1

A Brief Introduction to Special Relativity and Light

Special relativity is a theory that explains the space and time relationships between bodies moving at different velocities. Special relativity allows for situations where observers of the kinetics of bodies are moving at significantly different velocities to the bodies. A significantly different velocity needs to be usually within an order of magnitude or so of the very high velocity of light in vacuum.

The theory of special relativity supercedes the laws of motion developed by Isaac Newton (1643–1727). Bodies behave in a classical or Newtonian manner if the observer has a velocity close to that of the body and, for many applications, Newtonian physics is more than adequate. Newton's first law states that an object will remain at rest or in uniform motion in a straight line unless acted upon by an external force. His second law gives the manner of change of the velocity, or in other words, the acceleration of the body when acted upon by a force. When subject to a force $\mathbf{F}$, the acceleration $\mathbf{a}$ of a body of mass m is related to the force by $\mathbf{F} = m\mathbf{a}$. Newton's second law is often more generally expressed in terms of the momentum $\mathbf{p}$ of a body with the force $\mathbf{F}$ on the body related to the rate of change in time of the momentum $\mathbf{p}$ by $\mathbf{F} = d\mathbf{p}/dt$. Allowing $m\mathbf{a}$ or equivalently $d\mathbf{p}/dt$ to be a force, Newton's third law states that forces occur in equal but oppositely directed pairs.

Special relativity is based on the postulates that: (a) the laws of physics are the same for all observers,[1] and (b) the speed of light in a vacuum is the same for all observers regardless of their relative motion or the motion of the light source. Consequences of the constant speed of light are that to a stationary observer, the passing of time on a moving body appears to slow (known as time dilation) and distances in the direction of the moving body velocity appear shorter (known as length contraction). The logical progression to show that the constant speed of light

[1] There is a caveat for both Newtonian mechanics and special relativity regarding the constancy of the laws of physics for all observers. An accelerating observer experiences "fictional" forces such as the centrifugal force which depend on their acceleration. The laws of physics are taken to be the same for all observers if the observer is not subject to fictional forces.

leads to time dilation and length contraction is first outlined in Section 1.2 and further explored for different scenarios throughout this book.

There are theoretical deductions and a vast array of experimental measurements showing that the speed of light in vacuum is constant. For example, Maxwell's equations were developed to summarize relationships between electric charges and electric and magnetic fields. Electromagnetic waves are predicted by Maxwell's equations to have a velocity of propagation in vacuum solely dependent on some constants of proportionality between electric charges and oscillating electric and magnetic fields (the dielectric constant ϵ_0 and magnetic permeability μ_0). The electromagnetic wave velocity proportionality to electrical and magnetic quantities gives a *primae facie* case for the velocity to be a constant of nature (see Section 1.1).

This book deals particularly with the relativistic aspects of light interactions with plasma material, where "light" refers to any electromagnetic wave. The word "light" is popularly used to refer to visible electromagnetic waves (wavelengths 400−700 nm) and sometimes ultraviolet and infrared electromagnetic radiation. Other forms of electromagnetic radiation such as X-rays, microwaves, and radio waves exhibit many identical properties in vacuum, including the velocity of propagation. Differences between electromagnetic waves anywhere on the spectrum from X-ray to radio waves mainly arise when the waves interact with plasma or other states of matter. A plasma is created when a gas is heated and produces negatively charged free electrons and positively charged ions. Electromagnetic waves interact strongly with the free electrons in particular.

Most of the observable universe is made up of plasma material. Densities range from stellar interiors to the low density of intergalactic space (with approximately one ion and one free electron per cubic meter). The astrophysical and cosmological properties of the universe have been determined using the emission, absorption, scattering, and interaction of light with plasma. We consider these processes in this book with a special emphasis on the effects of special relativity. Special relativistic effects associated with the interaction of light with astrophysical plasmas are important when astrophysical objects are hot and when objects have large relative velocities. The expanding universe produces important relative velocity effects on cosmological length scales, for example, between galaxies.

Laboratory plasmas created when high-power lasers are focused onto solid, liquid, or gas targets produce expanding plasmas with densities ranging from many times solid state material densities down to densities obtained with laboratory vacuum equipment. Lasers can accelerate electrons to high velocity where relativistic effects are important. Some of the phenomena seen in astrophysical plasmas can be replicated in laboratory plasmas, enabling laboratory investigation of astrophysical phenomena [16], [78]. Relativistic laboratory plasmas are also useful sources of high-energy particles and high photon energy radiation [13], [73], [112]. Inertial confinement fusion driven by lasers seeks to compress the deuterium (D) and tritium (T) isotopes of hydrogen to high densities. Compression and heating of DT mixtures may produce "fusion-burning" plasmas exhibiting some of the relativistic effects seen in the early universe [99].

The theory of special relativity was developed by Albert Einstein in the early 20th century. Einstein is probably the most significant physicist of all time, though some may still credit Newton with this accolade. Einstein is certainly the most readily recognized physicist of all time. Einstein's name and image are instantaneously identified by almost everyone. Anyone with a passing interest in science has heard of Einstein's famous equation relating the mass m of a body with an energy: $E = mc^2$. Einstein's equivalence of mass and energy arises because the momentum of a body moving at a velocity $\mathbf{v}$ measured by a stationary observer increases due to time dilation and length contraction to a value above the momentum $m\mathbf{v}$ predicted by Newtonian mechanics. The new relationship between energy and momentum shows that even with zero momentum, there is a residual energy mc^2 (see Section 2.8).

Albert Einstein was born on March 14, 1879, in Ulm, Germany. He was 26 in 1905 and employed by the Swiss Patent Office in Bern when he published two papers setting out the theory of special relativity [32], [35]. Another paper in that year helped confirm the idea that light comes in "quanta," now referred to as photons [34]. In his "photons paper," Einstein explained the photo-electric effect where a threshold light frequency is required to ionize an atom and release an electron. A fourth paper dealt with the Brownian motion of atoms and led physicists to accept the existence of atoms [33]. As these fundamental papers in the development of modern physics appeared in the same year, the year 1905 has been termed Einstein's "annus mirabilis" ("the miracle year" in English). Einstein became director of the Kaiser Wilhelm Physical Institute and professor at the University of Berlin in 1914. By 1916, in order to explain gravitational fields, Einstein had extended the theory of special relativity to the theory of general relativity. In 1917, he applied the general theory of relativity to model the structure of the universe.

