



RESEARCH ARTICLE

A question of Frohardt on 2-groups, skew translation quadrangles of even order and cyclic STGQs

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Abstract

We solve a fundamental question posed in Frohardt’s 1988 paper [6] on finite 2-groups with Kantor families, by showing that finite groups K with a Kantor family $(\mathcal{F}, \mathcal{F}^*)$ having distinct members $A, B \in \mathcal{F}$ such that $A^* \cap B^*$ is a central subgroup of K and the quotient $K/(A^* \cap B^*)$ is abelian cannot exist if the center of K has exponent 4 and the members of \mathcal{F} are elementary abelian. Then we give a short geometrical proof of a recent result of Ott which says that finite skew translation quadrangles of even order (t, t) (where t is not a square) are always translation generalized quadrangles. This is a consequence of a complete classification of finite cyclic skew translation quadrangles of order (t, t) that we carry out in the present paper.

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1. Introduction

In a seminal paper of 1988 [6], Frohardt solved a big part of a famous question of Kantor which asks whether a finite group admitting a Kantor family is a p -group. Such groups are very important since they produce generalized quadrangles (GQs), the central Tits buildings of rank 2. In fact, groups with Kantor families yield the most important tool to construct GQs, and a large literature exists on these objects – see, for instance, [12, 14, 16, 3, 22]. GQs constructed from Kantor families – by a procedure which is explained in Section 2.4 – carry an interesting automorphism group which locally fixes some point line-wise, and are called “elation generalized quadrangles” (EGQs). The theory of EGQs was initially devised through the 1970s and 1980s (see [12, chapters 8–10] and [16]) in order to set up a framework to study automorphisms, and especially Moufang conditions, of GQs from a local point of view, much like translation planes were studied in the theory of projective planes. Together with a deep combinatorial theory, this enabled an almost entirely synthetic-geometric proof of the celebrated results of Fong and Seitz [4, 5] on split BN-pairs of rank 2, in the case of type B_2 BN-pairs – see [12, chapters 8 and 9], [14, chapters 10 and 11] and also the related recent note [18].

In [6], Frohardt also studied groups K with Kantor families $(\mathcal{F}, \mathcal{F}^*)$ which have distinct members $A, B \in \mathcal{F}$ such that $A^* \cap B^* \leq Z(K)$ (where $Z(K)$ denotes the center of K and $K/(A^* \cap B^*)$ is abelian. Such Kantor families are fundamental since almost all examples of Kantor families have this natural property – see, for example, the monograph [16] or Payne’s famous paper [10]. We also refer to Section 2.5 for more background information on this matter. Only a very few classes of finite groups are known which have Kantor families; the most used ones are elementary abelian groups, and Heisenberg groups defined over finite fields in odd characteristic, of dimension 3 or 5 (see, e.g., the monograph [16]). They all satisfy Frohardt’s condition.

Theorem 1.1 (D. Frohardt [6], 1988). *Let K be a finite group with a Kantor family $(\mathcal{F}, \mathcal{F}^*)$ having distinct members $A, B \in \mathcal{F}$ such that $A^* \cap B^* \leq Z(K)$ and $K/(A^* \cap B^*)$ is abelian. Then one of the following three cases occurs:*

- (1) $Z(K)$ and the elements of \mathcal{F} are elementary abelian.
- (2) $Z(K)$ is an elementary abelian 2-group and the elements of \mathcal{F} have exponent 4.
- (3) $Z(K)$ has exponent 4, and the elements of \mathcal{F} are elementary abelian 2-groups.

In *loc. cit.*, Frohardt asked whether cases (2) and (3) actually occur.

In 2006, Rostermundt [13] and, independently Thas [15], constructed an infinite class of examples (related to Hermitian varieties in projective dimension 3), which fitted in class (2), so class (3) became the final challenge. Frohardt’s problem resisted many years of tries (see, for instance, the work of Hachenberger [7]), and the problem also occurs in the large literature of translation nets. Besides, the existence of elements in class (3) presented one of the main obstacles in classifying so-called “skew translation generalized quadrangles” (STGQs) – see [19, 20] and also [21, 1]. We also refer to Section 2.5 for some milestones in that classification theory.

Payne’s first paper on STGQs [9] from 1975 marked the birth of the theory of STGQs. Pretty soon it became clear though that the “parity” of lines and points played a crucial role: In its long history, no examples were found of STGQs of even order (t, t) for which the elation group was *not* abelian, while when t is odd it is easy to prove that the elation group cannot be abelian at all. Since one expects that such STGQs would have to meet the requirements of Frohardt’s theorem and since for STGQs of order (t, t) the elements of \mathcal{F} are always elementary abelian (see [20] for those details), such examples would conjecturally live in class (3) of Theorem 1.1. And indeed they do, if they would exist. The best result until 2020 was the same result when $(t, t) = (8, 8)$ [1], with a very long and computer-aided technical proof. In the meanwhile, Ott has published a long paper [8] in which he showed that finite skew translation quadrangles of order (t, t) with t not a square are translation GQs; that is, the associated elation group is an elementary abelian 2-group, and hence the quadrangle has a projective representation in some projective space. This essentially means that there is no “proper” theory of STGQs in this case.

The present paper contains an alternative approach to the one of Ott, although we use one lemma of his paper. Our proof, which is geometric in nature, is very short in comparison with Ott’s proof, and we first completely classify STGQs of order (t, t) which have a cyclic group of symmetries with center the elation point. We then use the lemma of Ott to reduce the general problem to the case of “cyclic STGQs.” In this paper, we also finally resolve Frohardt’s question by showing that class (3) of Theorem 1.1 is empty, and this will be done in Sections 4 and 5. The proof consists of a blend of synthetic geometrical and group theoretical techniques, which allows us to sufficiently understand the behavior of involutions to end the whole thing.

