X) \leq n$.

An analogous proof to that of part (1) proves part (2). □

Proposition 3.14. *Let R be a left coherent ring and let X be a homologically bounded above complex. Then* $\text{fd}_R(X^+) = \text{FP-id}_R(X)$.

Proof. If X is exact, then the result clearly holds. Now suppose that X is not exact. Since X is a homologically bounded above complex we can assume that $\sup H(X) = 0$.

Let $X \longrightarrow I$ be a DG-injective resolution of X and $\sup I = 0$. Then $I^+ \longrightarrow X^+$ is a DG-flat resolution and $\inf I^+ = 0$. If $\text{FP-id}_R(X) = n \leq \infty$, then $Z_j(I)$ is FP-injective for all $j \leq -n$ and $\inf H(X) \geq -n$. Thus we have an exact sequence

$$0 \longrightarrow Z_{-n}(I) \longrightarrow I_{-n} \longrightarrow\xrightarrow{\delta_{-n}} I_{-n-1} \longrightarrow \cdots ,$$

which yields the sequence

$$\cdots \longrightarrow I_{-n-1}^+ \xrightarrow{\delta_{-n}^+} I_{-n}^+ \longrightarrow Z_{-n}(I)^+ \longrightarrow 0,$$

which is exact. Hence

$$C_{-n}(I^+) = \operatorname{Coker}(\delta_{-n}^+) \cong Z_{-n}(I)^+.$$

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Since I_j and $Z_{-n}(I)$ are FP-injective, I_j^+ and $Z_{-n}(I)^+$ are flat and, consequently, $C_{-n}(I^+) \cong C_{-n}(I)^+$ is flat. As $\inf H(I) = \inf H(X) \geq -n$, we deduce that $\sup H(I^+) \leq n$. Thus $\mathrm{fd}_R(X^+) \leq \text{FP-id}_R(X)$. By analogy with the above discussion, we deduce that $\text{FP-id}_R(X) \leq \mathrm{fd}_R(X^+)$. Therefore $\mathrm{fd}_R(X^+) = \text{FP-id}_R(X)$ and our result is established. □

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