The theories of relativity are the towering monuments to Einstein, but he also developed other ideas important to radiation and statistical mechanics. Einstein proposed the idea of stimulated emission and the relationships between radiation emission and absorption processes [36], [37]. We treat these concepts in Chapter 7. Stimulated emission stayed as a process sometimes important in radiation theory until the invention of the laser in 1960 when its practical use started to become evident. Laser is an acronym for light amplification by the stimulated emission of radiation. Later Einstein supported the Indian physicist Satyendra Bose (1894–1974) in the development of the distribution function for photons, now known as the Bose-Einstein distribution function (see Section 6.1). In 1924 Einstein translated

Bose's article on particle distributions into German for the *Zeitschrift für Physik* after it was rejected by an English-language journal, added his own supporting paper, and ensured publication of the work.

The Nobel prize in physics for 1921 was awarded to Albert Einstein with the citation "for his services to Theoretical Physics, and especially for his discovery of the law of the photoelectric effect." The prize rules do not allow an award unless it has been "tested by time" which meant ironically that the prize was not awarded for special and general relativity. The Nobel judges did not fully accept the theories of relativity despite measurements in 1919 by Arthur Eddington (1882–1944) showing that the light from a star was influenced by the solar gravity during a solar eclipse as predicted by the general theory of relativity [28]. Antisemitism from other German physicists may have also influenced the Nobel committee [12].

Einstein traveled extensively in the 1920s, visiting and lecturing in the United States, the United Kingdom, Japan, Palestine, and South America. He became a "celebrity" in the full modern sense of the term. Although celebrated in most of the world, in Germany the rise of the Nazi Party led to campaigns against Einstein and bizarrely his relativity theories. The scientific campaign against Einstein and relativity was orchestrated largely by earlier Nobel prize winners Philipp Lenard and Johannes Stark.[2] Adolf Hitler achieved sufficient electoral success that he was appointed chancellor of Germany in January 1933. Einstein left Germany in December 1932 to lecture as a visiting professor at the California Institute of Technology. He returned to Europe in March 1933 by ship, berthing at Antwerp and stayed first in Belgium and then the United Kingdom. Both of these countries organized armed protection of Einstein as there were fears he might be assassinated. In 1933–1935, the German government officially seized all of Einstein's assets in Germany [98].

In October 1933 Einstein moved permanently to the United States and took up a position as professor of theoretical physics at Princeton University.[3] He became an American citizen in 1940. On the eve of World War II, Einstein endorsed a letter to President Franklin D. Roosevelt alerting him to the potential German nuclear weapons program and recommending that the United States begin similar

[2] Philipp Lenard was awarded the Nobel prize for work on "cathode rays" (electron beams) in 1905 and Johannes Stark was given the award in 1919 for showing that electric fields split the spectral lines of atoms, now known as the Stark effect. The Stark effect, Stark line broadening (see Section 10.3 of Tallents [110]), and the motional Stark effect (see Section 3.2) all now carry Stark's name. In 1933, Stark become president of the Physical and Technical Institute of the German Reich and removed all Jewish people from academic posts. There have been calls for the Stark effect to be renamed and for Lenard and Stark to be stripped of their Nobel prizes [72].

[3] In April 1933 while in Belgium, Einstein received offers of professorships from the Collège de France in Paris; the University of Madrid; universities in Holland, India, and Turkey; and the new Institute for Advanced Studies in Princeton University. He accepted the Princeton offer provided he could fulfill his lecture commitments in England [98].

https://doi.org/10.1017/9781009236072.002 Published online by Cambridge University Press

research.[4] Roosevelt authorized the creation of an advisory committee that led to the setting up of the Manhattan project and the development of weapons based on nuclear fission. In nuclear fission some of the mass of, for example, the uranium-235 isotope is converted to energy.[5]

Einstein was denied a security clearance to work on the Manhattan project due to a perceived security risk because of his pacifist leanings and celebrity status. In 1952 Einstein was offered the largely ceremonial post of president of Israel, but he declined. In the 1950s up until his death, Einstein was vocal in his criticism of Senator Joseph McCarthy who zealously chaired the Senate Un-American Activities Committee investigating unevidenced communist subversion in universities, the motion-picture industry and the U.S. federal government.

After 1925, Einstein's scientific interest centered on what he termed "unified field theory." This was an attempt to unify general relativity and electromagnetism. Einstein accepted the evidence for quantum mechanics, but had difficulty with the idea of probability distributions and action at a distance (as wavefunctions have spatial extent). In 1926 Einstein famously wrote that "*He* (God) *is not playing at dice*." In 1952 Einstein wrote, "*The conviction prevails that the experimentally assured duality of nature (corpuscular and wave structure) can be realized only by ... a weakening of the concept of reality*" [98].

One of Einstein's approaches to a unified field theory survives, in a much modified form, in today's string theory [98]. String theory is a theoretical framework in which point-like particles are replaced by one-dimensional objects called strings with different vibrational states determining the mass, charge, and other properties of the particle as observed on distance scales much larger than the string. String theory has led to mathematical developments, but it seems unclear (at least at the time of writing this book) if it is useful for physics predictions due to the freedom in string theory to choose a range of parameters to fit experimental observations.

[4] In 1966 U.S. president Lyndon Johnson recognized the discoverers of nuclear fission with the Enrico Fermi prize awarded to Otto Hahn (1879–1968), Friedrich Strassmann (1902–1980), and Lise Meitner (1878–1968) who had been based in Germany. Awarded in 1945 after the end of the war in Europe, Hahn alone had received the 1944 Nobel chemistry prize for the discovery of nuclear fission. He received news of the Nobel award while held in England where secret tape recording of German scientists revealed details of the German wartime effort to develop a nuclear bomb [8]. Meitner, who like Einstein was of Jewish descent, had fled to Sweden from Germany in 1938, while Strassmann remained in Germany, but was blacklisted by the Nazi regime. Lise Meitner was the first woman to receive the Enrico Fermi prize.