Remark 1.2. On the combinatorial level, there is also an other interesting aspect to this result. If every line incident with some point in a finite GQ of even order (t, t) is *regular* (see Section 2.2 for a formal definition of this very important notion), then it can be shown that the point itself is also regular ([12, section 1.5.2]). Ott’s result provides a group-theoretical converse if t is not a square: If a finite EGQ of even order (t, t) with t not a square, has a regular elation point (this is another way of defining STGQs in this specific case), then it is a translation quadrangle, so all the lines incident with x are regular.

Organization

In Sections 2 and 3, we introduce the basic notions that we will need. In Section 4, we make a number of synthetic observations; we essentially show that in case (3) of Frohardt’s theorem, it is sufficient to work with Kantor families of type (t, t) . This is done in the geometrical language of GQs. Then, in Section 5, we show that Frohardt’s class (3) is empty. And finally, in the last section, we classify cyclic STGQs (of finite order (t, t)). As a consequence, STGQs of even order (t, t) with t not a square are always translation quadrangles.

2. Synopsis of definitions

Let Γ be a thick *generalized quadrangle* (GQ). It is a rank 2 geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ (where we call the elements of \mathcal{P} “points” and those of \mathcal{L} “lines”) such that the following axioms are satisfied:

- (a) there are no ordinary digons and triangles contained in Γ ;
- (b) each two elements of $\mathcal{P} \cup \mathcal{L}$ are contained in an ordinary quadrangle;
- (c) there exists an ordinary pentagon.

It can be shown that there are constants s and t such that each point is incident with $t + 1$ lines and each line is incident with $s + 1$ points. We say that (s, t) is the *order* of Γ . Note that an ordinary quadrangle is just a (necessarily thin) GQ of order $(1, 1)$ – we call such a subgeometry also “apartment” (of Γ).

2.1. Subquadrangles

A *subquadrangle* (subGQ) $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathbf{I}')$ of a GQ $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$, is a GQ for which $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}' \subseteq \mathcal{L}$, and \mathbf{I}' is the incidence relation which is induced by \mathbf{I} on $(\mathcal{P}' \times \mathcal{L}') \cup (\mathcal{L}' \times \mathcal{P}')$.

2.2. Regularity

Let Γ be a thick GQ of order (s, t) . If X and Y are (not necessarily different) points, or lines, $X \sim Y$ denotes the fact that there is a line incident with both, or a point incident with both. Then $X^\perp := \{Y \mid Y \sim X\}$, and if S is a point set, or a line set, $S^\perp := \bigcap_{s \in S} s^\perp$ and $S^{\perp\perp} := (S^\perp)^\perp$.

A particularly important example is the case where $S = \{X, Y\}$ is a set of distinct noncollinear points (or nonconcurrent lines, but this is merely the dual situation, which we leave to the reader); then each line incident with X is incident with precisely one point of $\{X, Y\}^\perp$ (so if Γ is finite, $|\{X, Y\}^\perp| = t + 1$). The set $\{X, Y\}^{\perp\perp}$ consists of all points which are collinear with every point of $\{X, Y\}^\perp$, so

$$\{X, Y\} \subseteq \{X, Y\}^{\perp\perp}. \tag{1}$$

Let Z be any point of $\{X, Y\}^\perp$; if each line incident with Z is incident with exactly one point of $\{X, Y\}^{\perp\perp}$, then this property is independent of the choice of Z , and we say that $\{X, Y\}$ is *regular*. In the finite case, we could equivalently have asked that $|\{X, Y\}^{\perp\perp}| = t + 1$.

We call a point/line X *regular* if $\{X, Y\}$ is regular for all $Y \not\sim X$.

2.3. Symmetry

Isomorphisms and automorphisms of GQs are defined in the usual manner. See chapter 1 of [12]. The automorphism group of a GQ Γ will be denoted by $\text{Aut}(\Gamma)$.

Let X be a point or a line in a thick GQ Γ . A *symmetry* with *center* X (in the case of a point) or *axis* X (in the case of a line) is an element of $\text{Aut}(\Gamma)$ that fixes each element of X^\perp . We say that X is a *center of symmetry* (point case) or an *axis of symmetry* (line case) if there exist Y and Z in X^\perp such that $Y \not\sim Z$, for which the group of all symmetries $\mathcal{S}(X)$ with center/axis X acts transitively on $\{Y, Z\}^\perp \setminus \{X\}$. This definition does not depend on the choice of (Y, Z) , and one easily shows that “transitive” implies

“sharply transitive.” In the finite case, we could also have asked that

$$|S(X)| = t \tag{2}$$

if the order of Γ be (s, t) .

Note that a center/axis of symmetry is necessarily regular.

We end this section with a very useful result of Benson.

Theorem 2.1 (C. T. Benson [2]). *Let Γ be a thick finite GQ of order (s, t) , and let γ be an automorphism of Γ . If N is the number of fixed points of γ , and M is the number of points x for which $x^\gamma \neq x \sim x^\gamma$, then*

$$(t + 1)N + M \equiv st + 1 \pmod{(s + t)}. \tag{3}$$

2.4. Kantor families

In this section, we will recall the notion of Kantor family. Groups with Kantor families produce GQs, and vice versa, a certain type of GQ gives rise to Kantor families. We will only define *finite* Kantor families, as that is the case we will need in due course.

So suppose K is a finite group of order u^2v for positive integers u and v , both at least 2. A *Kantor family* (of type (u, v)) in K is a pair $(\mathcal{F}, \mathcal{F}^*)$ of sets of subgroups of K for which the properties below are satisfied:

- (a) $|\mathcal{F}| = |\mathcal{F}^*| = v + 1$, and there is a bijection $*$: $\mathcal{F} \mapsto \mathcal{F}^*$ which maps each $A \in \mathcal{F}$ to $A^* \in \mathcal{F}^*$ such that $A \leq A^*$;
- (b) for each $A \in \mathcal{F}$, $|A| = u$ and $|A^*| = uv$;
- (c) if A, B, C are different elements in \mathcal{F} , then $AB \cap C = \{\text{id}\}$;
- (d) if A and B are different elements in \mathcal{F} , then $A \cap B^* = \{\text{id}\}$.