[5] Uranium 235 has 92 protons and 143 neutrons and an atomic mass of 235. The uranium 235 nucleus disintegrates to less massive elements if excited, releasing a neutron which, in turn, collides with other uranium nuclei, causing them to disintegrate. The lighter elements proceed to decay to still lighter elements with more neutron production in a "chain reaction." There is a mass decrease of the product particles which is converted to kinetic energy of the particles following Einstein's equation $E = mc^2$. Developed during and after World War II, nuclear reactors [103] typically use a low concentration of uranium 235 mixed with the more common uranium isotope of atomic mass 238 which in a reactor transmutes to the readily fissionable plutonium 239. The fission rate of a reactor is controlled using neutron-absorbing media (to reduce the fission rate) and "moderators" such as "heavy water" D_2O or graphite which slow the neutrons closer to the maximum cross-section for fission (to increase the fission rate). Techniques used to make sure that fission in a nuclear weapon occurs rapidly and with complete fission of uranium or plutonium are understandably classified.

https://doi.org/10.1017/9781009236072.002 Published online by Cambridge University Press

Einstein died in 1955 (aged 76) in Princeton, New Jersey, due to a ruptured abdominal aortic aneurysm [84]. His life and his brain have been extensively analyzed and – literally for his brain – dissected. Einstein's brain was removed and dissected shortly after his death by a pathologist who preserved it in pieces which were distributed to a number of medical researchers. The fate of Einstein's brain was discovered by reporters in 1979 and efforts were made to repatriate some of the brain to Einstein's grand-daughter [12], [86].

1.1 The Propagation of Light

Electromagnetic waves comprise oscillating electric and magnetic fields which are directed transversely to the direction of propagation of the wave. The frequency of the oscillation determines the spectral name of the radiation, with a range from low-frequency radio waves, through the infrared, the visible, ultra-violet, and X-rays to gamma rays occupying the highest frequencies. Common usage often restricts "light" to mean visible light (wavelength 400–700 nm) and perhaps ultraviolet light (wavelength 50–400 nm). However, as the different spectral regions for electromagnetic waves only become important when electromagnetic radiation interacts with matter, we will use the word "light" to describe electromagnetic radiation of any frequency, particularly when there is no radiation interaction with matter.

The propagation of light at all frequencies can be modeled using a wave equation obtained by a manipulation of Maxwell's equations. In this section, we derive the wave equation and show that the speed of light in a vacuum in the International System of Units (SI) is determined by two quantities (the dielectric constant ϵ_0 and the magnetic permeability μ_0) used with SI units to get the correct relationships in Maxwell's equations. The simple relationship between the speed of light and some electric and magnetic field constants strongly supports the idea that the speed of light is a "constant of nature." A constant value of the speed of light in vacuum is a fundamental assumption of the theories of both special and general relativity.

Maxwell's equations describe the observed behavior of electrical and magnetic fields. The propagation behavior of light in vacuum is developed using Faraday's law, which describes the way a magnetic field oscillating in time produces an electric field, and Ampere's law, which describes the way an electric field oscillating in time produces a magnetic field. We have respectively:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.1}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{1.2}$$

https://doi.org/10.1017/9781009236072.002 Published online by Cambridge University Press

Magnetic **B** and electric **E** fields can be produced by temporal changes of each field (and equivalently a magnetic field can be created by a flow of current density **J**). Here ϵ_0 is the vacuum dielectric constant and μ_0 is the vacuum magnetic permeability.

Taking the curl ($\nabla\times$) of Faraday's law and substituting Ampere's law for $\nabla \times \mathbf{B}$ means that

$$\nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\left(\mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}\right). \tag{1.3}$$

In vacuum, the density of charge ρ_c and current flow **J** is zero. The third of Maxwell's equations, known as Gauss's law, describes the production of an electric field from a charge density. In vacuum, we can write

$$\nabla.\mathbf{E} = \frac{\rho_c}{\epsilon_0} = 0 \tag{1.4}$$

as the charge density ρ_c is zero in vacuum. Using the vector identity $\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla.\mathbf{E}) - \nabla^2\mathbf{E}$, setting $\mathbf{J} = 0$ and $\nabla.\mathbf{E} = 0$ produces the wave equation for light. We have

$$\nabla^2\mathbf{E} = \epsilon_0\mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{1.5}$$

The wave equation has solutions for the electric and magnetic fields which oscillate in time and space and which are known generally as electromagnetic radiation. Substituting into the wave equation (Equation 1.5) verifies that variations of electric field of the following form satisfy the wave equation at position **r**:

$$\mathbf{E} = \mathbf{E}_0 \exp[i(\mathbf{k}.\mathbf{r} - \omega t)], \tag{1.6}$$

where $\mathbf{E}_0$ is a field amplitude, **k** is a wavevector representing the rapid spatial variation of the field with position vector **r**, and ω is an angular frequency representing the rapid temporal variation of the field with time t. The angular frequency ω is the phase change of the field in radians per second and is related to the frequency ν in cycles per second (i.e. hertz) by $\omega = 2\pi\nu$.

Points of constant phase in the oscillating electric field solution of the wave equation (Equation 1.6) have $\mathbf{k}.\mathbf{r} - \omega t$ constant. If the wave propagates in the z-direction, points of constant phase on the wave have $kz - \omega t =$ constant. Differentiating $kz - \omega t =$ constant with respect to time yields $kdz/dt - \omega = 0$ and hence the phase velocity of the wave is $dz/dt = \omega/k$. The phase velocity of light in vacuum in the direction of **k** is given the symbol c, where

$$c = \frac{\omega}{k}. \tag{1.7}$$

https://doi.org/10.1017/9781009236072.002 Published online by Cambridge University Press

The wavelength λ of the light is the distance in the direction of propagation z where the spatial phase term in $\exp(i\mathbf{k}\cdot\mathbf{r}) = \exp(ikz)$ does a complete cycle and changes by 2π. The wavelength $\lambda = 2\pi/k = 2\pi c/\omega = c/\nu$.

Using Equation 1.6 as a solution of the wave equation suggests a plane wave oscillation that is uniform in directions perpendicular to the wave vector $\mathbf{k}$ with, consequently, energy flow due to the wave in the direction of $\mathbf{k}$. Real examples of electromagnetic waves such as a beam of light have a finite width with spatial variations of $\mathbf{E_0}$ perpendicular to $\mathbf{k}$. Practical electromagnetic waves can be regarded as a superposition of many plane waves, with the electric field of the plane waves canceling outside the volume of the radiation field. To a good approximation, spatial variations of $\mathbf{E_0}$ perpendicular to $\mathbf{k}$ result in energy flow in the direction of $\mathbf{k}$, provided the spatial variation scalelength is large compared to $1/k$.