From the data $(K, (\mathcal{F}, \mathcal{F}^*))$, one constructs a thick GQ $\Gamma = \Gamma(K, (\mathcal{F}, \mathcal{F}^*))$ of order (u, v) as follows. Its points are a symbol (∞) , left cosets of type gA^* ($A \in \mathcal{F}$ and $g \in K$) and the elements of K . Its lines are symbols $[A]$ ($A \in \mathcal{F}$) and left cosets of type gA ($A \in \mathcal{F}$ and $g \in K$). The point (∞) is incident with all lines of the first type and no other lines. A line $[A]$ is also incident with all points gA^* . All other incidences are (reversed) containment. Elements of the group K act naturally as automorphisms of Γ , by left multiplication on the cosets, while fixing the symbolic point and symbolic lines. Then K fixes all the lines incident with (∞) and acts sharply transitively on the points of Γ which are not collinear with (∞) .

Conversely, let Ω be a thick GQ of order (u, v) , with an automorphism group L which fixes some point x linewise while acting sharply transitively on the points not collinear with x . We call such GQs *elation generalized quadrangles* (EGQs) with *elation point* x and *elation group* L . Then $|L| = u^2v$. Now, take one arbitrary point y not collinear with x (due to the transitivity of L on these points, this choice is not important). For each line UIy , let \tilde{u} be the point which is incident with U and collinear with x ; let $L_U^* := L_{\tilde{u}}$. Then with $\mathcal{F} := \{L_U \mid UIy\}$ and $\mathcal{F}^* := \{L_U^* \mid UIy\}$, $(\mathcal{F}, \mathcal{F}^*)$ is a Kantor family of type (u, v) in L . Also, we have a natural isomorphism

$$\Gamma(L, (\mathcal{F}, \mathcal{F}^*)) \mapsto \Omega \tag{4}$$

which maps L to itself and (∞) to x .

2.5. Skew translation quadrangles

If (Γ, K) is an EGQ with elation point x and x is a center of symmetry such that the corresponding symmetry group \mathcal{S} is a subgroup of K , then we call (Γ, K) a *skew translation generalized quadrangle* (STGQ). In terms of Kantor families (using the notation of the previous subsection), a Kantor family $(\mathcal{F}, \mathcal{F}^*)$ gives rise to an STGQ if and only if there is a normal subgroup C of K such that for each $A \in \mathcal{F}$, we have $A^* = AC$ (see [9] or [12, chapter 10]); C then corresponds to the group of symmetries

with center the elation point. Note that for different A, B in \mathcal{F} , we have $A^* \cap B^* = C$. This type of GQ is very general: Each known finite GQ which is not isomorphic to the Hermitian quadrangle $\mathcal{H}(4, q^2)$ for some finite field \mathbb{F}_q , is – modulo point-line duality – either an STGQ, or the Payne-derived quadrangle of an STGQ. More details on this statement can be found in [20]; for a more or less recent but rather detailed census on the known GQs, we also refer to [11].

We recall some basic (“decisive”) classification results in the large theory of STGQs. Much more can be found in [20]. The first result was first obtained by the author in [20] (see [19] for chronological details in its conception). It was also obtained in [1] with virtually the same proof.

Theorem 2.2. *An STGQ of odd order (t, t) is isomorphic to the symplectic quadrangle $\mathcal{W}(t)$.*

The symplectic quadrangle $\mathcal{W}(t)$ arises as the \mathbb{F}_t -rational points in a projective space $\mathbb{P}^3(\mathbb{F}_t)$, together with the absolute lines with respect to a symplectic polarity of $\mathbb{P}^3(\mathbb{F}_t)$ (see [12, chapter 3]); it is one of the natural geometric modules of the group $\mathbf{PSp}_4(t)$.

The most investigated type of STGQ is arguably the class of “flock quadrangles.” All the known flock quadrangles arise through Kantor families in one and the same type of elation group: finite Heisenberg groups of dimension 5 (over a finite field). The next result proves the converse and is taken from [17].

Theorem 2.3. *If (Γ, K) is an EGQ of order (s, t) and K is isomorphic to a Heisenberg group of dimension 5 over \mathbb{F}_t , then Γ is a flock quadrangle.*

The following result is one of the only known results on STGQs of even order (t, t) ; its proof takes up more than half of the paper [1] and is computer-aided.

Theorem 2.4. *An STGQ of order $(8, 8)$ is a translation generalized quadrangle.*

Finally, we have Ott’s beautiful result:

Theorem 2.5 (Ott [8]). *An STGQ (Γ, K) of even order (t, t) is a TGQ if t is not a square. In that case, K is an elementary abelian 2-group.*

3. Notation and setting for Sections 4 and 5

Let K be a finite group with a Kantor family $(\mathcal{F}, \mathcal{F}^*)$ of type (s, t) . The associated GQ with parameters (s, t) is denoted Γ^x , or Γ , where the elation point is x . The Kantor family is defined relative to the point $z \rightsquigarrow x$ (unless we mention it otherwise). In Section 4 and Section 5, we assume the main assumptions of Frohardt’s Theorem 1.1 — this means that for certain distinct elements $A, B \in \mathcal{F}$, we suppose that $K/(A^* \cap B^*)$ is abelian and that $A^* \cap B^* \leq Z(K)$, the center of K . Put $A^* \cap B^* =: \mathcal{S}$. We want to work solely in Theorem 1.1(3) in Sections 4 and 5 so that we furthermore assume the following properties:

The elements of \mathcal{F} are elementary abelian, and $Z(K) =: Z$ has exponent 4.

In particular, this means that Γ is not a TGQ (and hence K is not abelian). On the other hand, we do know that s and t are powers of 2 as a direct consequence of Theorem 1.1.