As well as verifying that Equation 1.6 is a solution of the wave equation, substituting Equation 1.6 into the wave equation (Equation 1.5) produces a relationship between the amplitudes of the wavevector k and frequency ω. We have

$$\nabla^2\mathbf{E} = -k^2\mathbf{E},$$

$$\frac{\partial^2\mathbf{E}}{\partial t^2} = -\omega^2\mathbf{E}$$

so that

$$k^2 = \epsilon_0\mu_0\omega^2.$$

It follows that the vacuum speed of light c is related to the vacuum dielectric constant ϵ_0 and vacuum magnetic permeability μ_0 by

$$c = \frac{\omega}{k} = \frac{1}{\sqrt{\epsilon\mu_0}}. \tag{1.8}$$

The wave equation (Equation 1.5) can be rewritten as

$$\nabla^2\mathbf{E} = \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}. \tag{1.9}$$

The vacuum velocity of light c depends on two constants (ϵ_0 and μ_0) used to get the correct proportionality between a changing electric and magnetic field. There is a strong suggestion here that the speed of light in a vacuum is a constant of nature.

The fundamental and constant value of the speed of light in vacuum has been recognized since 1983 by the specification of an exact speed of light in vacuum in SI units. The vacuum speed of light has an exact numerical value $c = 2.99792458 \times 10^8$ ms^{-1} because the meter unit of length is defined in terms of the vacuum speed of light as the distance traveled by light in $1/2.99792458 \times 10^{-8}$ seconds in vacuum.[6]

There is a corresponding oscillating magnetic field arising from the oscillating electric field for an electromagnetic wave in vacuum. Ampere's law (Equation 1.2) is given by

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} = -i\omega \epsilon_0 \mu_0 \mathbf{E} \qquad (1.10)$$

upon differentiating $\mathbf{E}$ and assuming vacuum propagation with $\mathbf{J} = 0$. Equation 1.10 requires that the magnetic field varies as $\mathbf{B} = \mathbf{B}_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$ in phase with the electric field variation. The definition of $\nabla \times$ in Cartesian coordinates determines that $\mathbf{B}$ is perpendicular to $\mathbf{E}$ and gives $-ikB = (-i\omega\epsilon_0\mu_0)E$. Using the equivalences $1/c^2 = \epsilon_0\mu_0$ and $c = \omega/k$, we have that

$$\frac{E}{B} = c. \qquad (1.11)$$

Taking the divergence of $\mathbf{E}$ in Equation 1.6.

$$\nabla \cdot \mathbf{E} = i\mathbf{k} \cdot \mathbf{E},$$

which from Equation 1.4 is identically equal to zero. With $\mathbf{k} \cdot \mathbf{E} = 0$ we have that the electric field $\mathbf{E}$ is perpendicular to the wavevector $\mathbf{k}$. A similar argument using the fourth of Maxwell's equations ($\nabla.\mathbf{B} = 0$) shows that the magnetic field $\mathbf{B}$ is perpendicular to $\mathbf{k}$. We have already determined in the discussion for Equation 1.11 that the magnetic field $\mathbf{B}$ is perpendicular to $\mathbf{E}$. The direction of energy propagation of the wave follows the $\mathbf{k}$ vector if spatial variations of the electric and magnetic fields have scalelengths $>>1/k$.

The orientation of the electric field $\mathbf{E}$ in the plane perpendicular to $\mathbf{k}$ is the polarization direction of the electromagnetic wave. Any electromagnetic wave can be decomposed into polarization components in two orthogonal directions both perpendicular to the wavevector $\mathbf{k}$. Unpolarized light can be regarded as the independent supposition of two polarized beams with polarization directions at angle $\pi/2$ to each other. For unpolarized light there is a random phase $\mathbf{k}.\mathbf{r} - \omega t$ relationship between the polarization directions. Elliptically polarized light can be regarded as consisting of polarization components in two orthogonal directions where the phase $\mathbf{k} \cdot \mathbf{r} - \omega t$ between components in the two orthogonal directions always differs by $\pi/2$.

The classical energy density of an electric field is $(1/2)\epsilon_0 E^2$ and for a magnetic field is $(1/2)B^2/\mu_0$. These quantities for the energy density of the fields can be

[6] The second has been defined since 1967 by an atomic physics measurement – the frequency of the ground state hyperfine transition for cesium 133 atoms (see https://physics.nist.gov/cuu/Units/).

obtained by, for example, considering the charging of a capacitor (for the electric field) and the increase of current into a solenoid (for the magnetic field). For an electromagnetic wave at any instant of time, the energies in the electric and magnetic fields are equal. Using $E/B = c$ and $1/c^2 = \epsilon_0\mu_0$, we have $(1/2)\epsilon_0 E^2 = (1/2)B^2/\mu_0$. The total instantaneous energy density of an electromagnetic wave is consequently $\epsilon_0 E^2$ or equivalently B^2/μ_0. The electric field expression for the energy density of an electromagnetic wave is usually used.

The average of the sinusoidal variation of the electric and magnetic fields in time is given by the temporal average of the real and imaginary components of $\exp[i(\mathbf{k}.\mathbf{r} - \omega t)]$, that is,

$$\frac{\int_{\omega t=0}^{2\pi} \cos^2(\omega t)d(\omega t)}{\int_{\omega t=0}^{2\pi} d(\omega t)} = \frac{\int_{\omega t=0}^{2\pi} \sin^2(\omega t)d(\omega t)}{\int_{\omega t=0}^{2\pi} d(\omega t)} = 1/2.$$

The time-averaged electric and magnetic energies are equal and together have a total energy density $(1/2)\epsilon_0 E_0^2$. Imagine a unit cross-section volume of unit area and length c along the direction $\mathbf{k}$ of propagation of the beam. In unit time, the energy from the volume of c will pass through the unit area. This energy passing per unit time per unit area normal to the direction of the beam is known as the intensity or irradiance I and is given by

$$I = \frac{1}{2}\epsilon_0 c E_0^2. \tag{1.12}$$

The proportionality of the intensity of light to the square of the electric field of the light enables the intensity of light to be significantly enhanced when the electric fields from different sources of light add coherently. Coherent addition requires the sources of light to have the same phase relationship, so that addition of the electric fields, rather than subtraction, occurs. With coherent addition of light, adding n sources of coherent light together causes an instantaneous n^2 increase in light intensity when all the sources are in phase.