4. Preliminary properties and reduction to central STGQs of even order t

In this section, we observe some initial properties which narrow down the possibilities for elements in Theorem 1.1(3). In the final part of the proof of the main theorem, we will not need all these properties, but it is interesting to see how far one can go “synthetically.”

Below, we say that a group $G \leq K$ satisfies (*) if for any element α of G , we have that if α fixes some point $y \sim x \neq y$, α also fixes xy pointwise. We say that $G \leq K$ satisfies (*) at $U\mathbf{I}x$ if this property is locally fulfilled at U .

First, note that for any line $W\mathbf{I}x$, K acts transitively on $W \setminus \{x\}$. As $\mathcal{S} \leq Z(K)$, it hence follows that \mathcal{S} fixes $[A]$ and $[B]$ pointwise (so \mathcal{S} is in the kernel of the action of K on the point set of $[A]$, and $[B]$). As K/\mathcal{S} is abelian and transitive on both $[A] \setminus \{x\}$ and $[B] \setminus \{x\}$, we have that K has (*) at $[A]$ and $[B]$.

Let $\eta \in Z^\times$; if η would fix an affine line (which is by definition a line not incident with x), η is in some conjugate of an element of \mathcal{F} , implying that $\eta^2 = 1$. Assume that η does not fix an affine line. Now, suppose there is an affine line U such that U^η does not intersect U . Take any element $V \in \{U, U^\eta\}^\perp$ which is not incident with x ; choose the element β in K_V which sends U^η to U ; then $\eta\beta$ fixes U and $[\eta, \beta] = \text{id}$, so $\text{id} = (\eta\beta)^2 = \eta^2\beta^2 = \eta^2$. So we may suppose that η fixes each point of x^\perp , that is, η is a symmetry with center x . In its turn, this implies that $\eta \in \mathcal{S}$.

4.1. Reduction of parameters

In this subsection, we will show that $s = t$, that is, the number of points incident with a line equals the number of lines incident with a point.

Now, suppose that η is a (nontrivial) symmetry with center x , and suppose that $U \sim [A]$ is an affine line. Let $\alpha \in A^*$ be an element which sends U^η to U such that $\eta\alpha \neq \text{id}$ (and note that such α exist, of course). Then $\eta\alpha$ fixes U . As $[\eta, \alpha] = \text{id}$, η is an involution if and only if α is an involution. As $(\mathcal{F}, \mathcal{F}^*)$ is a Kantor family, we can write $A^* = AS$ (with $A \cap \mathcal{S} = \{\text{id}\}$). We can write $\alpha = \eta^{-1}a$, with $a \in K_U$ so that the fixed points structure of α is that of a – precisely the points incident with $[A]$. Suppose α does not fix affine lines. Applying Theorem 2.1, we conclude

$$(t + 1)(s + 1) + st \equiv st + 1 \pmod{(s + t)}, \tag{5}$$

which gives a contradiction unless $s = t$ since s and t are powers of 2. In the other cases, α must fix affine lines, so that it is an involution, and hence η also is. So if $s \neq t$, each element of Z is an involution, showing that case (3) of Frohardt’s theorem cannot occur under these assumptions. **It follows that $s = t$.**

4.2. The centralizers $C_B(\alpha)$

Now, consider the case where $s = t$ and t is even. Note that \mathcal{S} only consists of symmetries and involutions. Suppose $\beta \in \mathcal{S}^\times$ is not an involution (it is then a symmetry); compose β with a hypothetical involution ι in Z which is not a symmetry. Then $\beta\iota \in Z$ is not a symmetry, so it is an involution, and $\text{id} = (\beta\iota)^2 = \beta^2\iota^2 = \beta^2$, a contradiction. So all involutions in \mathcal{S} are symmetries, and whence all elements of \mathcal{S} are symmetries. It follows easily that x is a regular point.

If K is abelian, the theory of translation GQs gives us that K is elementary abelian (see [14, chapter 3]), so we may assume that K is not abelian. **By Proposition 3.1 of Hachenberger [7], it follows that $Z(K) = \mathcal{S}$.**

Let $C \neq D$ be in \mathcal{F} ; then $[K, K] = [C, D]$ (as $K = CDS$) = $\langle (cd)^2 \mid c \in C, d \in D \rangle$. As this group is a subgroup of \mathcal{S} , it follows that $[K, K]$ is elementary abelian. As $\Phi(K) = K^2[K, K]$ (where $\Phi(K)$ denotes the Frattini subgroup of K), it easily follows that $\Phi(K)$ is the elementary abelian subgroup of \mathcal{S} of all squares. Then on the other hand, if $s = \gamma^2 \in \mathcal{S}$ is a square, there is an $e \in E \in \mathcal{F}$ and $c \in \mathcal{S}$ such that $\gamma = ec$ (as $K = \cup_{A \in \mathcal{F}} A^*$), and $s = ecec = c^2$. So $\Phi(K) = \mathcal{S}^2$. Also, for $A \in \mathcal{F} \setminus \{E\}$, there is a unique $B \in \mathcal{F}$ such that $\gamma \in AE$ — write γab with $a \in A$ and $b \in B$. Then $(ab)^2 = \gamma^2 = s$ so that $s \in [K, K]$. **We have obtained that $[K, K] = \mathcal{S}^2 = \Phi(K)$.**

Let $A \neq B$ be arbitrary in \mathcal{F} . Now, let $\alpha \in A^\times$ be arbitrary, and let $\beta \in B^\times$ be such that $[\alpha, \beta] = \text{id}$. Put $\mathcal{F} \setminus \{A, B\} = \{C_1, \dots, C_{t-1}\}$. Then for each $i \in \{1, \dots, t - 1\}$ there is precisely one triple $(c_i, \beta_i, s_i) \in C_i \times B \times \mathcal{S}$ such that $\alpha\beta = c_i\beta_i s_i$. Note that the maps $\mu : \{1, \dots, t - 1\} \rightarrow \mathcal{S}^\times : j \rightarrow s_j$ and $\mu' : \{1, \dots, t - 1\} \rightarrow B \setminus \{\beta\} : j \rightarrow \beta_j$ are surjective. Suppose that $s_e^2 = \text{id}$; then as $(\alpha\beta)^2 = \text{id}$, $[c_e, \beta_e] = \text{id}$, and so $[\beta_e, c_e\beta_e s_e] = \text{id} = [\beta_e, \alpha\beta]$, implying that $[\alpha, \beta_e] = \text{id}$. The converse is also true. So

$$|\text{involutions in } \mathcal{S}|(\text{including id}) = |C_B(\alpha)| =: \ell. \tag{6}$$