This section has illustrated the wave-like properties of light. Light also has a particle-like nature.[7] Light can be shown to comprise discrete quanta of radiation energy known as photons. The particle nature of light is experimentally apparent

[7] Both light and matter exhibit wave-like and particle-like properties depending on the measurement being undertaken. This is the essence of the "complementarity" principle proposed by Neils Bohr (1885–1962). With the complementarity principle, wave or particle models are not just useful approximations for a physical situation, but intrinsic models of light and matter with one description or the other applicable depending on the measurement. With a particle treatment, position can be regarded as being known accurately, with a larger uncertainty on the velocity or momentum. With a wave treatment, the velocity/momentum is known accurately, but with a large uncertainty on the position. The uncertainty principle first developed by Werner Heisenberg (1901–1976) showed that measurements of complementary parameters such as position/momentum, frequency/time could not both be made, even in principle, with infinite precision. Many assume that the complementarity principle is equivalent to the uncertainty principle, but the complementary principle has a deeper meaning (see [31]).

in the photo-electric effect as discussed originally by Einstein [34]. In the photo-electric effect, light irradiated material only emits an electron when the energy of a photon of the light exceeds the binding energy of the material. Light is absorbed by matter in units of energy and if the unit of light energy is less than the quantum differences for the energies in the material irradiated, the light is not absorbed.

The term "photon" was introduced in a paper published in 1926 [66]. The quantization of light in photons of energy $h\nu$, where $h = 6.626 \times 10^{-34}$ Js is Planck's constant and ν is the frequency of the light, is discussed later in Section 2.9, where the wave-like property of matter and radiation is introduced, and in Section 6.4, where the equilibrium radiation field (black-body radiation) is considered. Calculations of the trajectories of light rays in vacuum as seen by observers with different velocities can be made using the treatments developed for particles by setting the particle velocity to the vacuum speed of light (see Section 2.5) and for propagation in a medium by setting the particle velocity to the phase velocity of light in the medium (see Chapter 9).

1.2 Time Dilation and Length Contraction

In this section, a "thought experiment" involving light propagating in a moving frame of reference is used to introduce two consequences of special relativity: (i) that time in a moving frame of reference runs slower as measured in the stationary or rest frame, and (ii) that lengths in a moving frame of reference are shorter in the direction of motion when measured in the stationary or rest frame. We will quantify more precisely the transformations between time and distance for different inertial frames of reference in Chapter 2.

Maxwell's equations lead to the wave equation which has solutions indicating that the vacuum speed of light c is constant (see Section 1.1). The theories of relativity take as an assumption that the vacuum speed of light c is constant in all frames of reference. The theory of special relativity applies for frames of reference which are "inertial." An inertial frame of reference is one where free particles move in straight lines at constant speed. An inertial frame of reference with free particles moving at constant velocity implies that there are no forces, known as "fictitious forces," which deviate the path of a body due to the acceleration of the frame of reference. Examples of fictitious forces include the centrifugal and Coriolis forces associated with a rotating body.

An inertial frame of reference is often specified as one which is not undergoing acceleration. However, it is possible to treat acceleration using the equations of special relativity as long as the acceleration is considered for an infinitesimally short period of time (see, e.g., Section 3.3).

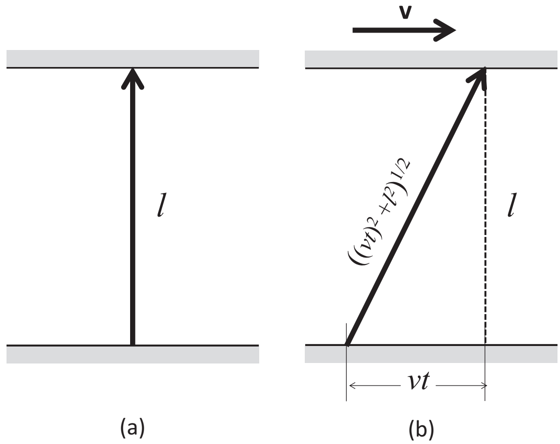

Figure 1.1 A thought experiment examining the time for light to propagate between two parallel mirrors in (a) a frame of reference moving with the mirrors, and (b) a stationary frame of reference. The mirrors are moving parallel to the mirror surfaces at a velocity **v** relative to the stationary observer. The distances for light propagation as seen by (a) a moving observer in a frame of reference S' and (b) a stationary observer in a frame of reference S are annotated.

We consider here a brief argument showing that to an observer in a stationary frame of reference, time in a moving inertial frame of reference appears to run more slowly. This is known as time dilation. Distances measured by a stationary observer of lengths in the moving inertial frame of reference in the direction of the velocity are also shown to be shortened.

Consider a frame of reference with a pulse of light reflecting between two parallel mirrors separated by a distance l which is moving at constant velocity v in an orthogonal direction to the light beam (see Figure 1.1). We label this moving frame of reference as S' and use primed symbols to designate time and distance in the frame of reference. The light propagation between the two mirrors is also observed by a nominally stationary observer in a frame of reference which we label as S with distances and time designated with unprimed symbols (see Figure 1.2). In the S' frame of reference moving at velocity v, the light takes a time $t' = l/c$ to pass from one mirror to the other. To the stationary observer, the light travels a distance $\sqrt{l^2 + v^2t^2}$ in a time given by $t = \sqrt{l^2 + v^2t^2}/c$. The important issue now is that to the observers in both the stationary S and moving S' frames of reference, the speed of light c is the same. The only way to resolve the different distances that the light travels according to the two observers is for the measurement of time to be different in the two frames of reference. Substituting $l = ct'$ gives

$$t = \frac{\sqrt{c^2t'^2 + v^2t^2}}{c}.$$

https://doi.org/10.1017/9781009236072.002 Published online by Cambridge University Press

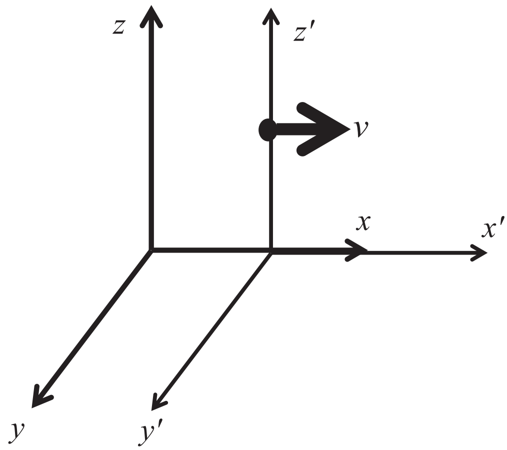

Figure 1.2 Two inertial frames of reference. The frame of reference S' with primed coordinates (x', y', z') has a velocity $\mathbf{v}$ in the x-direction relative to the frame of reference S with coordinates (x, y, z).