On the other hand, we have $t = |\alpha^B| \times |C_B(\alpha)|$, so for any such α , the size of α^K (which is the same as α^B) is a constant ($\frac{t}{\ell}$) independent of the choice of α . As $\alpha^K = \alpha^B$, it also follows that α fixes precisely $|C_B(\alpha)| = \ell$ points of $[A] \setminus \{x\}$ linewise.

Note at this point that if we prove that \mathcal{S} is an elementary abelian 2-group, then each A^* is also elementary abelian, and as K is covered by the A^* s, it follows that K itself is elementary abelian, contradiction with the assumption that K has exponent 4.

In the rest of this paper, we will denote the subgroup of \mathcal{S} that consists of all its involutions by \mathcal{I} .

4.3. Extra info on \mathcal{I} , and ℓ

Keeping the same notation as before, we recall that $[K, K] = [C, D] = \{(cd)^2 | c \in C, d \in D\}$. Each $(cd)^2 = cdcd = [c, d]$ is an element in \mathcal{S} which obviously is an involution if it is nontrivial, so $[K, K] \leq \mathcal{I}$. Also, note that the surjective morphism

$$\omega : \mathcal{S} \mapsto \mathcal{S}^2 : s \mapsto s^2 \tag{7}$$

has kernel \mathcal{I} , so that $\mathcal{S}^2 \cong \mathcal{S}/\mathcal{I}$. It follows that $|[K, K]| = |\mathcal{S}^2| = t/\ell$. We conclude that

$$\ell = |\mathcal{I}| \geq |[K, K]| = \frac{t}{\ell}, \tag{8}$$

so that $\ell \geq \sqrt{t}$.

5. Solution of Frohardt’s problem

Recall that we suppose that Γ is not a TGQ. For this part of the paper, we will invoke a result of Ott [8]. On page 413 of his paper (Proposition 4.2), Ott considers a maximal subgroup M of $\Phi(K)$ (in his paper, the notation G is used for our K and U_0 is used for \mathcal{S}). Then [8, Corollary 4.5, p. 414] says that $|M| = t/4$. In this section, we only will use that observation of his paper. We do want to obtain a full solution of Frohardt’s problem, so we cannot rely on the main result of [8]: There, the case that t is a square is excluded, and we want to handle that too. Besides, after [8, Corollary 4.5, p. 414], the proof of the main result still takes about 20 pages, and we can rely on Equation (8) instead to obtain a very short approach to Frohardt’s problem.

So let $|M| = t/4$; then as a maximal subgroup of the 2-group $\Phi(K)$, it has index 2 so that $|\Phi(K)| = t/2$. But as $\Phi(K) = [K, K]$, Equation (8) tells us that $|\Phi(K)| = t/\ell$ so that $\ell = 2$. As $\sqrt{t} \leq \ell$ by the previous section, it follows that $t \leq 4$. For the cases $t = 2$ and $t = 4$, it is well known that only the classical quadrangles $\mathcal{W}(2)$ and $\mathcal{W}(4)$ occur (see [12, chapter 6]), which means that Γ is a TGQ.

This shows that the class (3) in Frohardt’s Theorem 1.1 is empty, as all hypothetical members are TGQs, implying that K is elementary abelian.

6. STGQs of even order t , t not a square and cyclic STGQs

In this section, we will classify finite cyclic STGQs of order (t, t) . As a corollary, we will give a short geometric proof of Ott’s main result [8]. In the latter case, we will furthermore assume that t is even and not a square (but we will be clear about the assumptions in each part of the section).

We will also rely on one extra (nontrivial) result of Ott’s paper [8], namely the fact that $\Phi(K) = \Phi(\mathcal{S})$ if Γ is not a TGQ (under the assumptions of [8, Theorem 4.11]). It is our goal to give a very short alternative geometric approach to what follows after Theorem 4.11 in Ott’s paper (which is a long argument of 12 pages). Since we invoke that theorem, we also need to assume that t is not a square. Instead of “directly” obtaining the result, we choose a more interesting approach: We will completely classify cyclic STGQs of order (t, t) , cf. Theorem 6.2 (without any restrictions on t).

The following lemma reduces Ott’s theorem to a problem on cyclic STGQs.

Lemma 6.1. *Let (Γ^x, K) be an STGQ of order (t, t) with t even, and t not a square. Let \mathcal{S} be the group of symmetries with center x . Then we have that \mathcal{S} is cyclic.*

Proof. As the group M of the previous section has size $t/4$ and is a maximal subgroup of $\Phi(K)$ (cf. Ott’s paper), we have that $|\Phi(K)| = t/2$, and so $|\Phi(\mathcal{S})| = t/2$. As $[\mathcal{S} : \Phi(\mathcal{S})] = 2$, we have that \mathcal{S} is generated by one element (a finite p -group P is cyclic if and only if the Frattini quotient $P/\Phi(P)$ is cyclic). □

Angles in general STGQs

Let $(\tilde{\Gamma}, \tilde{K})$ be a general (not necessarily finite) STGQ with elation point x and associated group of symmetries (with center x) \mathcal{S} . Let $(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}^*)$ be the associated Kantor family, and let $A \in \tilde{\mathcal{F}}$ be arbitrary. Let A fix the line $U \sim [U]\mathbf{I}x$, where U is not incident with x . Suppose that A fixes each point incident with $[U]$. Define the *angle spectrum* of each element $a \in A^\times$ as follows. Let $V \sim [U]$, with V not incident with x ; then the *angle* of a at V is the unique element s of \mathcal{S} for which $V^a = V^s$. Then the *angle spectrum* of a is the set of all angles of a at V , where V varies over the lines in $[U]^\perp$ which are not incident with x .