Squaring this expression and re-arranging gives

$$t^2 = t'^2 + \frac{v^2}{c^2}t^2,$$

which simplifies to

$$t = \frac{t'}{\sqrt{1 - v^2/c^2}}. \tag{1.13}$$

The observer in the stationary frame of reference S observes a longer time for the light to reflect from one mirror to the other. The numerator in Equation 1.13 is less than one, so time t in the stationary frame S is greater than time t' in the moving frame S'. We can imagine that the reflection of the light pulse acts as a clock. Time in the moving frame of reference as observed from the stationary frame of reference is running more slowly. A "moving" and a "stationary" observer are equally valid observers, so to a nominally moving observer, the stationary clocks also appear to run more slowly. In summary, the time measured in the two frames of reference is different because the speed of light is the same in the two frames of reference – but the light needs to travel different distances in the two frames of reference.

The slowing down of time to a stationary observer of the time in a frame of reference moving with velocity approaching the speed of light is a real effect. For example, muons created by cosmic rays[8] interacting with the Earth's upper atmosphere

[8] Cosmic rays striking the Earth's atmosphere are particles comprising 90% protons (hydrogen nuclei), 9% alpha particles (helium nuclei), and 1% electrons with a small number of higher atomic number nuclei and antiparticles. With particle numbers peaking at energies ≈0.3 GeV, cosmic rays interact with air molecules to produce a cascade of particles traveling close to the direction of the initial cosmic ray. Muons make up more than half the cosmic radiation at sea level with the remainder comprising mainly electrons, positrons, and

https://doi.org/10.1017/9781009236072.002 Published online by Cambridge University Press

decay with a half-life of 1.56 μs and so would decay and not be observed on the Earth's surface if time in the muon frame of reference was the same as the passage of time on the Earth's surface [30]. Traveling close to the speed of light, high-energy muons would only travel close to 470 m in the time of their half-life from their creation, so almost all would have decayed before reaching the ground as they pass through 15 km or so of atmosphere. They are observed on the surface of the Earth because as observed on the ground, the muon time is passing much more slowly.

To quantify, let the distance x that a muon travels in the Earth's frame of reference be set so $x = vt$. The length of atmosphere moving past the muon in its rest frame is $x' = vt'$. We have

$$x' = vt' = vt\sqrt{1 - v^2/c^2} = x\sqrt{1 - v^2/c^2}.$$

Lengths of atmosphere x' as measured by the muon are reduced by a factor $\sqrt{1 - v^2/c^2}$ from the lengths x as measured in the Earth's frame of reference. For $v > 0.95c$, many muons created by cosmic rays reach the ground (see Exercise 1.8).

Distances in a frame of reference moving at velocity v as measured by stationary observers are reduced or contracted in the direction of the velocity. The muon in our example sees the Earth's atmosphere contracted by a factor $\sqrt{1 - v^2/c^2}$ as it moves past it at velocity v. This effect applies to any length measurement made in a stationary frame in the direction of velocity of a moving object, but only, of course, becomes significant when the velocity is a significant fraction of the speed of light.

As all observers in inertial frames of reference are equivalent, we can think of, for example, the length contraction of the Earth's atmosphere in the frame of reference of a muon even though the nominally stationary Earth may be regarded as a "better" frame of reference. To avoid confusion, the "proper time" of an event is used to mean the time measured by an observer in the frame of reference of the event. The "proper length" is the length of an object or the distance between two points measured by an observer in the same frame of reference as the object.

An observer in a stationary frame of reference measures times in a moving frame slower than the proper time and distances in the direction of the relative velocity of the two frames of reference shorter than the proper length. Distances measured orthogonal to the direction of the velocity are measured to have the same value in the two frames of reference. This change in time and measurement of distance is illustrated as a cartoon in Figure 1.3.

photons. Cosmic rays originate from solar flares and coronal mass ejections and from supernovae explosions in our galaxy and beyond. The sources and acceleration mechanisms for cosmic rays, particularly those with extreme energy, are the subjects of much research (see Section 5.2.3). Galactic cosmic ray energies up to 10^{15} eV and extragalactic up to 3×10^{20} eV have been observed [9], [22]. Cosmic rays with energies $>10^{20}$ eV are scattered by background intergalactic photons (the cosmic microwave background) and so are rarely detected (see Exercise 6.15).

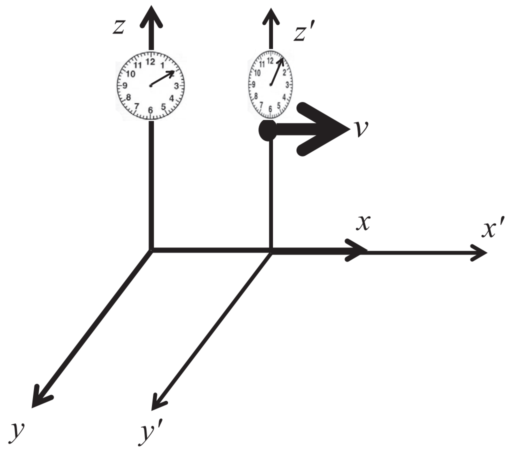

Figure 1.3 The two inertial frames of reference illustrated in Figure 1.2 with superimposed clocks as seen from the stationary frame S as the velocity $\mathbf{v}$ approaches the speed of light c. Time appears slower, and distances in the direction of the velocity appear shorter in the moving frame as observed from the stationary frame.

We quantify more precisely the transformations between time and distance for different inertial frames of reference in Chapter 2. We show that it is possible to have a transformation between time and distance (the Lorentz transformation) which ensures (i) that the speed of light is the same in different frames of reference, and (ii) that with a simple change of sign of the relative velocity of the observers, a "moving" observer sees dilated time and contracted lengths in the stationary frame, and the stationary observer measures dilated time and contracted lengths in the moving frame. The shortening of lengths in the moving frame as measured in the stationary frame of reference is often referred to as Lorentz contraction.

1.3 Light Emission from Charge Acceleration

Emission and absorption of electromagnetic radiation occurs when charges are accelerated. A charged body of any mass radiates when accelerated, but the most common emitters and absorbers are electrons as their low mass results in larger acceleration. The acceleration is apparent for unbound particles such as free electrons in a plasma, but is also present when dealing with the time-dependent quantum mechanics of electrons bound in atoms and ions interacting with electromagnetic waves. In this section, we follow a relatively simple treatment by Purcell [94] to determine the radiation produced by an accelerating charge.