We now prove the following theorem (in whose statement we do not require extra assumptions for t).

Theorem 6.2 (Cyclic STGQs). *Suppose (Γ, K) is an STGQ of order t , and suppose that \mathcal{S} is cyclic. Then $t = p$ is a prime and $\Gamma \cong \mathcal{W}(p)$.*

Proof. For t odd, we already know that $\Gamma \cong \mathcal{W}(t)$ and as \mathcal{S} is elementary abelian in this case, it follows that t is prime (in which case $\mathcal{S} \cong C_t$).

So now suppose that $s = t$ is even, with $t = 2^h$, and put $\mathcal{S} = \langle \alpha \rangle$. We suppose that $t \geq 4$. Let $y \neq x$ be some fixed point of Γ , and if $U\mathbf{I}y$, let $x_U := [U] \cap U$, where $[U]$ is the projection of U on x . Define the elements of the Kantor family of (Γ, K) as $\mathcal{F} = \{K_U \mid U\mathbf{I}y\}$ and $\mathcal{F}^* = \{K_{x_U} \mid U\mathbf{I}y\}$. Let $A \in \mathcal{F}$ be arbitrary, with $A = K_U$ and $U\mathbf{I}y$, and let A act on \mathcal{S} by conjugation. Define $N(A)$ to be the kernel of this action, and note that each element of $N(A)$ commutes with \mathcal{S} and hence fixes x_U linewise. Then $A/N(A)$ is a subgroup of the automorphism group of \mathcal{S} , which is well known to be isomorphic to $C_2 \times C_{2^{h-2}}$. But as $A/N(A)$ is elementary abelian, this implies that

$$A/N(A) \leq C_2 \times C_2, \tag{9}$$

where the latter is the 2-torsion subgroup of $C_2 \times C_{2^{h-2}}$. It follows that $|N(A)| \in \{t/4, t/2, t\}$.

Let $U, x_U, [U], A = K_U$ be as before, with U not incident with x arbitrary; then $N(A) \leq A$ has size $t, t/2$ or $t/4$. Now, let $y \neq x_U$ be any other point on $[U]$ different from x . □

Any element of $N(A)^\times$ fixes $t/2 + 1$ points on $[U]$ linewise, and all angles are given by id or $\alpha^{t/2}$

Let $a \in N(A)^\times$. As a fixes the point x_U linewise, it commutes with every element in \mathcal{S} . Suppose β is an angle of a ; then βa fixes a line (not incident with x), so it is an involution. It follows that $(\beta a)^2 = \beta^2 = \text{id}$. So β is the unique involution in \mathcal{S} , $\alpha^{t/2}$. Now, suppose a fixes λ points on $[U] \setminus \{x\}$ linewise, so that $t - \lambda$ points remain in $[U] \setminus \{x\}$ which are all incident with precisely one fixed line (which is $[U]$). If we would have that $\lambda < t/2$, then $t/2 < t - \lambda < t$. Considering the element $a\alpha^{t/2}$, we see that it fixes $t - \lambda$ points in $[U] \setminus \{x\}$ linewise. As we will observe in the next paragraph, this is a contradiction as $t - \lambda$ has to be a power of 2.

The map ζ

Now, let $a \in N(A)^\times$ be arbitrary; then a fixes $t/2$ points on $[U]$ (besides x) linewise. Call this point set $\mathcal{B}(a)$. Let $B \neq A$ in \mathcal{F} ; then it is easy to see that $|C_B(a)| = t/2$, and the set $\mathcal{B}(a)$ can be identified with a unique subgroup of B (which acts sharply transitively on $\mathcal{B}(a)$). As B is an elementary abelian 2-group (of order $t = 2^h$), we can associate an $(h - 1)$ -dimensional projective space over \mathbb{F}_2 with B , which we

denote by $\Pi(B)$; we then have that $\mathcal{B}(a)$ defines a hyperplane $\Pi(a)$ in $\Pi(B)$. Consider the map

$$\zeta : N(A) \mapsto \Pi(B) : a \mapsto \Pi(a), \tag{10}$$

and note that $\zeta(\text{id}) = \Pi(B)$.

For now, we suppose that ζ is an injection (if it is not, we will later easily show that a contradiction arises).

Note that the dual spaces of $\zeta(N(A)^\times)$ are contained in (in fact, form) a subspace of $\Pi(B)^* \simeq \Pi(B)$ (where $\Pi(B)^*$ is the dual space of $\Pi(B)$). If a and b are in $N(A)^\times$, then $ab = c$ is such that $\zeta(c)$ is the third hyperplane on $\zeta(a) \cap \zeta(b)$; in the dual space $\Pi(B)^*$, $\zeta(c)^*$ is the third point of the line $\zeta(a)^* \zeta(b)^*$. It follows that $\zeta(N(A)^\times)^* \simeq \zeta(N(A)^\times)$ is a subspace of $\Pi(B)^* \simeq \Pi(B)$.

Subgroups of $N(A)$ which fix at least one arbitrary point $x' \neq x_U, x$ linewise have at least size $t/8$

Let A be arbitrary in \mathcal{F} as before, and keep using the same notation. Let $x' \neq x_U$ be arbitrary in $[U] \setminus \{x\}$. Then the subgroup of A of elements which fix both x_U and x' linewise has at least size $t/8$. This follows from properties of the map ζ as we will see below. Since the dual spaces of the elements in $\zeta(N(A)^\times)$ form a subspace of $\Pi(B)^* \simeq \Pi(B)$, it follows that precisely one extra point \tilde{x} in $[U] \setminus \{x_U, x\}$ is fixed linewise by $N(A)$ if $|N(A)| = t/2$ ($\zeta(N(A)^\times)^*$ then is a hyperplane), or precisely three extra points $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ when $|N(A)| = t/4$ ($\zeta(N(A)^\times)^*$ then is a subspace of codimension 2). Suppose that x' is not one of the extra points. We distinguish three cases. Below, we put $t = 2^h$ so that $\Pi(B)$ is a projective \mathbb{F}_2 -space of dimension $h - 1$.