The production of electromagnetic radiation when a charge such as an electron accelerates arises due to a dislocation of the radially symmetric electric field from the charge. Immediately before an acceleration impulse, the electric field radiates

https://doi.org/10.1017/9781009236072.002 Published online by Cambridge University Press

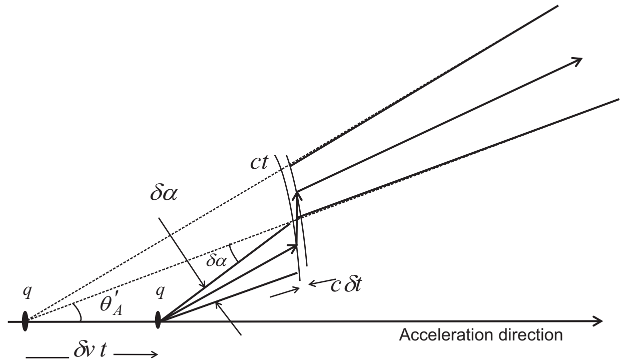

Figure 1.4 The electric field disruption at time t when a charge is accelerated to velocity δv over a time δt at $t = 0$ for an angle θ'_A to the acceleration direction in the frame of reference of the charge. All electric field lines from the new charge position within the angle $\delta\alpha$ need to move transversely at distance ct to link to the electric field lines established when the charge was in its initial position. The transverse electric field forms a pulse of electromagnetic radiation propagating away from the accelerated charge.

from the initial position in a reference frame moving with the initial charge velocity. Immediately after the acceleration, the electric field propagates in a radial direction from the new position in the reference frame. As the electric field lines from the charge are continuous, there is an electric field transverse jump propagating at the speed of light. For the purposes of evaluating an expression for the transverse electric field, we can assume that the velocity jump occurs on a very short timescale so that the frame of reference remains inertial.

We now consider more quantitatively the production of electromagnetic radiation by thinking about the electric field around a charged particle if it suffers an impulse of acceleration to a velocity δv in a time δt at some initial time $t = 0$. At a later time $t > 0$, the electric field lines associated with the initial position extend radially out from the initial position at distances greater than ct, which is the distance that light and the information about the disturbance in the electric field from the charge can travel in the time t. However, at the distance ct, there is a dislocation of the electric field associated with the charge having accelerated to a new position and the consequent need for a transverse movement of the electric field radially emanating from the new positions to join up with the electric field lines which started before time $t = 0$. The concept is schematically illustrated in Figure 1.4.

The acceleration of a charge q by a velocity δv in a time δt at time $t = 0$ causes a dislocation in the electric field distribution from the charge at a later time t at a distance ct from the charge and produces a transverse electric field. All electric field lines emanating from the later position of the charge move transversely within an annulus of thickness $c\delta t$ to match the field lines emitted before the charge accel-

https://doi.org/10.1017/9781009236072.002 Published online by Cambridge University Press

eration. Provided that the velocity increase δv is much less than the speed of light c, the electric field lines from the charge in the reference frame of the charge are directed radially from the position of the charge, except at the distance ct. We can calculate the enhancement of the electric field transversely over the distance ct to $c(t+\delta t)$ by considering a small angular range $\delta\alpha$ for a particular angle θ'_A to the acceleration direction of the charge. Using Figure 1.4, by simple geometry

$$\frac{\sin\delta\alpha}{\delta v\, t} = \frac{\sin(\pi - \theta'_A - \delta\alpha)}{c\, t}.$$

If $\delta v << c$, the angle $\delta\alpha$ is small and we have that

$$\delta\alpha = (\delta v/c)\sin\theta'_A.$$

The enhancement of the electric field in moving transversely through a thickness $c\delta t$ rather than being spread emanating radially from the charge over an angle $r\,\delta\alpha$ is given by

$$\frac{r\,\delta\alpha}{c\delta t} = \frac{r\delta v}{c^2\delta t}\sin\theta'_A.$$

The radial electric field from the charge q at distance r is given by

$$E_r = \frac{q}{4\pi\epsilon_0 r^2},$$

so the transverse field at distance r is given by

$$E_t = \frac{q\ \sin\theta'_A}{4\pi\epsilon_0 c^2 r}\frac{\delta v}{\delta t}.$$

The quantity $\delta v/\delta t$ is the acceleration of the charge which we can write as dv'/dt' (with the primes indicating a measurement of acceleration in the frame of the charge). The transverse electric field can then be written as

$$E_t = \frac{q}{4\pi\epsilon_0 c^2 r}\left(\frac{dv'}{dt'}\right)\sin\theta'_A. \tag{1.14}$$

The acceleration direction of a charge oscillates or rotates in many situations. For example, charged particles orbit around magnetic fields in a plasma in Larmor or gyro-orbits. An electron approaching an ion undergoes a hyperbolic trajectory with a change of direction and hence change of acceleration direction due to the electron-ion Coulomb attraction. In these cases the electric field in a particular direction will oscillate sinusoidally (possibly at many frequencies) and consequently create sinusoidally oscillating magnetic fields (from Ampere's law). Sinusoidally oscillating electric and magnetic fields are simply electromagnetic radiation. The

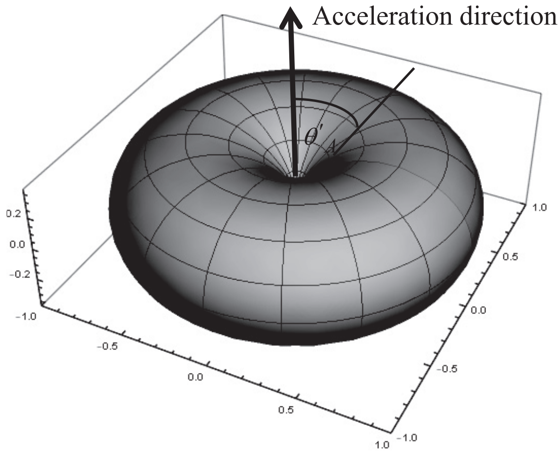

Figure 1.5 Three-dimensional profile of the power $dP'/d\Omega'$ per unit solid angle radiated by an accelerating charge in the frame of reference of the charge. The radiated power varies as $\sin^2\theta'_A$, where θ'_A is the angle to the acceleration direction.

power $dP'/d\Omega'$ radiated per unit solid angle is related to the intensity $I = \epsilon_0 c E_t^2$ at a distance r by $dP'/d\Omega' = Ir^2$, so we have that

$$\frac{dP'}{d\Omega'} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{4\pi c}\left(\frac{dv'}{dt'}/c\right)^2 \sin^2\theta'_A. \tag{1.15}$$

This expression for the radiation emitted by an accelerating charge is known as the Larmor formula after the Northern Irish physicist Joseph Larmor (1857–1942). Larmor was the first to determine the power radiated by an accelerated charge (Equation 1.15). He also developed the theoretical understanding of the splitting of spectral lines due to a magnetic field (now known as the Zeeman effect). Prior to Einstein's special relativity, Larmor worked on transformations similar to the relativistic Lorentz transformation (to be discussed in Section 2.2).

The radiation produced by an accelerating charge is polarized. The **k** vector of the radiation is directed radially away from the charge with the electric field directed transversely (in the plane formed by the **k** vector and the acceleration vector). The polarization of light is defined by the direction of the electric field so an accelerated charge produces light with a polarization in the plane formed by the light **k** vector and the charge acceleration vector. The power radiated by an accelerated charge q is proportional to the square of the acceleration and varies with angle θ'_A to the direction of acceleration (see Figure 1.5).

The power output (Equation 1.15) is valid in the frame of reference of the charged particle. If the charged particle has a velocity v approaching the speed of light c, we show later that there is a "beaming effect" in the stationary frame of reference with emission concentrated in the stationary frame in the direction of the charge velocity over a range of angles $1/\gamma$ to the velocity direction, where $\gamma = 1/\sqrt{1 - v^2/c^2}$ (see Section 3.4). Equation 1.15 is sometimes known as the "dipole" distribution of radiation as an oscillating electric dipole radiates with an identical angular variation ($\propto \sin^2 \theta'_A$).

Integrating Equation 1.15 for nonrelativistic emission over all solid angles gives the total radiated power

$$\begin{aligned} P &= \int \frac{dP'}{d\Omega'} d\Omega' \\ &= \int_0^{\pi} \frac{dP'}{d\Omega'} 2\pi \, \sin\theta'_A \, d\theta'_A \\ &= \frac{q^2}{4\pi\epsilon_0} \frac{1}{4\pi c} \left(\frac{dv'}{dt'} / c \right)^2 \int_0^{\pi} 2\pi \, \sin^3\theta'_A \, d\theta'_A \\ &= \frac{q^2}{4\pi\epsilon_0} \frac{2}{3c} \left(\frac{dv'}{dt'} / c \right)^2 . \end{aligned} \tag{1.16}$$

The total power radiated in the frame of reference of the charged particle given by Equation 1.15 is the same as the total power detected in the stationary rest frame, so Equation 1.16 also applies to the total power emitted for charged particles moving at relativistic velocities provided the acceleration dv'/dt' is measured in the frame of the charged particle. The expression for the total power emitted by an accelerating charged particle after integration over 4π steradian when the acceleration is measured in the rest frame of reference is evaluated in Chapter 3 (see, e.g., Equation 3.44).

Exercises

1.1 The foot is a unit of length in the British Imperial and U.S. customary system of units defined as 0.3048 m exactly. How long in nanoseconds does it take for light to travel a distance of one foot? [1.016 ns]

1.2 How far does a pulse of light travel in one year? [9.46×10^{15} m]

1.3 Consider visible light of wavelength 500 nm (a cyan or light blue color). Determine the frequency and photon energy of light of wavelength 500 nm. [6×10^{14} Hz, 2.48 eV]

1.4 A spectral line at wavelength 500 nm has a spectral width of 1 nm. Determine the spectral width in frequency and photon energy of this spectral line. [1.2×10^{12} Hz, 5×10^{-3} eV]

1.5 Evaluate Equation 1.12 to show that the electric field E_0 and magnetic field B_0 associated with light of irradiance I measured in Wm^{-2} are given by

$$E_0 = 27.4\ I^{1/2} \quad \mathrm{Vm}^{-1}$$
$$B_0 = 9.13 \times 10^{-8}\ I^{1/2} \quad \text{tesla.}$$

1.6 The clocks illustrated in Figure 1.3 show that time in the moving frame S' as measured in the stationary frame S is running at half the rate of time measured in the frame S. Calculate the relative velocity of frame S' to frame S. [$0.86c = 2.6 \times 10^8$ ms^{-1}]

1.7 The average lifetime of a π meson in the frame of reference of the meson is 26.0 ns. If a π meson has a velocity of $0.95c$ in a laboratory experiment, determine the lifetime of the π meson in the laboratory frame of reference and the average distance the π meson travels in the laboratory before decaying. [83.3 ns, 24 m]

1.8 Muons decay with a half-life of 1.56 μs in their frame of reference. They are created in the Earth's atmosphere when cosmic rays interact with air. For muons created in the upper atmosphere with a velocity of $0.98c$, calculate the observed half-life of the muons and determine the percentage of an initial population of muons which do not decay after traveling 10 km through the atmosphere. [6.8 ns, 4.9%]

1.9 Laser cavities consist of two parallel mirrors as illustrated in Figure 1.1. In the frame of reference of the mirrors, the laser output consists of "modes" of frequencies separated by $\Delta\nu' = 2l/c$, where l is the distance between the mirrors. If the laser mirrors move at a velocity **v** parallel to the surface of the mirrors, show that the mode spacing $\Delta\nu$ according to a stationary observer is given by

$$\Delta\nu = \frac{2}{c}(l^2 + (vt)^2)^{1/2}.$$

Hence, show that

$$\Delta\nu = \frac{\Delta\nu'}{(1 - v^2/c^2)^{1/2}}.$$

[The frequency spacing is Doppler shifted by an amount that is consistent with Equation 2.47 for light moving at angle $\pi/2$ to the velocity direction (as derived in Section 2.6).]

1.10 Evaluate Equation 1.16 to show that the power radiated by an accelerated electron is given by

$$P = 5.7 \times 10^{-30} \left(\frac{dv}{dt}\right)^2 \quad \text{watts,}$$

where the electron acceleration dv/dt is measured in units of m s^{-2}.

1.11 An isolated electron is accelerated by an electric field of 1 Vm^{-1} for a short time. Determine the power radiated by the electron. [1.76×10^{-7} watts]

1.12 Ignoring radiation losses for an isolated electron accelerated in a uniform electric field of 1 Vm^{-1}, determine the length of time needed to accelerate an isolated electron to a speed $c/100$, where c is the speed of light. [17 μs]

1.13 Considering the numerical answers for Exercises 1.11 and 1.12, discuss if it is valid to neglect radiation energy losses for an isolated electron accelerated to a speed of $c/100$ in an electric field. [The effect of radiated energy loss on the motion of charged particles is discussed in Section 7.12.]

1.14 A charge of one coulomb is accelerated at a rate c ms^{-2} for a short time. Determine the total instantaneous power of the electromagnetic radiation created by the acceleration. [20 watts]

https://doi.org/10.1017/9781009236072.002 Published online by Cambridge University Press