- (0) $|N(A)| = t$. Since all angles are in $\{\text{id}, \alpha^{t/2}\}$, the number of elements in $N(A)^\times$ which fix x_U and x' linewise is at least $t/2$. (Let $\forall x', V \neq [U]$; then $|N(A)_V| \leq 2$.)
- (1) $|N(A)| = t/2$. In this case, all elements of $\zeta(N(A)^\times)$ contain one fixed point r in $\Pi(B)$ (and all hyperplanes in $\Pi(B)$ containing that point are contained in $\zeta(N(A)^\times)$). The number of elements in $N(A)^\times$ which fix x_U and x' linewise is the number of sets $\mathcal{B}(a)$ ($a \in N(A)^\times$) which contain besides r a second point r' , and this equals the number of hyperplanes in $\Pi(B)$ which contain two different points, hence a line: $2^{h-2} - 1 = t/4 - 1$.
- (2) $|N(A)| = t/4$. In this case, all elements of $\zeta(N(A)^\times)$ contain one fixed line R in $\Pi(B)$ (and all hyperplanes in $\Pi(B)$ containing that line are contained in $\zeta(N(A)^\times)$). The number of elements in $N(A)^\times$ which fix x_U and x' linewise is the number of sets $\mathcal{B}(a)$ ($a \in N(A)^\times$) which contain both x_U and x' , and this equals the number of hyperplanes in $\Pi(B)$ which contain a given point and an extra line R , hence a plane: $2^{h-3} - 1 = t/8 - 1$.

Alternative proof using angles

We have seen that all elements in $N(A)$ only allow angles contained in $\{\text{id}, \alpha^{t/2}\}$. Take any line Y on x' different from $[U]$. Then $|N(A)_Y| \geq |N(A)|/2$ since at most one nontrivial angle can occur. And since $[N(A), S] = \{\text{id}\}$, it also follows that $N(A)_Y$ fixes all lines on x' . This gives us the same cases (0), (1), (2) as above.

In the rest of the proof, we suppose that $t \geq 16$ (if $t = 8$, we refer to [1]).

Each element in A , where $A \in \mathcal{F}$ is arbitrary, has an angle spectrum which is a subset of $\{\text{id}, \alpha^{t/2}\}$; as a consequence, we have that $[k, a] \in \{\text{id}, \alpha^{t/2}\}$ for all $k \in K, a \in A, A \in \mathcal{F}$

First, suppose that $a \in N(A)^\times$. Then we already know that only the angles id and $\alpha^{t/2}$ can occur. Now, let $k \in K^\times$ be arbitrary, and consider V^k . If a fixes V^k , we have that $[k, a] = \text{id}$ since it is a symmetry with center x which fixes V^k ; if a does not fix V^k , we have that $[k, a] = \alpha^{t/2}$ since it is a symmetry with center x which sends V^k to $(V^k)^{\alpha^{t/2}}$. So if $N(A) = A$, the statement is already proved.

Suppose that $a \in A^\times \setminus N(A)^\times$; then a induces (by conjugation) a nontrivial involutory automorphism of \mathcal{S} . Let $\gamma \neq \text{id}$ be an angle of a . Then γa fixes a line not incident with x , so that it is an involution. Then $\gamma a \gamma a = \text{id}$, so that $\gamma^a = \gamma^{-1}$. If a and γ commute, it follows that γ is an involution, so $\gamma = \alpha^{t/2}$. Now, suppose a and γ do not commute. Consider the line $U \perp x_U$, $U \neq [U]$, which is supposed to be fixed by a . We identify the action of a on \mathcal{S} with the action of a on the lines incident with x_U and different from $[U]$ (which we can do since \mathcal{S} acts sharply transitively on these lines: If $W \perp x_U$ and $W \neq [U]$ with $W = V^\alpha$, then we identify W with $\alpha \in \mathcal{S}$). It is easy to calculate that the only nontrivial involutions in $\text{Aut}(\mathcal{S})$ are given by the following images of α :

$$\begin{aligned} \nu_1 : \alpha &\mapsto \alpha^{t-1}; \\ \nu_2 : \alpha &\mapsto \alpha^{t/2+1}; \\ \nu_3 : \alpha &\mapsto \alpha^{(t-1)(t/2+1)} = \alpha^{t/2-1}. \end{aligned}$$

In the first case, ν_1 fixes precisely two elements (id and $\alpha^{t/2}$) of \mathcal{S} ; in the second case $t/2$ elements (all α^n with n even), and in the last case again two elements (id and $\alpha^{t/2}$). If a is of the first type or the third, at most two lines different from $[U]$ are fixed per point in $[U] \setminus \{x\}$ (in fact, one can easily show that it is *precisely* two per such point, but we do not need this equality). If a is of the second type, it fixes precisely $t/2$ lines incident with x_U and different from $[U]$ (and many more on other points as well, as we will later see).

Now, suppose $a \in A^\times$ is of type 1 or type 3. Suppose a fixes the line U (not incident with x), and let $u^a = v$ with $u \perp U$, u not collinear with x . We suppose without loss of generality that u “defines the Kantor family” – that is, if $W \perp u$, then K_W is an element of \mathcal{F} (and in particular, $K_U = A$). For any such $W \neq U$, if $B = K_W$ in \mathcal{F} and $[W]$ is the line incident with x which meets W , define $U(B) = W$ as the line incident with u which meets $[W]$, and let $V(B)$ be the line incident with v which meets $[W]$. Let $u_B := U(B) \cap [W]$ and $v_B := V(B) \cap [W]$. Then as we have seen in a previous step, we have that $B_{\{u_B, v_B\}}$, the subgroup of B which fixes u_B and v_B linewise, has size at least $t/8$. Note that $U^{B_{\{u_B, v_B\}}}$ consists of lines in $\{[U], U(B), V(B)\}^\perp$. As a consequence, since a maps u to v and fixes all points incident with $[U]$, it follows that a fixes all lines in $U^{B_{\{u_B, v_B\}}}$. Letting B vary in $\mathcal{F} \setminus \{A\}$, we see that a has at least $t(t/8 - 1) + 1$ fixed lines which are not collinear with x . Supposing that $t \geq 32$ (the case $t = 16$ will be handled separately below), we see that type 1 and 3 are indeed not possible (since such elements would fix at most $2t$ such lines).

Finally, suppose that a is of type 2. Suppose $\gamma \neq \text{id}$ is an angle of a ; then we have seen that $\gamma^a = \gamma^{-1}$. If we inspect the specific form of a and suppose that $\gamma = \alpha^n$, we get that

$$\alpha^{n(t/2+1)} = \alpha^{-n} \tag{11}$$

so that $\alpha^{n(t/2+2)} = \text{id}$. This is only possible if $n = t$ ($\gamma = \text{id}$) or $n = t/2$ ($\gamma = \alpha^{t/2}$). Now, let $k \in K^\times$ be arbitrary, and consider V^k . Similarly as in the case $a \in N(A)^\times$, we have that either $[k, a] = \text{id}$ or $[k, a] = \alpha^{t/2}$.

The claim is proved.

The case $t = 16$

We use the notation of the preceding part of the proof. Suppose again that $a \in A^\times$ is of type 1 or type 3. Suppose that $|N(B)| = t/4$; then B contains elements of every type. Suppose $\beta \in B$ is an element of type 2. We have seen moments ago that for such an element, only angles in $\{\text{id}, \alpha^{t/2}\}$ can arise. Whence the same is true for all elements in $\langle N(B), \beta \rangle$ (which is a group of size $2|N(B)|$). It follows that $|B_{U(B), V(B)}| \geq |\langle N(B), \beta \rangle|/2 = |N(B)|$. Note that $U^{B_{U(B), V(B)}}$ consists of lines in $\{[U], U(B), V(B)\}^\perp$. Hence in the counting argument above, we can actually replace $t/8$ by $t/4$ (such as in the cases $|N(B)| = t/2$ and $|N(B)| = t$), and we obtain that a has at least $t(t/4 - 1) + 1$ fixed lines which are not collinear with x . When $t = 16$, this amount is strictly larger than $2t$, so we have a contradiction. When a is of type 2, the argument is the same as above. So if $k \in K$, we have that either $[k, a] = \text{id}$ or $[k, a] = \alpha^{t/2}$ for any $a \in A$.

We have that $[K, K] \leq C_2$ (up to isomorphism)

This is easy now: Consider any commutator $[u, v]$ with $u, v \in K$; then u and v are products of elements contained in the members of \mathcal{F} . It follows that such a commutator is a product of (conjugates of) elements $[e, f]$, with e, f elements of members in \mathcal{F} . Whence $[u, v]$ is a product of copies of $\alpha^{t/2}$. The claim follows.

We have that $\Gamma \cong W(p)$ with $s = p$ a prime

Let $\alpha \in \mathcal{S}$. Then we can write α as $a_1 a_2 \cdots a_k$ with each a_k contained in some element of \mathcal{F} ; it follows that

$$\alpha^2 = a_1^2 a_2^2 \cdots a_k^2 \ell = \ell, \tag{12}$$

with ℓ a product of commutators, implying that the order of each such α is at most 4. This means that $s = |\mathcal{S}| \leq 4$ as \mathcal{S} is cyclic. By [1], we conclude that $\Gamma \cong \mathcal{W}(s)$ with $s \in \{2, 4\}$. Only in the case $\Gamma \cong \mathcal{W}(2)$, we have that \mathcal{S} is cyclic.

Now, suppose that ζ is not an injection. Let a, \tilde{a} be different nontrivial elements of $N(A)$ for which $\zeta(a) = \zeta(\tilde{a})$. Then as they have the same spectrum, it follows that $a\tilde{a}$ is a symmetry with axis $[U]$. Let S be the “kernel” of ζ : this is the subgroup of symmetries of $N(A)$ with axis $[U]$. Now, $(N(A)/S)^\times$ induces a subspace of $\Pi(B)^*$ (instead of $N(A)^\times$). Putting $|N(A)| = 2^{h-m}$, $m \in \{0, 1, 2\}$ and $|\mathcal{S}| = 2^k$, and using the same notation as before, we obtain that $2^{h-m-k-1} - 1$ elements of $(N(A)/S)^\times$ fix x_U and x' linewise (note that elements of $N(A)/S$ act naturally on the lines in $[U]^\perp$). As each element of S also fixes x_U and x' linewise, we obtain that $(2^{h-m-k-1} - 1)2^k + 2^k - 1 = 2^{h-m-1} - 1$ nontrivial elements of $N(A)$ fix x_U and x' linewise, which is the exact same result as in the injective case. Note that the short alternative approach using angles also works here! Now, the rest of the proof is the same. It follows that $\Gamma \cong \mathcal{W}(2)$.

The proof is complete.

Theorem 6.2 immediately leads to a very short proof of Ott’s result:

Corollary 6.3 (Ott’s result). *Let (Γ, K) be an STGQ of even order t . If t is not a square, then Γ is a TGQ and K is elementary abelian.*

Proof. Suppose Γ is not a TGQ, so that K is not elementary abelian. Then by Lemma 6.1 we know that \mathcal{S} is cyclic. So by Theorem 6.2, we conclude that $\Gamma \cong \mathcal{W}(2)$, a contradiction (since that is a TGQ). □

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